Filomat 31:19 (2017), 6015–6019
https://doi.org/10.2298/FIL1719015M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The DMP Inverse for Rectangular Matrices

Lingsheng Meng[a]

[a]*College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, PR China*

Abstract. The definition of the DMP inverse of a square matrix with complex elements is extended to rectangular matrices by showing that for any A and W, m by n and n by m, respectively, there exists a unique matrix X, such that

$$XAX = X,\ XA = WA_{d,w}WA \ \text{ and } \ (WA)^{k+1}X = (WA)^{k+1}A^{\dagger},$$

where $A_{d,w}$ denotes the W-weighted Drazin inverse of A and $k = \mathrm{Ind}(AW)$, the index of AW.

1. Introduction

Let $C^{m\times n}$ denotes the set of complex $m \times n$ matrices and $C_r^{m\times n}$, the subset of all rank r matrices in $C^{m\times n}$. The symbols A^*, $\mathbf{R}(A)$ and $\mathbf{N}(A)$ respectively stand for the conjugate transpose, the column space and null space of a matrix $A \in C^{m\times n}$. As usual, I_m denotes the identity matrix of order m. Moreover, $P_{L,M}$ denotes the projector onto L along M, where L and M are two complementary subspaces of C^n. For a given matrix $A \in C^{n\times n}$, this notation will be reduced to P_A when $L = \mathbf{R}(A)$ and M is the subspace orthogonal to L.

The Moore-Penrose inverse of a matrix $A \in C^{m\times n}$ is the unique matrix $A^{\dagger}$ satisfying the four equations:

$$AA^{\dagger}A = A,\ A^{\dagger}AA^{\dagger} = A^{\dagger},\ (AA^{\dagger})^* = AA^{\dagger},\ (A^{\dagger}A)^* = A^{\dagger}A,$$

as described by Penrose [9]. A matrix X that satisfies the equality $AXA = A$ is called a g-inverse of A and if X satisfies $XAX = X$, it is called an outer inverse of A.

For each square matrix $A \in C^{n\times n}$, the index of A, written as $\mathrm{Ind}(A)$, is the smallest non-negative integer k for which $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$. The Drazin inverse of a square matrix A, denoted by A^D, is the unique matrix satisfying the following equations:

$$A^kXA = A,\ \ XAX = X \ \text{ and } \ AX = XA,$$

where $k = \mathrm{Ind}(A)$. In particular, if the $\mathrm{Ind}(A) \leq 1$, the Drazin inverse is called the group inverse $A^{\#}$. In [5], Cline and Greville extended the Drazin inverse of square matrix to rectangular matrix. If $A \in C^{m\times n}$ and $W \in C^{n\times m}$, then $X = \left((AW)^D\right)^2 A \in C^{m\times n}$ is the unique solution to the equations:

$$(AW)^{k+1}XW = (AW)^k,\ \ XWAWX = X,\ \ AWX = XWA,\ k = \mathrm{Ind}(AW). \tag{1.1}$$

2010 *Mathematics Subject Classification*. 15A09
Keywords. DMP inverse, W-weighted Drazin inverse, Moore-Penrose inverse
Received: 22 March 2017; Accepted: 27 April 2017
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No. 11701458) and the Science and Technology Project of Northwest Normal University (No. NWNU-LKQN-16-15)
Email address: menglsh@nwnu.edu.cn (Lingsheng Meng)

The matrix X is called the W-weighted Drazin inverse of A and is written as $A_{d,w}$.

Recently, Malik and Thome [6] defined a new generalized inverse, namely the DMP inverse, of a square matrix $A \in C^{n\times n}$ of an arbitrary index. The DMP inverse of $A \in C^{n\times n}$, denoted by $A^{D,\dagger}$ is defined to be the matrix $A^{D,\dagger} = A^D AA^\dagger$, which is the unique solution of the following equations:

$$XAX = X, \;\; XA = A^D A, \;\; A^k X = A^k A^\dagger, \quad k = \text{Ind}(A).$$

Specially, if $\text{Ind}(A) = 1$, the DMP inverse is reduced to the core inverse $A^{\ominus}$ (see [1]), which is the unique matrix satisfying $AX = P_A$ and $R(X) \subseteq R(A)$.

The Moore-Penrose inverse, the Drazin inverse and the DMP inverse of a matrix always exist, while the group inverse as well as the core inverse of a square matrix A exist if and only if $\text{Ind}(A) = 1$. We refer the readers to [1, 2, 11] for basic results on these generalized inverses. All of these generalized inverses are known to be used in important applications. For example, the Moore-Penrose inverse is used to solve the least-squares problems, the group inverse has applications in Markov chain theory, the Drazin inverse has applications in singular differential equations and iterative methods, and the core inverse has applications in partial order theory (see for example [2–4, 7, 8, 10, 11]).

Motivated by the extension of Drazin inverse to W-weighted Drazin inverse, we extend the DMP inverse of square matrix to rectangular matrix in this paper. First, a canonical form for the new generalized inverse is established. Second, the equivalence of the algebraic definition (Definition 2.1) and the geometrical approach (Theorem 3.2) has been stated. We finally give some of the properties that this new generalized inverse possesses.

2. The W-weighted DMP Inverse of A

In this section, we give the definition of W-weighted DMP inverse of rectangular matrix $A \in C^{m\times n}$ and present the canonical form of it by using the singular value decompositions of A and $W \in C^{n\times m}$.

Theorem 2.1. *Let $A \in C^{m\times n}$ and $W \in C^{n\times m}$, then the matrix $X = WA_{d,w}WAA^\dagger$ is the unique solution to the equations*

$$XAX = X, \; XA = WA_{d,w}WA \;\text{ and }\; (WA)^{k+1}X = (WA)^{k+1}A^\dagger, \tag{2.1}$$

where $k = Ind(AW)$.

Proof. From the definition of $A_{d,w}$, we have

$$WA_{d,w}WAA^\dagger AWA_{d,w}WAA^\dagger = WA_{d,w}WAA^\dagger,$$

$$WA_{d,w}WAA^\dagger A = WA_{d,w}WA$$

and

$$(WA)^{k+1}WA_{d,w}WAA^\dagger = W(AW)^{k+1}A_{d,w}WAA^\dagger = (WA)^{k+1}A^\dagger,$$

i.e., $WA_{d,w}WAA^\dagger$ satisfies the three equations in (2.1).

To show uniqueness, suppose both X_1 and X_2 are two solutions to (2.1). Then using repeated applications of the three equations in (2.1) and (1.1), we have

$$\begin{aligned} X_1 &= X_1AX_1 = WA_{d,w}WAX_1 = (WA_{d,w})^2(WA)^2X_1 \\ &= \cdots = (WA_{d,w})^{k+1}(WA)^{k+1}X_1 = (WA_{d,w})^{k+1}(WA)^{k+1}A^\dagger \\ &= (WA_{d,w})^{k+1}(WA)^{k+1}X_2 = WA_{d,w}WAX_2 = X_2AX_2 = X_2. \end{aligned}$$

We have just finished the proof. □

It should be noted in Theorem 2.1 that $A^{D,\dagger} = A^D AA^\dagger = WA_{d,w}WAA^\dagger$ when A is square and $W = I_n$. In view of the correspondence between the defining equations for $A^{D,\dagger}$, and those in Theorem 2.1, we define the DMP inverse of a rectangular matrix in the following manner:

Definition 2.2. *For any matrices A and W, m by n and n by m, respectively, the matrix $X = WA_{d,w}WAA^\dagger$ is called the W-weighted DMP inverse of A, and is written as $X = A_w^{D,\dagger}$.*

Remark 2.3. *Obviously, when $A \in C^{m\times m}$ and $W = I_m$, then $A_w^{D,\dagger}$ reduces to $A^{D,\dagger}$. When $A \in C^{m\times m}$, $W = I_m$ and $Ind(A) = 1$, then $A_w^{D,\dagger}$ reduces to the core inverse of A. Moreover, when A is a nonsingular square matrix and $W = I_m$, then $A_w^{D,\dagger} = A^{-1}$.*

We now give the canonical form for the W-weighted DMP inverse of A by using the singular value decompositions of A and W. Let $A \in C_r^{m\times n}$, $W \in C_s^{n\times m}$ respectively have the following singular value decompositions:

$$A = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}V^* \text{ and } W = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix}\tilde{U}^*, \tag{2.2}$$

where $U = (U_1, U_2)$, $\tilde{U} = (\tilde{U}_1, \tilde{U}_2) \in C^{m\times m}$ and $V = (V_1, V_2)$, $\tilde{V} = (\tilde{V}_1, \tilde{V}_2) \in C^{n\times n}$ are unitary matrices, $U_1 \in C^{m\times r}$, $\tilde{U}_1 \in C^{m\times s}$, $V_1 \in C^{n\times r}$, $\tilde{V}_1 \in C^{n\times s}$, $\Sigma_1 = diag(\sigma_1, \cdots, \sigma_r)$, $\tilde{\Sigma}_1 = diag(\tilde{\sigma}_1, \cdots, \tilde{\sigma}_s)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_s > 0$. After a series of complicated calculation, we can get the following theorem.

Theorem 2.4. *Let $A \in C^{m\times n}$ and $W \in C^{n\times m}$ have the singular value decompositions (2.2). Then*

$$A_w^{D,\dagger} = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 S_{11}\Lambda\tilde{\Sigma}_1 S_{11} & 0 \\ 0 & 0 \end{pmatrix}U^*,$$

where $S_{11} = \tilde{U}_1^* U_1$ and $\Lambda = (\Sigma_1 T_{11})_{d,\tilde{\Sigma}_1 S_{11}}$ with $T_{11} = V_1^*\tilde{V}_1$. Here $(\Sigma_1 T_{11})_{d,\tilde{\Sigma}_1 S_{11}}$ denotes the $\tilde{\Sigma}_1 S_{11}$-weighted Drazin inverse of matrix $\Sigma_1 T_{11}$.

Proof. Denote $S = \tilde{U}^* U$, $T = V^*\tilde{V}$, and assume that S and T have the following block forms

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \text{ and } T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $S_{11} \in C^{s\times r}$, $T_{11} \in C^{r\times s}$, then $S^*S = I_m$ and $T^*T = I_n$, i.e., S and T are unitary matrices. Then, we have

$$A = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}V^* = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}V^*\tilde{V}\tilde{V}^* = U\begin{pmatrix} \Sigma_1 T_{11} & \Sigma_1 T_{12} \\ 0 & 0 \end{pmatrix}\tilde{V}^*$$

and

$$W = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix}\tilde{U}^* = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix}\tilde{U}^* UU^* = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 S_{11} & \tilde{\Sigma}_1 S_{12} \\ 0 & 0 \end{pmatrix}U^*.$$

Let $X = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\tilde{V}^*$ be the W-weighted Drazin inverse of A, where $X_1 \in C^{r\times s}$. From $AWX = XWA$, $XWAWX = X$ and $(AW)^{k+1}XW = (AW)^k$, we can get the following equalities after some tedious manipulation

$$X_1 = X_1\tilde{\Sigma}_1 S_{11}\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11}X_1, \;\; X_1\tilde{\Sigma}_1 S_{11}\Sigma_1 T_{11} = \Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11}X_1,$$
$$X_2 = X_1\tilde{\Sigma}_1 S_{11}X_1\tilde{\Sigma}_1 S_{11}\Sigma_1 T_{12}, \;\; X_3 = 0, \;\; X_4 = 0,$$
$$(\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11})^{k+1}X_1\tilde{\Sigma}_1 S_{11} = (\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11})^k$$

and

$$(\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11})^{k+1}X_1\tilde{\Sigma}_1 S_{12} = (\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{11})^{k-1}\Sigma_1 T_{11}\tilde{\Sigma}_1 S_{12}.$$

These equations show that the W-weighted Drazin inverse of A is the matrix

$$A_{d,w} = U\begin{pmatrix} \Lambda & \Lambda\tilde{\Sigma}_1 S_{11}\Lambda\tilde{\Sigma}_1 S_{11}\Sigma_1 T_{12} \\ 0 & 0 \end{pmatrix}\tilde{V}^*,$$

where $\Lambda = (\Sigma_1 T_{11})_{d,\tilde{\Sigma}_1 S_{11}}$.

Furthermore, it is easy to check that

$$A^\dagger = \tilde{V}\begin{pmatrix} T_{11}^*\Sigma_1^{-1} & 0 \\ T_{12}^*\Sigma_1^{-1} & 0 \end{pmatrix}U^*.$$

Thus, we have

$$A_w^{D,\dagger} = WA_{d,w}WAA^\dagger = \tilde{V}\begin{pmatrix} \tilde{\Sigma}_1 S_{11}\Lambda\tilde{\Sigma}_1 S_{11} & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

The proof of this theorem is now complete. □

3. Properties of the W-weighted DMP Inverse

In this section, we study the properties of the W-weighted DMP inverse.

Theorem 3.1. *Let $A \in C^{m\times n}$ and $W \in C^{n\times m}$. Then the following statements hold:*

(*i*). *$AA_w^{D,\dagger}$ is a projector onto $\mathbf{R}(AWA_{d,w})$ along $\mathbf{N}(A_{d,w}A^\dagger)$.*

(*ii*). *$A_w^{D,\dagger}A$ is a projector onto $\mathbf{R}((WA)^l)$ along $\mathbf{N}((WA)^l)$, where $l = Ind(WA)$.*

Proof. (1). Since $A_w^{D,\dagger}$ is an outer inverse of A, it follows that $AA_w^{D,\dagger}$ is a projector. It follows from

$$AA_w^{D,\dagger} = AWA_{d,w}WAA^\dagger \text{ and } AWA_{d,w} = (AWA_{d,w}WAA^\dagger)AWA_{d,w}$$

that $\mathbf{R}(AA_w^{D,\dagger}) \subseteq \mathbf{R}(AWA_{d,w})$ and $\text{rank}(AA_w^{D,\dagger}) = \text{rank}(AWA_{d,w})$, which implies that $\mathbf{R}(AA_w^{D,\dagger}) = \mathbf{R}(AWA_{d,w})$. Similarly, we can get $\mathbf{N}(AA_w^{D,\dagger}) = \mathbf{N}(A_{d,w}A^\dagger)$.

(2). We can immediately prove the result (ii) in this theorem by using the fact that

$$WA_{d,w}WA = P_{\mathbf{R}((WA)^l),\mathbf{N}((WA)^l)} \text{ (see [5]) and } A_w^{D,\dagger}A = WA_{d,w}WA.$$

The proof is complete. □

In Definition 2.1 the W-weighted DMP inverse has been introduced from an algebraic approach. Next result presents a characterization of the W-weighted DMP inverse from a geometrical point of view.

Theorem 3.2. *Let $A \in C^{m\times n}$ and $W \in C^{n\times m}$. Then $A_w^{D,\dagger}$ is the unique matrix X that satisfies*

$$AX = P_{\mathbf{R}(AWA_{d,w}),\mathbf{N}(A_{d,w}A^\dagger)}, \quad \mathbf{R}(X) \subseteq \mathbf{R}((WA)^l), \tag{3.1}$$

where $l = Ind(WA)$.

Proof. According to Theorem 3.1, it follows that $A_w^{D,\dagger}$ is the solution to (3.1). It remains to prove that the system (3.1) has only one solution.

Suppose that both X_1 and X_2 are solutions to (3.1). Then

$$A(X_1 - X_2) = 0, \ \mathbf{R}(X_1) \subseteq \mathbf{R}((WA)^l) \text{ and } \mathbf{R}(X_2) \subseteq \mathbf{R}((WA)^l).$$

Consequently,

$$\mathbf{R}(X_1 - X_2) \subseteq \mathbf{N}(WA) \text{ and } \mathbf{R}(X_1 - X_2) \subseteq \mathbf{R}((WA)^l).$$

Therefore, we have $\mathbf{R}(X_1 - X_2) \subseteq \mathbf{R}((WA)^l) \cap \mathbf{N}((WA)^l) = \{0\}$ since WA has index l. Thus, $X_1 = X_2$. □

Obviously, $A_w^{D,\dagger}$ is the only matrix that satisfies (2.1) and (3.1). Hence, both algebraic and geometrical approaches are equivalent. The W-weighted DMP inverse also has the following properties.

Theorem 3.3. *Let $A \in C^{m\times n}$ and $W \in C^{n\times m}$. Then*

(a). $A_w^{D,\dagger} = WA_{d,w}WP_A$;

(b). $A_w^{D,\dagger}$ *is an outer inverse of* A.

Proof. The proof follows from the definitions and properties of the Moore-Penrose inverse and W-weighted Drazin inverse. □

The following example shows that in general the W-weighted DMP inverse, the Moore-Penrose inverse and the W-weighted Drazin inverse are different.

Example. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then simple computations give

$$A^\dagger = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}, \quad A_{d,w} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_w^{D,\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the W-weighted DMP inverses provide a new class of generalized inverses for rectangular matrices because in general the W-weighted DMP inverse of a matrix is different from each of its Moore-Penrose inverse and W-weighted Drazin inverse.

Acknowledgement

I'm grateful to the handing editor and referees for their helpful comments and suggestions.

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (2011) 681–697.
[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, (2nd edition), Springer, New York, 2003.
[3] S.L. Campbell, C.D. Meyer Jr, Generalized Inverse of Linear Transformation, Dover, New York, 1991.
[4] S.L. Campbell, C.D. Meyer Jr, N.J. Rose, Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, SIAM J. Appl. Math. 31 (1976) 411–425.
[5] R.E. Cline, T.N.E. Greville, A Drazln inverse for rectangular matrices, Linear Algebra Appl. 29 (1980) 53–62.
[6] S.B. Malik, N. Thome, On a new generalized inverse for matrices of an arbitrary index, Appl. Math. Comput. 226 (2014) 575–580.
[7] C.D. Meyer Jr, The role of the group generalized inverse in the theory of finite Markov chains, SIAM Rev. 17 (1975) 443–464.
[8] S.K. Mitra, P. Bhimasankaram, S.B. Malik, Matrxi Partial Orders, Shorted Operators and Applications, World Scientific, Singapore, 2010.
[9] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 52 (1955) 406–413.
[10] C.R. Rao, S.K. Mitra, S.B. Malik, Generalized Inverse of Matrices and its Applications, John Wiley and Sons, New York, 1971.
[11] G. Wang, Y. Wei, S. Qiao, Generalized inverses: Theory and Computations, Science Press, Beijing, 2004.