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A New Method for Approximation of Closed Convex Subsets of B(H)

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Abstract.
In this paper we use the concept of numerical range to characterize best approximation points in closed convex subsets of $B(H)$. Finally by using this method we give also a useful characterization of best approximation in closed convex subsets of a C^*-algebra $\mathbb{A}$.

1. Introduction

The theory of best approximation by elements of convex sets in normed linear spaces, have been studied by many investigators. Some results on existence and uniqueness of best approximation and co-approximation in general Banach spaces can be found in [1, 4, 6, 8, 11, 14, 17, 19]. The Noncommutative approximation and, in particular, approximation in the space of Hilbert operator has a long history (see for example, [2, 9, 10, 12, 20]). This topic for the case of C^*-algebra $\mathbb{A}$ and its C^*-subalgebras in terms of state functions of $\mathbb{A}^*$ is done by Rieffel in [16], where $\mathbb{A}^*$ is the dual space of $\mathbb{A}$. These works are mainly about the existence of best approximation and we can not found any matter for characterization of best approximation.

In this paper, we give some results to characterize best approximation of convex sets in $B(H)$. Finally This characterization will be extend to convex subsets of C^*-algebras. Our main tools is using the concept of numerical range and Gelfand-Naimark theorem.

2. Characterization of Approximation Points

In this section, first we give some definitions and lemmas which will be used later. Then we present various characterizations of best approximation and co-approximation of elements of C^*-algebras.

Let $f, g \in B(H)$. The numerical range of f relative to g which is denoted by $W(g^* f)$ is defined as follows:

$$W(g^*f) := \{\lambda \in \mathbb{C} : \lambda = \lim_{n\to\infty} \langle f(x_n), g(x_n)\rangle, \quad \{x_n\} \in Z_f\}, \tag{1}$$

where

$$Z_f := \{\{x_n\} \in H : \|x_n\| = 1, \ \lim_{n\to\infty} \|f(x_n)\| = \|f\|\}. \tag{2}$$

It is well Known that $W(g^*f)$ is a compact convex subset of the complex plane [13]. An interesting special case is when g be the identity operator, see[3, 18].

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We denote the directional derivative norm in point f along g by

$$\tau_2(f,g) := \limsup_{t\to 0^+} \frac{\|f+tg\|^2 - \|f\|^2}{2t}.$$

In the following, the relationship between the numerical range of f relative to g and its norm derivatives will be investigated.

Lemma 2.1. *Let $f, g \in B(H)$. Then*

$$-\tau_2(f,-g) \le \min ReW(g^* f) \le \max ReW(g^* f) \le \tau_2(f,g).$$

Proof. Suppose $\{x_n\} \in Z_f$, we get

$$\begin{aligned} \|f+tg\|^2 &\geq \lim_{n\to\infty} \|f(x_n)+tg(x_n)\|^2 \\ &= \lim_{n\to\infty} (\|f(x_n)\|^2 + t^2\|g(x_n)\|^2 + 2tRe\langle f(x_n), g(x_n)\rangle) \\ &= \|f\|^2 + t^2 \lim_{n\to\infty} \|g(x_n)\|^2 + 2t \lim_{n\to\infty} Re\langle f(x_n), g(x_n), \end{aligned}$$

hence

$$\frac{\|f+tg\|^2 - \|f\|^2}{t} \geq t \lim_{n\to\infty} \|g(x_n)\|^2 + 2 \lim_{n\to\infty} Re\langle f(x_n), g(x_n)\rangle,$$

setting $t \to 0^+$, and taking lim sup then

$$\tau_2(f,g) \geq \lim_{n\to\infty} Re\langle f(x_n), g(x_n)\rangle.$$

Thus $\tau_2(f,g) \geq \max ReW(g^* f)$. For the other inequality replace g by $-g$. □

Let W be a nonempty subset of a normed vector space $B(H)$ and $f \in B(H)$. The set of all best approximation to f from W is denoted by $\mathbf{P}_W(f)$. Thus

$$\mathbf{P}_W(f) := \{h \in W | \; \|f-h\| = \inf_{g\in W} \|f-g\|\}. \tag{3}$$

Theorem 2.2. *Let U be a closed convex subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{P}_U(f)$.

ii) *For each $h \in U$,*

$$\max ReW((h-g_0)^*(f-h)) \le 0. \tag{4}$$

Proof. $i \to ii$. Since $g_0 \in \mathbf{P}_U(f)$ for $h \in U$ and $t = 1$ we have

$$\|f-h+t(h-g_0)\|^2 - \|f-h\|^2 \le 0.$$

As the function φ defined by $\varphi(t) = \frac{\|f+tg\|^2-\|f\|^2}{2t}$ is non-decreasing, setting $t \to 0^+$, and taking lim sup therefore $\tau_2(f-h, h-g_0) \le 0$. Now by Lemma 2.1 we get (4).
$ii \to i$. It is not restrictive to assume $g_0 = 0$. Let inequality (4) holds but $g_0 \notin \mathbf{P}_U(f)$, then there exist $h_1 \in U \setminus \{0\}$, such that $\|f-h_1\| < \|f\|$. By applying (4) to $h_\lambda = \lambda h_1$, for $0 < \lambda \le 1$ we get

$$\max ReW(h_\lambda^*(f-h_\lambda) = \max ReW(\lambda h_1^*(f-\lambda h_1) \le 0.$$

Since $0 < \lambda$, then

$$\max ReW(h_1^*(f-\lambda h_1) \le 0.$$

and

$$\begin{aligned} \max ReW(-h_1^*(f-\lambda h_1)) &\geq \min ReW(-h_1^*(f-\lambda h_1)) \\ &= -\max ReW(h_1^*(f-\lambda h_1)) \geq 0. \end{aligned}$$

Now by Lemma 2.1 we get

$$\tau_2((f-\lambda h_1),-h_1) \geq \max ReW(-h_1^*(f-\lambda h_1)) \geq 0.$$

Since τ_2 is upper semi-continuous in its arguments, we have

$$\tau_2(f,-h_1) \geq \limsup_{\lambda\to o^+} \tau_2((f-\lambda h_1),-h_1) \geq 0.$$

This implies that there exist ε_1 such that for $t \in (0,\varepsilon_1]$ we have $\frac{\|f-th_1\|^2-\|f\|^2}{2t} \geq 0$. Again since φ is non-decreasing, we have $\|f\| \leq \|f-h_1\|$, which is a contradiction. □

Lemma 2.3. *Let g_0, h and $f \in B(H)$. If*

$$\max ReW((g_0-h)^*(f-g_0)) \geq 0, \tag{5}$$

then $\|f-g_0\| \leq \|f-h\|$.

Proof. Let (5) be true and $\{x_n^h\}_{n\in\mathbb{N}}$ be a sequence of Z_{f-g_0} such that

$$\lim_{n\to\infty} Re\langle (f-g_0)(x_n^h), (g_0-h)(x_n^h)\rangle \geq 0.$$

Therefore

$$\begin{aligned} \|f-h\| \geq \lim_{n\to\infty} \|f(x_n^h)-h(x_n^h)\|^2 &= \lim_{n\to\infty} \|f(x_n^h)-g_0(x_n^h)+g_0(x_n^h)-h(x_n^h)\|^2 \\ &= \lim_{n\to\infty} (\|f(x_n^h)-g_0(x_n^h)\|^2 + \|g_0(x_n^h)-h(x_n^h)\|^2) \\ &+ 2Re\langle (f-g_0)(x_n^h), g_0(x_n^h)-h(x_n^h))\rangle \\ &\geq \lim_{n\to\infty} \|f(x_n^h)-g_0(x_n^h)\|^2 = \|f-g_0\|^2. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *Let U be a closed convex subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$, If* $\max ReW((h-g_0)^*(f-h)) \geq 0$ *for each $h \in U$, then*

$$\max ReW((g_0-h)^*(f-g_0) \geq 0, \ (h \in U).$$

Proof. Suppose, on the contrary, it is possible to find an element $h \in U$ such that

$$\max ReW((g_0-h)^*(f-g_0) = -\delta < 0.$$

Since $Re\langle (f-g_0)(x), (g_0-h)(x)\rangle$ is a continuous function on H, there exist an open set $G \subseteq H$ such that $Z_{f-g_0} \subseteq G$ and

$$\max_{x\in G} Re\langle (f-g_0)(x), (g_0-h)(x)\rangle < -\delta.$$

Then there exist $0 < \varepsilon_0$, such that for each $\varepsilon \in (0,\varepsilon_0]$ we have

$$X_{f-g_0}(\varepsilon) = \{\{x_n\}_{n\in\mathbf{N}} \in H : \lim_{n\to\infty} \|(f-g_0)(x_n)\| \geq \|f-g_0\| - \varepsilon\} \subseteq G$$

and

$$\max Re \lim_{n\to\infty} \langle (f-g_0)(x_n), (g_0-h)(x_n)\rangle < -\frac{\delta}{2}, \ \textit{where } \{x_n\}_{n\in\mathbf{N}} \in X_{f-g_0}(\varepsilon).$$

Put $D = X_{f-g_0}(\varepsilon_0)$, $\varepsilon_1 \leq \frac{1}{2}\varepsilon_0$ and

$$h_t = th + (1-t)g_0, \quad for\ 0 < t < 1.$$

Since $h_t \to g_0$ as $t \to 0$, there exist $t_0 > 0$ such that for any $0 < t \leq t_0$,

$$\|h_t - g_0\| \leq \varepsilon_1. \tag{6}$$

Now

$$\begin{aligned} Re \lim_{n\to\infty} \langle (f-h_t)(x_n), (h_t-g_0)(x_n)\rangle &= Re \lim_{n\to\infty} [\langle (f-(th+(1-t)g_0), (h_t-g_0)(x_n)\rangle] \\ &= Re \lim_{n\to\infty} [\langle (f-g_0)(x_n), t(h-g_0)(x_n)\rangle \\ &- t^2\|(h-g_0)(x_n)\|^2] \\ &> t\frac{\delta}{2} - t^2\|h-g_0\|^2, \end{aligned}$$

then for sufficiently small $t > 0$, we have

$$Re \lim_{n\to\infty} \langle (f-h_t)(x_n), (h_t-g_0)(x_n)\rangle > 0, \ for\ \{x_n\}_{n\in\mathbf{N}} \in D. \tag{7}$$

On the other hand, by (6) for each $\{x_n\}_{n\in\mathbf{N}} \in Z_{f-h_t}$,

$$\begin{aligned} \lim_{n\to\infty} \|(f-g_0)(x_n)\| &= \lim_{n\to\infty} \|(f-h_t)(x_n) - (g_0-h_t)(x_n)\| \\ &\geq \|f-h_t\| - \|g_0-h_t\| \\ &\geq \|f-g_0\| - 2\varepsilon_1 \\ &\geq \|f-g_0\| - \varepsilon_0. \end{aligned}$$

It follows that $Z_{f-h_t} \subseteq D$. Now by (7) we get

$$\begin{aligned} \max ReW((h_t-g_0)^*(f-h_t)) &= \max_{\{x_n\}_{n\in\mathbf{N}} \in Z_{f-h_t}} Re \lim_{n\to\infty} \langle (f-h_t)(x_n), (h_t-g_0)(x_n)\rangle \\ &\geq \min_{\{x_n\}_{n\in\mathbf{N}} \in Z_{f-h_t}} Re \lim_{n\to\infty} \langle (f-h_t)(x_n), (h_t-g_0)(x_n)\rangle \\ &\geq \min_{x\in D} Re\langle (f-h_t)(x), (h_t-g_0)(x)\rangle > 0. \end{aligned}$$

This contradiction completes the proof. □

Theorem 2.5. *Let U be a closed convex subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{P}_U(f)$.

ii) *For each $h \in U$,*

$$\max ReW((g_0-h)^*(f-g_0)) \geq 0. \tag{8}$$

Proof. $i \to ii$. Since $g_0 \in \mathbf{P}_U(f)$ by Theorem 2.2 for each $h \in U$, we have

$$\max ReW((h-g_0)^*(f-h)) \leq 0.$$

Now by Lemma 2.4 we obtain (8).
$ii \to i$. It is a consequence of Lemma 2.3. □

Corollary 2.6. *Let U be a convex set of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{P}_U(f)$.

ii) *For each $h \in U$,*

$$\max ReW((h-g_0)^*(f-h)) \leq \max ReW((g_0-h))^*(f-g_0). \tag{9}$$

Proof. $i \to ii$. It is a direct consequence of Theorems 2.2 and Lemma 2.4.
$ii \to i$. Suppose that (9) holds but g_0 is not a best approximation to f from U. By Theorem 2.2 there exist $h_1 \in U$ such that

$$\max ReW((h_1 - g_0)^*(f - h_1)) > 0. \tag{10}$$

The latter relation with Lemma 2.1 implies that $\tau_2(f - h_1, h_1 - g_0) \geq 0$. Thus there exist sufficiently small t_0 such that $\frac{\|f-h_1-t_0(h_1-g_0)\|^2-\|f-h_1\|^2}{2t_0} \geq 0$.

Since $\varphi(t) = \frac{\|f-h_1-t_0(h_1-g_0)\|^2-\|f-h_1\|^2}{2t}$ is non-decreasing, for $t = 1$ we have

$$\|f - h_1\| < \|f - g_0\|. \tag{11}$$

By (9) and (10) we get

$$\max ReW((g_0 - h_1)^*(f - g_0)) > 0.$$

Now by Lemma 2.3, we get $\|f - g_0\| < \|f - h_1\|$. This is a contradiction with (11). □

Corollary 2.7. *Let U be a subspace of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{P}_U(f)$.

ii) *For each $h \in U$,*

$$\min ReW(h^*(f - g_0)) \leq 0 \leq \max ReW(h^*(f - g_0)). \tag{12}$$

Proof. It is a consequence of Theorem 2.5 and Corollary 2.6. □

In the following we introduce existence and uniqueness of best approximation in $B(H)$.

Theorem 2.8. *Let U be a closed convex subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. Then the following statements are equivalent.*

i) $\mathbf{P}_U(f) = \{g_0\}$.

ii) *For each $h \neq g_0 \in U$,*

$$\max ReW((h - g_0)^*(f - h)) < 0. \tag{13}$$

Proof. $i \to ii$. Since $\mathbf{P}_U(f) = \{g_0\}$ then $\|f - h + t(h - g_0)\|^2 - \|f - h\|^2 < 0$ for $h \in U$. Dividing by t and let $t \to 0^+$ we obtain (13).
$ii \to i$. By Theorem 2.2 we have $g_0 \in \mathbf{P}_U(f)$. Now suppose $g_1 \neq g_0 \in \mathbf{P}_U(f)$ thus by Theorem 2.5 we obtain $\max ReW((g_1 - h)^*(f - g_1)) \geq 0$. But by applying (13) to g_1 it is impossible. This shows that $\mathbf{P}_U(f) = \{g_0\}$. □

Recall for $f \in B(H), \sigma(f), r(f)$, denote the spectrum and spectral radius of f and $con(A)$ = convex hull of A.

Example 2.9. *Let $H = \mathbb{C}^2$ and $f : H \to H$ defined by $(x_1, x_2) \to (-x_2, x_1)$ and $U = co(I)$ where I is the identity operator. Then $\mathbf{P}_U(f) = \{0\}$.*

Let $g_0 = \lambda_0 I \in \mathbf{P}_U(f)$. Since for $\lambda \in \mathbb{R}$, $f - \lambda I$ is normal operator we have

$$\begin{aligned} W((h - g_0)^*(f - h)) &= W(((\lambda - \lambda_0)I)(f - \lambda I)) \\ &= (\lambda - \lambda_0)con(\sigma(f - \lambda I)) \\ &= (\lambda - \lambda_0)con(\{-\lambda \pm i\}). \end{aligned}$$

By Theorem 2.2, $g_0 = \lambda_0 I \in \mathbf{P}_U(f)$ if and only if $\max ReW((h-g_0)^*(f-h)) \leq 0$, holds for every λ . But this inequality holds only if $\lambda_0 = 0$. Then $\mathbf{P}_U(f) = \{0\}$.
Also we can show this without applying Theorem 2.2. For $\lambda \in \mathbb{R}$ we have

$$\|f - \lambda I\| \geq r(f - \lambda I) = \sup\{|-\lambda \pm i|\} \geq 1.$$

Thus $\inf_{\lambda \in [0,1]} \|f - \lambda I\| \geq 1$, in the other hand $\|f - 0\| = 1$ therefore $\mathbf{P}_U(f) = \{0\}$.

Let U be a closed subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. For each $h \in H$ put $v_h = (h-g_0)^*(f-g_0)$. The following corollary follows immediately from Theorem 2.5.

Corollary 2.10. *Let U be a convex set of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. If $v_h - t\, id_H$ be an invertible element of $B(H)$ such that $\|(v_h - t\, id_H)^{-1}\| \leq t^{-1}$ for each $t > 0$ and $h \in U$, then $g_0 \in \mathbf{P}_U(f)$.*

Proof. Suppose that $\|(v_h - tid_H)^{-1}\| \leq \frac{1}{t}$ then $t\|x\| \leq \|(v_h - tid_H)(x)\|$, for each $x \in H$. Consequently for $\{x_n\} \in Z_{f-g_0}$ we have

$$t^2 \lim_{n\to\infty} \|x_n\|^2 \leq \lim_{n\to\infty} \|(v_h - tid_H)(x_n)\|^2,$$

hence

$$t^2 \lim_{n\to\infty} \|x_n\|^2 \leq \lim_{n\to\infty} [\|v_h(x_n)\|^2 + t^2\|x_n\|^2 - 2tRe\langle v_h(x_n), x_n\rangle],$$

or

$$\lim_{n\to\infty} Re\langle v_h(x_n), x_n\rangle \leq \frac{1}{2t} \lim_{n\to\infty} \|v_h(x_n)\|^2.$$

Now letting $t \to \infty$, we obtain $\lim_{n\to\infty} Re\langle v_h(x_n), x_n\rangle \leq 0$ which holds for all $\{x_n\} \in Z_{f-g_0}$. Therefore we have $\max ReW((h-g_0)^*(f-g_0)) \leq 0$. This implies

$$\max ReW((g_0 - h)^*(f - g_0)) \geq 0.$$

By applying part (ii) Theorem 2.5, we get $g_0 \in \mathbf{P}_U(f)$. □

Definition 2.11. *A subset U of $B(H)$is called semi-Chebyshev if every $f \in B(H)$ has at most one best approximation in U.*

Corollary 2.12. *The following statements are equivalent:*

i) U is a semi-Chebyshev convex subset of $B(H)$.

ii) for each $f \in B(H) \setminus U$, $g_0 \in U$ is a best approximation to f from U if and only if for each $h \neq g_0 \in U$,

$$\max ReW((h - g_0)^*(f - h)) < 0.$$

Proof. It is a consequence of Theorem 2.8. □

Another kind of approximation from W has been introduced by Franchetti and Furi [7] who have considered those elements (if any) $w_0 \in W$ satisfying

$$\|w - w_0\| \leq \|x - w\|, \ \forall w \in W. \tag{14}$$

Such an element is called best coapproximant of x in W. The set of all such elements, satisfying above inequality, is denoted by $\mathbf{R}_G(x)$.

if x, y are elements of a normed linear space X, then x is orthogonal to y in the Birkhoff-james sense, in short $x \perp_B y$, if $\|x\| \leq \|x + \lambda y\|$, $(\lambda \in \mathbb{R})$.

Lemma 2.13. *[7] Let G be a subspace of a normed linear space X, $g_0 \in G$ and $x \in X$. Then $g_0 \in \mathbf{R}_G(x)$ if and only if $G \perp_B x - g_0$.*

Lemma 2.14. *Let B be a closed subspace of $B(H)$, $f \in B(H) \setminus B$ and $g_0 \in B$. If $g_0 \in \mathbf{R}_B(f)$ then there exist a sequence $\{x_n^h\}_{n\in\mathbb{N}} \in Z_{h-g_0}$ for $h \in B$ such that*

$$\lim_{n\to\infty} \langle h(x_n^h), (f-g_0)(x_n^h)\rangle = 0.$$

Proof. Suppose fails. Then there exist $h_1 \in B$ such that $\lim_{n\to\infty}\langle h_1(x_n^{h_1}), (f-g_0)(x_n^{h_1})\rangle \neq 0$ for $\{x_n^{h_1}\}_{n\in\mathbb{N}} \in Z_{h_1}$. Put $A = W((f-g_0)^* h_1)$. Since A is convex and $0 \notin A$, then A is contained in the half-plane. By rotation, we may suppose A is contained in the right half-plane, therefore there is a line which separates 0 from A. Thus there exist $c > 0$ such that $Re\, z \geq c > 0$ for $z \in A$. Let

$$S = \{x \in H : \|x\| = 1, Re\langle h_1(x), (f-g_0)(x)\rangle \leq \frac{1}{2}c\}.$$

Let $\varrho = \sup_{t\in S}\|h_1(t)\|$ then $\varrho < \|h_1\|$ otherwise let $\{x_n^{h_1}\}_{n\in\mathbb{N}} \in Z_{h_1}$ be a sequence of elements of S. We may suppose that $\{\langle h_1(x_n^{h_1}), (f-g_0)(x_n^{h_1})\rangle\}$, $n \in \mathbb{N}$. is a convergence sequence since it is bounded. If we set

$$\lambda_0 = lim_{n\to\infty}\langle h_1(x_n^{h_1}), (f-g_0)(x_n^{h_1})\rangle,$$

then $Re(\lambda_0) \leq \frac{1}{2}c$ and, this is a contradiction.

Let $\mu = \min\{\frac{c}{\|f-g_0\|^2}, \frac{\|h_1\|-\varrho}{\|f-g_0\|}\}$ and x be a unit vector in H. We consider two cases. First, if $x \in S$ then

$$\|h_1(x) - \mu(f-g_0)(x)\| \leq \|h_1(x)\| + \mu\|f-g_0\| < \varrho + \frac{\|h_1\|-\varrho}{\|f-g_0\|}\|f-g_0\| = \|h_1\|.$$

Next, if $x \notin S$, since $\mu^2\|(f-g_0)(x)\|^2 - c\mu < 0$, then

$$\begin{aligned} \|h_1(x) - \mu(f-g_0)(x)\|^2 &= \|h_1(x)\|^2 + \mu^2\|(f-g_0)(x)\|^2 - 2Re\langle h_1(x), (f-g_0)(x)\rangle \\ &\leq \|h_1(x)\|^2 + \mu^2\|(f-g_0)(x)\|^2 - c\mu \\ &< \|h_1\|^2. \end{aligned}$$

Therefore we deduce that $\|h_1 + \mu(f-g_0)\| < \|h_1\|$ which contradicts the hypothesis $g_0 \in \mathbf{R}_B(f)$ because by Lemma 2.13, we have $\|h\| \leq \|h + \lambda(f-g_0)\|$ for all $h \in B$ and $\lambda \in \mathbb{R}$. □

Theorem 2.15. *Let W be a subspace of $B(H)$, $f \in B(H) \setminus W$ and $g_0 \in W$. Then the following statements are equivalent.*

i) $g_0 \in \mathbf{R}_W(f)$.

ii) For each $h \in W$,

$$\min ReW((f-g_0)^* h) \leq 0. \tag{15}$$

Proof. *ii) → i)* Let g be an arbitrary element of W and $\{x_n^{g-g_0}\} \in Z_{g-g_0}$

$$\begin{aligned} \|f-g\|^2 &\geq \|(f-g)(x_n^{g-g_0})\|^2 \\ &= \|(f-g_0)(x_n^{g-g_0})\|^2) + \|(g-g_0)(x_n^{g-g_0})\|^2 \\ &- 2Re\langle (g-g_0)(x_n^{g-g_0}), (f-g_0)(x_n^{g-g_0})\rangle \\ &\geq \|(g-g_0)((x_n^{g-g_0}))\|^2 = \|g-g_0\|^2. \end{aligned}$$

Therefore $\|f-g\| \geq \|g-g_0\|$, i,e. $g_0 \in \mathbf{R}_W(f)$.

$i) \to ii)$ Let $g_0 \in \mathbf{R}_W(f)$ by Lemma 2.14 there is $\{x_n^h\}_{n\in\mathbb{N}} \in Z_h$ such that

$$\lim_{n\to\infty} \langle (f-g_0)(x_n^h), h(x_n^h)\rangle = 0.$$

Thus we have (15). □

Let $\mathbb{A}$ be a C^*-algebra, then $\mathbb{A}$ has a faithful representation, i.e. $\mathbb{A}$ is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space H. This result is called the "Gelfand-Naimark Theorem". (For details about C^*-algebra we refer the reader to [5]).

Let $\mathbb{A}$ be a C^*-algebra, and (π, H) be a faithful representation for $\mathbb{A}$. Let $a, b \in \mathbb{A}$, the numerical range of a^*b relative to a, which is denoted by $W_{\mathbb{A}}(a^*b)$ is defined as follows

$$W_{\mathbb{A}}(a^*b) := \{\lambda \in \mathbb{C} : \lambda \in W(\pi(a^*b)) = W(\pi(a)^*\pi(b))\}. \tag{16}$$

By using this concept we have the following Corollaries.

Corollary 2.16. *Let $\mathbb{B}$ be a closed convex subset of a C^*-algebra $\mathbb{A}$, $a \in \mathbb{A} \setminus \mathbb{B}$ and $b_0 \in \mathbb{B}$. Then $b_0 \in \mathbf{P}_{\mathbb{B}}(a)$ if and only if for each $b \in \mathbb{B}$,*

$$\max ReW_{\mathbb{A}}((b-b_0)^*(a-b)) \leq \max ReW_{\mathbb{A}}((b_0-b)^*(a-b_0)). \tag{17}$$

Proof. Since for each $b \in \mathbb{B}$, we have the inequality (17), by Definition 16 for $\pi(b) \in \pi(\mathbb{B})$ we have

$$\max ReW(\pi(b-b_0)^*\pi(a-b)) \leq \max ReW(\pi(b_0-b)^*\pi(a-b_0)).$$

By Corollary 2.6, $\pi(b_0) \in \mathbf{P}_{\pi(\mathbb{B})}(\pi(a))$ and thus $b_0 \in \mathbf{P}_{\mathbb{B}}(a)$ since π is an isometrically isomorphism. □

Now consider μ as a positive Boreal measure on compact space X, then the map $\pi : L_\infty(X) \to B(L_2(\mu))$ defined by $\pi(f) = M_f$ is a representation [5]. where

$$M_f(h) = foh, \ \ \forall h \in L_2(\mu).$$

Corollary 2.17. *Let B be a closed convex subset of $L_\infty(X)$, and $f \in L_\infty(X) \setminus B$. Then $g_0 \in \mathbb{P}_B(f)$ if and only if for each $g \in B$,*

$$\max_{h_i \in Z_{M_{f-g}}} Re \int ((f-g)oh_i)(x)\overline{((g-g_0)oh_i)(x)}d\mu \leq 0.$$

Proof. It is a consequence of Corollary 2.16. □

Corollary 2.18. *Let $\mathbb{B}$ be a subspace of $\mathbb{A}$, $a \in \mathbb{A} \setminus \mathbb{B}$ and $b_0 \in \mathbb{B}$. Then $b_0 \in \mathbf{R}_{\mathbb{B}}(a)$ if and only if for each $b \in \mathbb{B}$,*

$$\min ReW_{\mathbb{A}}((a-b_0)^*b) \leq 0. \tag{18}$$

Proof. Its proof is similar to proof Corollary 2.16. □

Corollary 2.19. *Let B be a subspace of $L_\infty(X)$, $f \in X \setminus B$ and $g \in B$. Then $b_0 \in \mathbf{R}_B(a)$ if and only if for every function $g \in B$,*

$$\min_{h_i \in Z_{M_g}} Re \int (goh_i(x))((f-g_0)oh_i(x))d\mu \leq 0. \tag{19}$$

Proof. It is a consequence of Corollary 2.18. □

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