Filomat 32:4 (2018), 1313–1322
https://doi.org/10.2298/FIL1804313S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some General Coefficient Estimates for a New Class of Analytic and Bi-Univalent Functions Defined by a Linear Combination

H. M. Srivastava[a], F. Müge Sakar[b], H. Özlem Güney[c]

[a] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*

and

Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China

[b] *Department of Business Administration, Faculty of Management and Economics, Batman University, TR-72060 Batman, Turkey*

[c] *Department of Mathematics, Faculty of Science, Dicle University, TR-21280 Diyarbakır, Turkey*

Abstract. In the present paper, we introduce and investigate a new class of analytic and bi-univalent functions $f(z)$ in the open unit disk $\mathbb{U}$. For this purpose, we make use of a linear combination of the following three functions:

$$\frac{f(z)}{z}, \quad f'(z) \quad \text{and} \quad zf''(z)$$

for a function belonging to the normalized univalent function class $\mathcal{S}$. By applying the technique involving the Faber polynomials, we determine estimates for the general Taylor-Maclaurin coefficient of functions belonging to the analytic and bi-univalent function class which we have introduced here. We also demonstrate the not-too-obvious behaviour of the first two Taylor-Maclaurin coefficients of such functions.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the family of functions *analytic* in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

which are normalized by the condition:

$$f(0) = f'(0) - 1 = 0$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50.
Keywords. Analytic functions; Univalent functions; Bi-Univalent functions; Linear combination; Faber polynomials; Taylor-Maclaurin coefficients; Carathéodory Lemma.
Received: 16 March 2017; Accepted: 21 April 2017
Communicated by Dragan S. Djordjević
Email addresses: harimsri@math.uvic.ca (H. M. Srivastava), mugesakar@hotmail.com (F. Müge Sakar), ozlemg@dicle.edu.tr (H. Özlem Güney)

and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ of the form given by (1), which are univalent (or schlicht) in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. It is a well-known fact that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right). \tag{2}$$

In fact, according to the *Koebe One-Quarter Theorem* [7], the inverse function f^{-1} is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} b_n\, w^n. \end{aligned} \tag{3}$$

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1). Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\,\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ. We know also that, for $f \in \Sigma$ of the form (1), the inverse function f^{-1} has the Taylor-Maclaurin series expansion given by (3).

Lewin [13] was the first to investigate the bi-univalent function class Σ and showed that, if the function $f \in \Sigma$ is given by the Taylor-Maclaurin series expansion (1), then

$$|a_2| < 1.51.$$

Subsequently, Brannan and Clunie [7] conjectured that

$$|a_2| \leqq \sqrt{2}.$$

Netanyahu [14], on the other hand, showed that

$$\max_{f \in \Sigma} |a_2| = \frac{4}{3}.$$

Brannan and Taha [8] introduced certain subclasses of the function class Σ similar to the familiar subclasses of the univalent function class $\mathcal{S}$. Actually, the work of Srivastava *et al.* [22] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [22], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [5], [26] and [27]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1) were obtained in several recent papers. However, the problem to find the general coefficient bounds on $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2, 3\})$ for the function $f \in \Sigma$ is presumably still an open problem. In other words, not much is known about the bounds on the general coefficient $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2, 3\})$. In the existing literature, only a few works determine the general coefficient bounds for $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2, 3\})$ for analytic and bi-univalent functions in Σ (see, for example, [9], [12], [15] and [18]). Some other recent contributions to the subject of the bi-univalent function class Σ include (for example) [16], [17], [19], [20], [21], [23] and [24].

The results over simple expressions involving a function $f(z)$ in the normalized univalent function class $\mathcal{S}$ and its derivatives $f'(z)$ and $f''(z)$, such as

$$\frac{f(z)}{z}, \quad f'(z) \qquad \text{and} \qquad zf''(z),$$

play a significant rôle in the theory of univalent functions. In this paper, we propose to study on a subclass of the bi-univalent function class Σ, which involve a linear combination of the following three expressions:

$$\frac{f(z)}{z}, \quad f'(z) \qquad \text{and} \qquad zf''(z)$$

and use the Faber polynomial coefficient expansion in order to obtain bounds for the general coefficients $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2, 3\})$ of such functions. In particular, we investigate bounds for the first two coefficients

$$|a_2|, \quad |a_3| \qquad \text{and} \qquad |a_3 - 2a_2^2|$$

for such functions.

We begin by defining the aforementioned analytic and bi-univalent function class $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$ as follows.

Definition. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$ if the following condition is satisfied:

$$\Re\left((1-\lambda)(1-\nu)\,\frac{f(z)}{z} + [\nu + \lambda(1+\nu)]f'(z) + \lambda\nu\,[zf''(z) - 2]\right) > \alpha \tag{4}$$

$$\Big(0 \leqq \alpha < 1;\ \lambda \geqq 0;\ 0 \leqq \nu \leqq 1;\ z \in \mathbb{U}\Big).$$

By appropriately specializing the parameters λ and ν, we can get several known subclasses of the bi-univalent function class Σ. For example, for $\nu = 0$ and $\lambda \geqq 1$, we have the class given by

$$\mathcal{A}_\Sigma(\lambda, 0, \alpha) = \mathcal{D}(\alpha, \lambda),$$

whose elements satisfy the following condition:

$$f \in \Sigma \qquad \text{and} \qquad \Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha \qquad \Big(0 \leqq \alpha < 1;\ \lambda \geqq 1;\ z \in \mathbb{U}\Big),$$

which was studied by Jahangiri and Hamidi [12]. On the other hand, in the special case of (4) when $\lambda = 1$, if we make the following notational changes:

$$\frac{\nu}{2\nu+1} \mapsto \rho \qquad \text{and} \qquad \frac{2\nu+\alpha}{2\nu+1} \mapsto \alpha,$$

we arrive at the bi-univalent function class $\mathcal{N}_\Sigma^{(\alpha,\rho)}$ $(0 \leqq \alpha < 1;\ \rho \geqq 0)$ given by

$$f \in \Sigma \qquad \text{and} \qquad \Re\left\{f'(z) + \rho zf''(z)\right\} > \alpha \qquad \Big(0 \leqq \alpha < 1;\ \rho \geqq 0; z \in \mathbb{U}\Big),$$

which was studied by Srivastava *et al.* [18] (see also [6]). Finally, for $\lambda = 1$ and $\nu = 0$, we have the class given by

$$\mathcal{A}_\Sigma(1,0,\alpha) = \mathcal{D}(\alpha,1) = \mathcal{H}_\Sigma(\alpha),$$

that is, by

$$f \in \Sigma \qquad \text{and} \qquad \Re\{f'(z)\} > \alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

which was introduced and investigated in the pioneer work on the subject by Srivastava *et al.* [22] who derived the following initial coefficient bounds for the functions in $\mathcal{H}_\Sigma(\alpha)$.

Theorem 1. (see, for details, [22]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{H}_\Sigma(\alpha)$ $(0 \leqq \alpha < 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)}{3}}$$

and

$$|a_3| \leqq \frac{(1-\alpha)(5-3\alpha)}{3}.$$

Here, in our present investigation, we make use of the Faber polynomial expansions of functions $f \in \mathcal{A}$ of the form (1). Just as in the equation (3), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [3] and [4]; see also [12] and [18]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\, w^n, \tag{5}$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}\, a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}\, a_2^{n-3} a_3 \\ &\quad + \frac{(-n)!}{(-2n+3)!(n-4)!}\, a_2^{n-4} a_4 \\ &\quad + \frac{(-n)!}{(2(-n+2))!(n-5)!}\, a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &\quad + \frac{(-n)!}{(-2n+5)!(n-6)!}\, a_2^{n-6}\left[a_6 + (-2n+5)a_3 a_4\right] + \sum_{j \geqq 7} a_2^{n-j} V_j, \end{aligned} \tag{6}$$

where such expressions as (for example) $(-n)!$ are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right) \tag{7}$$

and V_j $(7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$ (see, for details, [3] and [4]). In particular, the first three terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2,$$

$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2 a_3 + a_4\right).$$

In general, an expansion of K_n^p is given by (see, for details, [3])

$$K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n \qquad (p \in \mathbb{Z}), \tag{8}$$

where

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\} \qquad \text{and} \qquad D_n^p = D_n^p (a_2, a_3, \cdots)$$

and, alternatively, by (see, for details, [25]; see also [1])

$$D_n^m(a_1, a_2, \cdots, a_n) = \sum_{\mu_1, \cdots, \mu_n \geqq 0} \left(\frac{m!}{\mu_1! \cdots \mu_n!} \right) a_1^{\mu_1} \cdots a_n^{\mu_n}, \tag{9}$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, \cdots, \mu_n$ satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m \\ \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that (see, for example, [2])

$$D_n^n(a_1, a_2, \cdots, a_n) = a_1^n.$$

2. Main Results and Their Consequences

Our first main result (Theorem 2 below) gives an upper bound for the general Taylor-Maclaurin coefficient $|a_n|$ of functions in the class $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$.

Theorem 2. *For* $0 \leqq \alpha < 1$, $\lambda \geqq 0$, $0 \leqq \nu \leqq 1$ *and* $z \in \mathbb{U}$, *let the function* f *given by* (1) *as well as the inverse function* $g = f^{-1}$ *be in the class* $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$. *If*

$$f(z) = z + a_n z^n + \cdots \qquad (n \in \mathbb{N} \setminus \{1\}),$$

so that the inverse function $g = f^{-1}$ *is given by*

$$g(w) = w + b_n w^n + \cdots = w - a_n w^n + \cdots,$$

then

$$|a_n| \leqq \frac{2(1-\alpha)}{(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Proof. We first let the function f given by (1) be in the class $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$. Then we have

$$\begin{aligned} &(1-\lambda)(1-\nu)\frac{f(z)}{z} + [\nu + \lambda(1+\nu)]f'(z) + \lambda\nu\,[zf''(z) - 2] \\ &\qquad = 1 + \sum_{n=2}^{\infty} [(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1] a_n z^{n-1} \end{aligned} \tag{10}$$

and, for its inverse map $g = f^{-1}$, it is seen that

$$\begin{aligned} &(1-\lambda)(1-\nu)\frac{g(w)}{w} + [\nu + \lambda(1+\nu)]g'(w) + \lambda\nu\,[wg''(w) - 2] \\ &\qquad = 1 + \sum_{n=2}^{\infty} [(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1] b_n w^{n-1}. \end{aligned} \tag{11}$$

On the other hand, since

$$f \in \mathcal{A}_\Sigma(\lambda, \nu, \alpha) \qquad \text{and} \qquad g = f^{-1} \in \mathcal{A}_\Sigma(\lambda, \nu, \alpha),$$

by hypothesis, there exist two functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A} \qquad \text{and} \qquad q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A}$$

with

$$\Re\big(p(z)\big) > 0 \qquad \text{and} \qquad \Re\big(q(w)\big) > 0$$

in $\mathbb{U}$, such that

$$\begin{aligned}(1-\lambda)(1-\nu)\frac{f(z)}{z} &+ [\nu + \lambda(1+\nu)]f'(z) + \lambda\nu\,[zf''(z) - 2] \\ &= \alpha + (1-\alpha)p(z) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1\,(c_1, c_2, \cdots, c_n)\,z^n \end{aligned} \tag{12}$$

and, similarly,

$$\begin{aligned}(1-\lambda)(1-\nu)\frac{g(w)}{w} &+ [\nu + \lambda(1+\nu)]g'(w) + \lambda\nu\,[wg''(w) - 2] \\ &= \alpha + (1-\alpha)q(w) \\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1\,(d_1, d_2, \cdots, d_n)\,w^n. \end{aligned} \tag{13}$$

Thus, by applying the Carathéodory Lemma (see [10]), we find that

$$|c_n| \leqq 2 \qquad \text{and} \qquad |d_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

If we now compare the corresponding coefficients in Eqs. (10) and (12) for any $n \in \mathbb{N} \setminus \{1\}$, we get

$$[(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1]a_n = (1-\alpha)K_{n-1}^1\,(c_1, c_2, \cdots, c_{n-1})\,. \tag{14}$$

Similarly, from Eqs. (11) and (13), we can find that

$$[(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1]b_n = (1-\alpha)K_{n-1}^1\,(d_1, d_2, \cdots, d_{n-1})\,. \tag{15}$$

Now, for the function $f(z)$ given by

$$f(z) = z + a_n z^n + \cdots \qquad (n \in \mathbb{N} \setminus \{1\}),$$

so that the inverse function $g = f^{-1}$ is given by

$$g(w) = w + b_n w^n + \cdots = w - a_n w^n + \cdots\,,$$

we have $b_n = -a_n$. Consequently, we obtain

$$[(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1]a_n = (1-\alpha)c_{n-1}$$

and

$$-[(n^2+1)\lambda\nu + (n-1)(\lambda+\nu) + 1]a_n = (1-\alpha)d_{n-1}.$$

Upon taking the moduli of either of the above equalities and using the Carathéodory Lemma once again, we get

$$|a_n| = \frac{(1-\alpha)|c_{n-1}|}{(n^2+1)\lambda\nu + (n-1)(\lambda+\nu)+1} \leqq \frac{2(1-\alpha)}{[(n^2+1)\lambda\nu + (n-1)(\lambda+\nu)+1]}$$

or, equivalently,

$$|a_n| = \frac{(1-\alpha)|d_{n-1}|}{(n^2+1)\lambda\nu + (n-1)(\lambda+\nu)+1} \leqq \frac{2(1-\alpha)}{(n^2+1)\lambda\nu + (n-1)(\lambda+\nu)+1},$$

which is the required result. This completes the proof of Theorem 2. □

Theorem 3 below gives the unpredictable behavior of the first two Taylor-Maclaurin coefficients of functions $f \in \mathcal{A}_\Sigma(\lambda, \nu, \alpha)$.

Theorem 3. *For* $0 \leqq \alpha < 1$, $\lambda \geqq 0$, $0 \leqq \nu \leqq 1$ *and* $z \in \mathbb{U}$, *let the function* f *given by* (1) *as well as the inverse function* $g = f^{-1}$ *be in the class* $\mathcal{A}_\Sigma(\lambda, \nu, \alpha)$. *Then*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}} & \left(0 \leqq \alpha < \dfrac{1+2(\lambda+\nu)(1-5\lambda\nu)+\lambda\nu(8-25\lambda\nu)-(\lambda^2+\nu^2)}{2[1+2(\lambda+\nu)+10\lambda\nu]}\right) \\[2ex] \dfrac{2(1-\alpha)}{1+\lambda+\nu+5\lambda\nu} & \left(\dfrac{1+2(\lambda+\nu)(1-5\lambda\nu)+\lambda\nu(8-25\lambda\nu)-(\lambda^2+\nu^2)}{2[1+2(\lambda+\nu)+10\lambda\nu]} \leqq \alpha < 1\right), \end{cases}$$

$$|a_3| \leqq \frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}$$

and

$$\left|a_3 - 2a_2^2\right| \leqq \frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}.$$

Proof. In the case when $n = 2$, Eqs. (14) and (15) yield

$$[1+(\lambda+\nu)+5\lambda\nu]a_2 = (1-\alpha)c_1, \tag{16}$$

$$[1+2(\lambda+\nu)+10\lambda\nu]a_3 = (1-\alpha)c_2, \tag{17}$$

$$-[1+(\lambda+\nu)+5\lambda\nu]a_2 = (1-\alpha)d_1 \tag{18}$$

and

$$[1+2(\lambda+\nu)+10\lambda\nu](2a_2^2 - a_3) = (1-\alpha)d_2. \tag{19}$$

Now, if we take the moduli in (16) and (18) and apply the Carathéodory Lemma, we find that

$$|a_2| = \frac{(1-\alpha)\,|c_1|}{1+\lambda+\nu(1+5\lambda)} = \frac{(1-\alpha)\,|d_1|}{1+\lambda+\nu(1+5\lambda)} \leqq \frac{2(1-\alpha)}{1+\lambda+\nu(1+5\lambda)}. \tag{20}$$

Upon adding the two equations (17) and (19) and solving for $|a_2|$, if we apply the Carathéodory Lemma once again, we obtain

$$2a_2^2[1+2(\lambda+\nu)+10\lambda\nu]=(1-\alpha)(c_2+d_2) \tag{21}$$

or, equivalently,

$$\left|a_2^2\right|=\frac{(1-\alpha)|c_2+d_2|}{2[1+2(\lambda+\nu)+10\lambda\nu]}.$$

Therefore, we have

$$|a_2|\leqq\sqrt{\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}}.$$

We now subtract Eq. (19) from Eq. (17) and solve for $|a_2|$ as follows:

$$[1+2(\lambda+\nu)+10\lambda\nu](-2a_2^2+2a_3)=(1-\alpha)(c_2-d_2),$$

which, when solved for $|a_3|$, yields

$$a_3=a_2^2+\frac{(1-\alpha)(c_2-d_2)}{2[1+2(\lambda+\nu)+10\lambda\nu]}. \tag{22}$$

Substituting from Eq. (16) into Eq. (22), we get

$$a_3=\frac{(1-\alpha)^2c_1^2}{[1+\lambda+\nu(1+5\lambda)]^2}+\frac{(1-\alpha)(c_2-d_2)}{2[1+2(\lambda+\nu)+10\lambda\nu]},$$

so that, by using the Carathéodory Lemma, we find that

$$|a_3|\leqq\frac{4(1-\alpha)^2}{[1+\lambda+\nu(1+5\lambda)]^2}+\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}.$$

On the other hand, if we substitute from Eq. (21) into Eq. (22), we obtain

$$a_3=\frac{(1-\alpha)c_2}{1+2(\lambda+\nu)+10\lambda\nu},$$

so that

$$|a_3|\leqq\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}. \tag{23}$$

Consequently, since

$$\begin{aligned}\min&\left\{\frac{4(1-\alpha)^2}{[1+\lambda+\nu(1+5\lambda)]^2}+\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu},\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}\right\}\\&=\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu},\end{aligned} \tag{24}$$

we readily find that

$$|a_3|\leqq\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu}.$$

Finally, from Eq. (19), we have

$$|a_3-2a_2^2|=\frac{(1-\alpha)|d_2|}{1+2(\lambda+\nu)+10\lambda\nu}\leqq\frac{2(1-\alpha)}{1+2(\lambda+\nu)+10\lambda\nu},$$

which completes the proof of Theorem 3. □

Upon setting $\nu = 0$ and $\lambda \geqq 1$ in Theorem 2 and Theorem 3, we deduce the Corollary 1 and Corollary 2, respectively.

Corollary 1. (see [12]) *For $0 \leqq \alpha < 1$ and $\lambda \geqq 1$, let the function $f \in \mathcal{D}(\alpha, \lambda)$ be given by* (1). *Also let $g = f^{-1} \in \mathcal{D}(\alpha, \lambda)$. If*

$$f(z) = z + a_n z^n + \cdots \qquad (n \in \mathbb{N} \setminus \{1\}),$$

so that the inverse function $g = f^{-1}$ is given by

$$g(w) = w + b_n w^n + \cdots = w - a_n w^n + \cdots,$$

then

$$|a_n| \leqq \frac{2(1-\alpha)}{1+\lambda(n-1)} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 2. (see [12]) *For $0 \leqq \alpha < 1$ and $\lambda \geqq 1$, let the function $f \in \mathcal{D}(\alpha, \lambda)$ be given by* (1). *Also let $g = f^{-1} \in \mathcal{D}(\alpha, \lambda)$. Then*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{1+2\lambda}} & \left(0 \leqq \alpha < \dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)}\right) \\[2ex] \dfrac{2(1-\alpha)}{1+\lambda} & \left(\dfrac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leqq \alpha < 1\right), \end{cases}$$

$$|a_3| \leqq \frac{2(1-\alpha)}{1+2\lambda}$$

and

$$\left|a_3 - 2a_2^2\right| \leqq \frac{2(1-\alpha)}{1+2\lambda}.$$

For $\nu = 0$ and $\lambda = 1$, we have Corollary 3 below, which shows that the coefficient estimates given in Theorem 3 are better than those given earlier by Srivastava *et al.* [22] and Frasin and Aouf [11].

Corollary 3. *For $0 \leqq \alpha < 1$, let $f \in \mathcal{A}_\Sigma(1, 0, \alpha)$ and $g \in \mathcal{A}_\Sigma(1, 0, \alpha)$. Then*

$$|a_2| \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{3}} & \left(0 \leqq \alpha < \dfrac{1}{3}\right) \\[2ex] 1-\alpha & \left(\dfrac{1}{3} \leqq \alpha < 1\right) \end{cases}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)}{3}.$$

Many other corollaries and consequences of our main results can be deduced similarly.

3. Concluding Remarks and Observations

The main objective in this paper has been to derive some Taylor-Maclaurin coefficient estimates for functions belonging to a new class of analytic and bi-univalent functions $f(z)$ in the open unit disk $\mathbb{U}$, which we have introduced here by means of a linear combination of the following three functions:

$$\frac{f(z)}{z}, \quad f'(z) \qquad \text{and} \qquad zf''(z).$$

Indeed, by using some techniques involving the Faber polynomials, we have successfully determined the bound for the general Taylor-Maclaurin coefficient. We have also found estimates for the first two Taylor-Maclaurin coefficients for functions belonging to this class. The results presented in this paper have been shown to generalize and improve some recent work of Srivastava *et al.* [22].

References

[1] H. Airault, Symmetric sums associated to the factorization of Grunsky coefficients, In: *Proceedings of the Conference on Groups and Symmetries* (Montréal, Canada; 27–29 April 2007).

[2] H. Airault, Remarks on Faber polynomials, *Internat. Math. Forum* **3** (2008), 449–456.

[3] H. Airault and A. Bouali, Differential calculus on the Faber polynomials, *Bull. Sci. Math.* **130** (2006), 179–222.

[4] H. Airault and J.-G. Ren, An algebra of differential operators and generating functions on the set of univalent functions, *Bull. Sci. Math.* **126** (2002), 343–367.

[5] Ş. Altınkaya and S. Yalçın, Initial coefficient bounds for a general class of biunivalent functions, *Internat. J. Anal.* **2014** (2014), Article ID 867871, 1–4.

[6] Ş. Altınkaya and S. Yalçın, Faber polynomial coefficient bounds for a subclass of bi-univalent functions, *C. R. Acad. Sci. Paris Sér. I*, **353** (2015), 1075-1080.

[7] D. A. Brannan and J. G. Clunie (Editors), *Aspects of Contemporary Complex Analysis*, In: Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1–20, 1979), pp. 79–95, Academic Press, New York and London, 1980.

[8] D. A. Brannan and T. S. Taha, On some classes of bi-unvalent functions, In: *Mathematical Analysis and Its Applications* (Kuwait; February 18–21, 1985) (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors), pp. 53–60, KFAS Proceedings Series, Vol. **3**, Pergamon Press (Elsevier Science Limited), Oxford, 1988; see also *Studia Univ. Babeş-Bolyai Math.* **31** (2) (1986), 70–77.

[9] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, *Novi Sad J. Math.* **43** (2) (2013), 59–65.

[10] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[11] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24** (2011), 1569–1573.

[12] J. M. Jahangiri and S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, *Internat. J. Math. Math. Sci.* **2013** (2013), Article ID 190560, 1–4.

[13] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18** (1967), 63–68.

[14] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.* **32** (1969), 100–112.

[15] P. Sharma, Faber polynomial coefficient estimates for a class of analytic bi-univalent functions involving a certain differential operator, *Asian-European J. Math.* **10** (1) (2017), Article ID 1750016, 1–11.

[16] H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egyptian Math. Soc.* **23** (2015), 242–246.

[17] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat* **27** (2013), 831–842.

[18] H. M. Srivastava, S. Sümer Eker and R. M. Ali, Coefficient Bounds for a certain class of analytic and bi-univalent functions, *Filomat* **29** (2015), 1839–1845.

[19] H. M. Srivastava, S. Gaboury and F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, *Acta Math. Sci. Ser. B Engl. Ed.* **36** (2016), 863–871.

[20] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, *Acta Univ. Apulensis Math. Inform. No.* **41** (2015), 153–164.

[21] H. M. Srivastava, S. B. Joshi, S. S. Joshi and H. Pawar, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, *Palest. J. Math.* **5** (Special Issue: 1) (2016), 250–258.

[22] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010), 1188–1192.

[23] H. M. Srivastava, G. Murugusundaramoorthy and N. Mangesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global J. Math. Anal.* **1** (2) (2013), 67–73.

[24] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, *Tbilisi Math. J.* **7** (2) (2014), 1–10.

[25] P. G. Todorov, On the Faber polynomials of the univalent functions of class Σ, *J. Math. Anal. Appl.* **162** (1991), 268–276.

[26] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), 990–994.

[27] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012), 11461–11465.