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Two Finite q-Sturm-Liouville Problems and their Orthogonal Polynomial Solutions

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Abstract. In this paper, we consider two new q-Sturm-Liouville problems and prove that their polynomial solutions are finitely orthogonal with respect to two weight functions which correspond to Fisher and T-student distributions as $q \to 1$. Then, we obtain the general properties of these polynomial solutions, such as orthogonality relations, three term recurrence relations, q-difference equations and basic hypergeometric representations, where all results in the continuous case are recovered as $q \to 1$.

1. Introduction

Let α_1, α_2 and β_1, β_2 be constant numbers, $K(x)$, $K'(x)$, and $w(x)$ be assumed continuous for $x \in [a, b]$. A boundary value problem in the form

$$\frac{d}{dx}\left(K(x)\frac{dy_n(x)}{dx}\right) + \lambda_n w(x) y_n(x) = 0, \tag{1}$$

where $K(x) > 0$, and $w(x) > 0$ which is defined in an open interval, say (a, b), with the boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0, \quad \alpha_2 y(b) + \beta_2 y'(b) = 0, \tag{2}$$

is referred to as a regular Sturm-Liouville problem of continuous type. Moreover, if one of the boundary points a or b is singular (*i.e.* $K(a) = 0$ or $K(b) = 0$), the problem is called a singular Sturm-Liouville problem of continuous type.

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Let $n \neq m$ and y_n, y_m be two eigenfunctions of the differential equation (1). According to Sturm-Liouville theory [18], these two functions are orthogonal with respect to the weight function $w(x)$ under the given boundary conditions (2) *i.e.*

$$\int_a^b w(x)y_n(x)y_m(x)dx = d_n^2\delta_{mn}, \tag{3}$$

where δ_{mn} denotes the Kronecker delta and d_n^2 the norm square of the functions y_n.

Two types of orthogonality can appear for relation (3), namely infinitely orthogonality and finitely orthogonality. In the infinite case, the positive integer number n is free up to infinite, while in the finite case some constraints on n must be imposed. Three sequences of hypergeometric polynomials which are finitely orthogonal have been studied in [17]. The first sequence satisfies the second order linear differential equation [17]

$$x(1+x)y_n''(x) + \Big((2-s)x + (1+t)\Big)y_n'(x) - n(n+1-s)y_n(x) = 0, \quad n = 0,1,2,\ldots. \tag{4}$$

According to [17], the orthogonal polynomial sequence of solutions of the latter equation, denoted by $\{y_n(x) = M_n^{(s,t)}(x)\}_n$, satisfies a finite orthogonality relation as

$$\int_0^\infty \frac{x^t}{(1+x)^{s+t}} M_n^{(s,t)}(x) M_m^{(s,t)}(x)dx = \frac{n!\,(s-n-1))!(t+n)!}{(s-2n-1))\,(s+t-n-1)!}\delta_{m,n},$$

if and only if $m,n = 0,1,2,\ldots,N < \frac{1}{2}(s-1)$ and $t > -1$. As well, the second finite sequence satisfies the second order linear differential equation [17]

$$(1+x^2)y_n''(x) + (3-2p)xy_n'(x) - n(n+2-2p)y_n(x) = 0, \tag{5}$$

and its symmetric polynomial solution, denoted by $y_n(x) = I_n^{(p)}(x)$, is finitely orthogonal as

$$\int_{-\infty}^\infty (1+x^2)^{\frac{1}{2}-p} I_n^{(p)}(x) I_m^{(p)}(x)dx = \frac{n!\,2^{2n-1}\sqrt{\pi}\Gamma^2(p)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+\frac{1}{2})\Gamma(2p-n-1)}\delta_{m,n},$$

if and only if $m,n = 0,1,2,\ldots,\ N < p-1$.

Similarly, we can consider regular or singular Sturm-Liouville problem in the form [12]

$$D_q\Big(K(x;q)D_q y_n(x;q)\Big) + \lambda_{n,q}w(x;q)y_n(x;q) = 0, \tag{6}$$

where $K(x;q) > 0$, $w(x;q) > 0$ and the q-difference operator is defined by

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} \qquad (x \neq 0,\ q \neq 1), \tag{7}$$

with $D_q f(0) := f'(0)$ (provided $f'(0)$ exists), and (6) satisfies a set of boundary conditions like (2). The solutions of the above equation are known as q-orthogonal functions. Therefore, for $n \neq m$, if we have two eigenfunctions of (6), denoted by $y_n(x;q)$ and $y_m(x;q)$, then these functions are orthogonal with respect to a weight function $w(x;q)$ on a discrete set [19].

As a particular case of q-orthogonal functions, the so-called q-orthogonal polynomials have been analyzed in detail (see e.g. [10, 14] and references therein) due to their applications to e.g. continued fractions [14], q-algebras and quantum groups [15, 16, 23] or q-oscillators [1, 2, 6].

Let $\varphi(x) = ax^2 + bx + c$ and $\psi(x) = dx + e$, $a,b,c,d,e \in \mathbb{C}$, $d \neq 0$ be two polynomials of degree at most 2 and 1. If $\{y_n(x;q)\}_n$ is a sequence of polynomials that satisfies the q-difference equation [14]

$$\varphi(x)D_q^2 y_n(x;q) + \psi(x)D_q y_n(x;q) + \lambda_{n,q}y_n(qx;q) = 0, \tag{8}$$

where the composition $D_q^2 = D_q(D_q)$ is given by

$$D_q^2(f(x)) = \frac{f(q^2x) - (1+q)f(qx) + qf(x)}{q(q-1)^2x^2},$$

$\lambda_{n,q} \in \mathbb{C}$, $n \in \{0,1,2,\dots\}$, $q \in \mathbb{R} \setminus \{-1,0,1\}$ and D_q is defined in (7), then the following orthogonality relation holds

$$\int_a^b w(x;q)y_n(x;q)y_m(x;q)d_qx = \left(\int_a^b w(x;q)y_n^2(x;q)d_qx\right)\delta_{n,m},$$

in which $w(x;q)$ is solution of the Pearson-type q-difference equation

$$D_q(w(x;q)\varphi(q^{-1}x)) = w(qx;q)\psi(x). \tag{9}$$

In what follows $w(x;q)$ is assumed to be positive and $w(q^{-1}x;q)\varphi(q^{-2}x)x^k$ for $k \in \mathbb{N}_0$ must vanish at $x = a, b$.

Let $P_n(x) = x^n + \cdots$ be a monic solution of equation (8). Then, by equating the coefficients of x^n in (8) it is possible to compute the eigenvalue $\lambda_{n,q}$ as

$$\lambda_{n,q} = -\frac{[n]_q}{q^n}(a[n-1]_q + d),$$

where the q-number $[z]_q$ is defined by

$$[z]_q := \frac{q^z - 1}{q-1}, \quad \text{and} \quad [0]_q := 0.$$

The orthogonality of all possible polynomial solutions of the q-hypergeometric equation (8) has been studied in [4], by means of a qualitative analysis of the q-Pearson equation (9). Also, the boundary condition [4]

$$\varphi(x)w(x;q)x^k\Big|_{a,b} = \varphi^*(q^{-1}x)w(q^{-1}x;q)x^k\Big|_{a,b} = 0, \tag{10}$$

for $k \in \mathbb{N}_0$ where

$$\varphi^*(x) := q\Big(\varphi(x) + (1-q^{-1})x\psi(x)\Big),$$

must be satisfied in all q-orthogonal polynomial solutions.

In order to determine the weight function $w(x;q) > 0$, Adigüzel [20] studied the rational function $w(qx;q)/w(x;q)$ in detail and obtained all possible cases of q-orthogonal polynomials from the behaviour of the aforesaid rational function. In this analysis it has been showed that some cases do not lead to any q-orthogonal polynomial solution, since the boundary condition (10) is not satisfied for them. In our approach, we reconsider this problem by replacing $y_m\, D_q y_n - y_n\, D_q y_m$ instead of x^k in (10) and after imposing some constraints on n to obtain two finite classes of q-orthogonal polynomials. In [3], Álvarez-Nodarse and Medem classified the q-orthogonal polynomial families of the q-Hahn tableau, and compared both q-Askey scheme and Nikiforov-Uvarov tableaus. Recently in [21], we have studied a class of finite q-orthogonal polynomials whose weight function corresponds to the inverse gamma distribution as $q \to 1$.

The main aim of this paper is to consider two specific q-difference equations of type (8), which give two q-analogues of the finite orthogonal polynomials $\{M_n^{(s,t)}(x)\}_n$ and $\{I_n^{(p)}(x)\}_n$ satisfying the equations (4) and (5) respectively. The paper is organized as follows. In section 2, we recall some basic definitions and notations. In section 3, we obtain the polynomial solutions of two specific q-difference equations of type (8) and prove that they are finitely orthogonal. We also obtain general properties of them and show that as $q \to 1$, all obtained results in the continuous case are recovered.

2. Basic Definitions and Notations

In what follows, we shall consider the notations as in [9, 14]. The rising factorial or Pochhammer symbol is defined by

$$(a)_k := a(a+1)\cdots(a+k-1), \quad \text{with} \quad (\mathrm{a})_0 := 1,$$

and its q-analogue, the q-shifted factorial, is defined by

$$(a;\, q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

with $(a;\, q)_0 := 1$. As an extension,

$$(a;\, q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad \text{for} \quad 0 < |q| < 1.$$

The hypergeometric series are defined as

$${}_rF_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} \middle| \, z \right) := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k}\frac{z^k}{k!},$$

where

$$(a_1,\ldots,a_r)_k := (a_1)_k \,\cdots\, (a_r)_k,$$

and $r, s \in \mathbb{Z}_+$ and $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, z \in \mathbb{C}$. We shall assume that $b_1, \ldots, b_s \neq -k$ $(k = 0, 1, \ldots)$, in order to have a well defined series. The basic hypergeometric series (or q-hypergeometric series) is defined by

$${}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} \middle| \, q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\frac{z^k}{(q;q)_k}[(-1)^k q^{\frac{k(k-1)}{2}}]^{1+s-r},$$

where

$$(a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k,$$

and $r, s \in \mathbb{Z}_+$ and $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, z \in \mathbb{C}$. In this case, we shall assume that $b_1, b_1, \ldots, b_s \neq q^{-k}$ $(k = 0, 1, \ldots)$, in order to have a well defined series. The following limit relation holds true [14]

$$\lim_{q\to 1} {}_r\phi_s\left(\begin{array}{c} q^{a_1},\ldots,q^{a_r} \\ q^{b_1},\ldots,q^{b_s} \end{array} \middle| \, q; (q-1)^{1+s-r}z \right) = {}_rF_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array} \middle| \, z \right). \tag{11}$$

We shall denote by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (0 < q < 1), \tag{12}$$

which is a q-analogue of the gamma function.

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subseteq \mathbb{C}$ is called a μ-geometric set if for $x \in A$, $\mu x \in A$. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{C}$. If $0 \in A$, we say that f has the q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n} \quad (x \in A),$$

exists and does not depend on x. We then denote this limit by $D_q f(0)$.

The following properties can be directly derived from the definition of the q-difference operator

$$D_qD_{q^{-1}} = q^{-1}D_{q^{-1}}D_q, \;\; D_q = D_{q^{-1}} + (q-1)xD_qD_{q^{-1}},$$
$$D_qf(q^{-1}x) = D_{q^{-1}}f(x), \;\; D_qD_{q^{-1}}f(x) = q^{-1}D_q^2 f(q^{-1}x),$$

Moreover, for two functions $f_1, f_2 \in A$ we have

$$D_q(f_1(x)f_2(x)) = D_q(f_1(x))f_2(x) + f_1(qx)D_q(f_2(x)).$$

The inverse of the q-difference operator is usually known as q-integral. It was introduced by Thomae [22] and F.H. Jackson [11] —see also [10, 14]— and it is defined as

$$\int_0^x f(t)d_qt = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x) \quad (x \in A),$$

provided that the series converges. Moreover,

$$\int_0^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

and

$$\int_{-\infty}^{\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n \left(f(q^n) + f(-q^n)\right).$$

The q^{-1}-integrals can be similarly defined.

Let A be a q-geometric set such that $0 \in A$, and let f be defined in A. The function f is said to be q-regular at zero if $\lim_{n\to\infty} f(xq^n) = f(0)$ for every $x \in A$. The q-analogue of integration by parts is given by [5, 13]

$$\int_0^a g(x)D_qf(x)d_qx = (fg)(a) - \lim_{n\to\infty}(fg)(aq^n) - \int_0^a D_qg(x)f(qx)d_qx. \tag{13}$$

Notice that if the functions f, g are q-regular at zero, then we have that $\lim_{n\to\infty}(fg)(aq^n)$ on the right-hand side of (13) can be replaced by $(fg)(0)$.

Let $0 < R \leq \infty$ and Ω_R denote the disc $\{z \in \mathbb{C} : |z| < R\}$. The q-analogue of the fundamental theorem of calculus says: If $f : \Omega_R \to \mathbb{C}$ is q-regular at zero and $\theta \in \Omega_R$ is fixed, then the function

$$F(x) = \int_{\theta}^{x} f(t)d_qt \quad (x \in \Omega_R),$$

is q-regular at zero, $D_qF(x)$ exists for any $x \in \Omega_R$ and $D_qF(x) = f(x)$. Conversely, If $a, b \in \Omega_R$ we have

$$\int_a^b D_qf(t)d_qt = f(b) - f(a).$$

The function f is q-integrable on Ω_R if $|f(t)|d_qt$ exists for all $x \in \Omega_R$.

In some particular cases, it is possible to derive a q-analogue of the theorem of change of variable. If $u(x) = \alpha x^{\beta}$, then [13]

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(x))D_{q^{\frac{1}{\beta}}}u(x)d_{q^{\frac{1}{\beta}}}x. \tag{14}$$

3. Two New Classes of Finite q-orthogonal Polynomials

In this section we consider two special cases of equation (8), providing a detailed analysis for the orthogonal polynomial solutions of these equations.

3.1. First class: q-orthogonal polynomials corresponding to Fisher distribution

Consider the q-difference equation

$$x(qx+1)D_q^2 y_n(x;q) - (q\,[s-2]_q\,x + [-t-1]_q)D_q y_n(x;q) + \lambda_{n,q} y_n(qx;q) = 0, \tag{15}$$

with

$$\lambda_{n,q} = [n]_q\,[s-n-1]_q,$$

for $n = 0,1,2,\ldots$ and $q \in \mathbb{R} \setminus \{-1,0,1\}$.

It is clear that

$$\lim_{q\to 1} \lambda_{n,q} = n(s-n-1),$$

which is the same value as in the continuous case (4).

An equivalent form of equation (15) is as

$$(x+1)y_n(qx;q) - \left((q^{s-1-n} + q^n)x + (q^{-t}+1)\right) y_n(x;q) + (q^{s-1}x + q^{-t})\, y_n(q^{-1}x;q) = 0.$$

Theorem 3.1. *Let $\{M_n^{(s,t)}(x;q)\}_n$ be a sequence of polynomials that satisfies the q-difference equation (15). For $n = 0,1,\ldots,N$ we have*

$$\int_0^\infty w_1(x;q) M_n^{(s,t)}(x;q) M_m^{(s,t)}(x;q) d_q x = \left(\int_0^\infty w_1(x;q)\left(M_n^{(s,t)}(x;q)\right)^2 d_q x\right)\delta_{n,m},$$

where $0 < q < 1$, $t > -1$, $N = \max\{m,n\}$, $N < \frac{1}{2}(s-1)$ and $w_1(x;q)$ is the solution of the Pearson-type q-difference equation

$$D_q\left(w_1(x;q)(q^{-1}x^2 + q^{-1}x)\right) = -w_1(qx;q)\left(q\,[s-2]_q\,x + [-t-1]_q\right),$$

which is equivalent to

$$\frac{w_1(x;q)}{w_1(qx;q)} = \frac{q^{s+t}x+1}{x+1}\,q^{-t}. \tag{16}$$

Proof. It can be verified that

$$w_1(x;q) = \frac{x^t}{(-x;q)_{s+t}} \quad \left(s,t \in \mathbb{R} \quad \text{and} \quad 0 < |q| < 1\right), \tag{17}$$

is a solution of the Pearson-type q-difference equation (16). It can be verified that

$$\lim_{q\to 1} w_1(x;q) = \frac{x^t}{(1+x)^{s+t}},$$

which coincides with [17]

Now we rewrite equation (15) in self-adjoint form

$$D_q[w_1(x;q)(q^{-1}x^2 + q^{-1}x)D_q M_n^{(s,t)}(x;q)] + \lambda_{n,q} w_1(qx;q) M_n^{(s,t)}(qx;q) = 0, \tag{18}$$

and for m as

$$D_q[w_1(x;q)(q^{-1}x^2+q^{-1}x)D_qM_m^{(s,t)}(x;q)]+\lambda_{m,q}w_1(qx;q)M_m^{(s,t)}(qx;q)=0. \tag{19}$$

Let us multiply (18) by $M_m^{(s,t)}(qx;q)$ and (19) by $M_n^{(s,t)}(qx;q)$ and substract each other in order to obtain

$$\begin{aligned}(\lambda_{m,q}-\lambda_{n,q})&w_1(x;q)M_m^{(s,t)}(x;q)M_n^{(s,t)}(x;q)\\&=q^2D_q[\varpi_1(x;q)D_qM_n^{(s,t)}(q^{-1}x;q)]M_m^{(s,t)}(x;q)-q^2D_q[\varpi_1(x;q)D_qM_m^{(s,t)}(q^{-1}x;q)]M_n^{(s,t)}(x;q).\end{aligned} \tag{20}$$

where $\varpi_1(x;q)=w_1(q^{-1}x;q)(q^{-3}x^2+q^{-2}x)$. By using q-integration by parts on both sides of (20) over $[0,\infty)$ it yields

$$\begin{aligned}&(\lambda_{m,q}-\lambda_{n,q})\int_0^\infty w_1(x;q)M_m^{(s,t)}(x;q)M_n^{(s,t)}(x;q)d_qx\\&=\int_0^\infty q^2\Big\{D_q[\varpi_1(x;q)D_qM_n^{(s,t)}(q^{-1}x;q)]M_m^{(s,t)}(x;q)-D_q[\varpi_1(x;q)D_qM_m^{(s,t)}(q^{-1}x;q)]M_n^{(s,t)}(x;q)\Big\}d_qx\\&\qquad=q^2\Big[\varpi_1(x;q)\Big(D_qM_n^{(s,t)}(q^{-1}x;q)M_m^{(s,t)}(x;q)-D_qM_m^{(s,t)}(q^{-1}x;q)M_n^{(s,t)}(x;q)\Big)\Big]_0^\infty.\end{aligned} \tag{21}$$

Since

$$\max\ \deg\{D_qM_n^{(s,t)}(q^{-1}x;q)M_m^{(s,t)}(x;q)-D_qM_m^{(s,t)}(q^{-1}x;q)M_n^{(s,t)}(x;q)\}=m+n-1,$$

if $t>-1$ and $N<\frac{1}{2}(s-1)$ for $N=\max\{m,n\}$, the following boundary conditions must hold

$$\begin{cases}\lim\limits_{x\to0}w_1(q^{-1}x;q)(q^{-3}x^2+q^{-2}x)x^{2N-1}=0,\\ \lim\limits_{x\to\infty}w_1(q^{-1}x;q)(q^{-3}x^2+q^{-2}x)x^{2N-1}=0.\end{cases}$$

In these conditions, the right hand side of (21) tends to zero and consequently

$$\int_0^\infty w_1(x;q)M_m^{(s,t)}(x;q)M_n^{(s,t)}(x;q)d_qx=0,$$

if and only if $m\neq n$, $t>-1$ and $N<\frac{1}{2}(s-1)$ for $N=\max\{m,n\}$. $\square$

Corollary 3.2. *The finite set* $\{M_n^{(s,t)}(x;q)\}_{n=0}^{N<\frac{1}{2}(s-1)}$ *for* $t>-1$ *is orthogonal with respect to the weight function* $w_1(x;q)=\dfrac{x^t}{(-x;q)_{s+t}}$ *on* $[0,\infty)$*. Especially, when* $q\to1$ *in this set, a q-analogue of [17] is obtained.*

Proposition 3.3. *The monic polynomial solution of equation (15) has the following basic hypergeometric representation*

$$\bar{M}_n^{(s,t)}(x;q)=\frac{q^{\frac{1}{2}\left(n^2+(1-2s-2t)n\right)}(q^{t+1};q)_n}{(q^{n-s+1};q)_n}\times{}_2\phi_1\left(\begin{matrix}q^{-n},q^{n-s+1}\\q^{t+1}\end{matrix}\ \Big|\,q;-q^{s+t}x\right). \tag{22}$$

To obtain (22), it is enough to expand the polynomial solution of equation (15) as

$$M_n^{(s,t)}(x;q)=\sum_{k=0}^n a_{n,k}\frac{x^k}{[k]_q!}\quad(a_{n,n}\neq0,\ n=0,1,2,\dots).$$

Then it can be verified that the coefficients $\{a_{n,k}\}_{k=0}^n$ satisfy the two-term recurrence relation

$$[n-k]_q(q^{s-1}-q^{n+k})a_{n,k}=(q^{-(t+1)}-q^k)q^{n-k}a_{n,k+1},$$

and they are determined uniquely up to the normalizing constant $a_{n,n} \neq 0$ as

$$a_{n,k} = \left(\prod_{i=1}^{n-k} \frac{q^i}{[i]_q} \frac{(q^{-(t+1)} - q^{n-i})}{(q^{s-1} - q^{2n-i})} \right) a_{n,n},$$

for $k = 0, 1, \ldots, n-1$. For $a_{n,n} = (1-q)^{-n}(q;q)_n$, the monic solution is finally derived as

$$\bar{M}_n^{(s,t)}(x;q) = q^{\frac{1}{2}n(n+1)} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \left(\prod_{i=1}^{n-k} \frac{(q^{-(t+1)} - q^{n-i})}{(q^{s-1} - q^{2n-i})} \right) (-x)^k,$$

which is equivalent to

$$\bar{M}_n^{(s,t)}(x;q) = \frac{q^{\frac{1}{2}\left(n^2+(1-2s-2t)n\right)}(q^{t+1};q)_n}{(q^{n-s+1};q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n-s+1};q)_k}{(q^{t+1};q)_k (q;q)_k} (-q^{s+t}x)^k.$$

Remark 3.4. *From (11), we can directly conclude that*

$$\lim_{q\to 1} \frac{q^{-\frac{1}{2}\left(n^2+(1-2s-2t)n\right)}(q^{n-s+1};q)_n}{(q^{t+1};q)_n} \bar{M}_n^{(s,t)}(x;q) = M_n^{(s,t)}(x),$$

where $M_n^{(s,t)}(x)$ is the polynomial solution of equation (4) given by

$$M_n^{(s,t)}(x) = (-1)^n n! \binom{t+n}{n} {}_2F_1\left(\begin{array}{c} -n, n+1-s \\ t+1 \end{array} \middle| -x \right).$$

Moreover, the q-difference equation (15) converges formally to equation (4) when $q \to 1$.

Proposition 3.5. *The following relation holds true*

$$D_q \bar{M}_n^{(s,t)}(x;q) = q^{1-n}[n]_q \bar{M}_{n-1}^{(s-2,t+1)}(qx;q).$$

Proof. From (22) we have

$$\bar{M}_n^{(s,t)}(x;q) - \bar{M}_n^{(s,t)}(qx;q) = \frac{q^{\frac{1}{2}(n^2+(1-2s-2t)n)}(1-q^{-n})(q^{t+2};q)_{n-1}}{(q^{n-s+2};q)_{n-1}} \sum_{k=1}^{n} \frac{(q^{1-n};q)_{k-1}(q^{n-s+2};q)_{k-1}}{(q^{t+2};q)_{k-1}(q;q)_{k-1}} (-q^{s+t})^k x^k,$$

and

$$x\bar{M}_{n-1}^{(s-2,t+1)}(qx;q) = \frac{q^{\frac{1}{2}(n^2+(1-2s-2t)n-(2-2s-2t))}(q^{t+2};q)_{n-1}}{(q^{n-s+2};q)_{n-1}} \sum_{k=1}^{n} \frac{(q^{1-n};q)_{k-1}(q^{n-s+2};q)_{k-1}}{(q^{t+2};q)_{k-1}(q;q)_{k-1}} (-q^{s+t})^{k-1} x^k.$$

Therefore

$$\bar{M}_n^{(s,t)}(x;q) - \bar{M}_n^{(s,t)}(qx;q) = q^{1-n}(1-q^n)x\bar{M}_{n-1}^{(s-2,t+1)}(qx;q),$$

which is equivalent to

$$D_q \bar{M}_n^{(s,t)}(x;q) = q^{1-n}[n]_q \bar{M}_{n-1}^{(s-2,t+1)}(qx;q).$$

3.2. Computing the Norm Square Value

It is straightforward to find that the monic polynomial solution of equation (15) satisfies the recurrence relation

$$\bar{M}_{n+1}^{(s,t)}(x;q) = (x - c_n)\,\bar{M}_n^{(s,t)}(x;q) - d_n \bar{M}_{n-1}^{(s,t)}(x;q),$$

with the initial terms $\bar{M}_0^{(s,t)}(x;q) = 1$, $\bar{M}_1^{(s,t)}(x;q) = x - c_0$, where

$$c_n = -\frac{1}{q^{2s-2}(1-q^{2n-s})(1-q^{2n-s+2})} \times \left(q^{n+s-1}(1-q^n-q^{n+1}+q^{2n-s+1}) + q^{2n-t}(q^{s-n-1}-q^{-1}-1+q^n)\right),$$

$$d_n = q^{2n-2s-t+1}\frac{[n]_q\,[n-s]_q\,[n-t-s]_q\,[n+t]_q}{[2n-s-1]_q\,([2n-s]_q)^2\,[2n-s+1]_q}.$$

Since $d_n > 0$ for $n = 1,2,\dots,N < \frac{1}{2}(s-1)$, applying the Favard theorem [7] it yields

$$\int_0^\infty w_1(x;q)\bar{M}_m^{(s,t)}(x;q)\bar{M}_n^{(s,t)}(x;q)d_qx = \left(\prod_{k=1}^n d_k \int_0^\infty w_1(x;q)d_qx\right)\delta_{n,m}.$$

To compute

$$\int_0^\infty w_1(x;q)d_qx = \int_0^\infty \frac{x^t}{(-x;q)_{s+t}}d_qx,$$

we can directly use [8] to get

$$\int_0^\infty \frac{x^t}{(-x;q)_{s+t}}d_qx = \frac{2\,\Gamma_q(t+1)\Gamma_q(s-1)}{(-1;q)_{t+1}(-1;q)_{-t}\,\Gamma_q(s+t)}, \tag{23}$$

where $\Gamma_q(x)$ is defined in (12). Hence, the norm square value of the monic q-polynomials (22) is computed as

$$\int_0^\infty \frac{x^t}{(-x;q)_{s+t}}(\bar{M}_n^{(s,t)}(x;q))^2d_qx = \frac{\Gamma_q(t+1)\Gamma_q(s-1)}{\Gamma_q(s+t)}\,\frac{2\,q^{\left(n^2+(2-t-2s)n\right)}(q,q^{1-s},q^{1-s-t},q^{1+t};q)_n}{(q^{1-s},q^{2-s},q^{2-s},q^{3-s};q^2)_n\,(-1;q)_{t+1}(-1;q)_{-t}}, \tag{24}$$

which is valid for $n = 0,1,...,N < \frac{s-1}{2}$.

Remark 3.6. *Relation (23) helps us compute the moments corresponding to the weight function (17) as*

$$\mu_k = \int_0^\infty \frac{x^{t+k}}{(-x;q)_{s+t}}d_qx = \frac{2\,\Gamma_q(t+1+k)\Gamma_q(s-1-k)}{(-1;q)_{t+1+k}(-1;q)_{-t-k}\,\Gamma_q(s+t)}.$$

Note that if $n > \frac{1}{2}(s-1)$ in (24), the above moments would be divergent and the norm square value will be divergent too.

3.3. Second class: q-orthogonal polynomials corresponding to T-student distribution

As a further special case of equation (8), let us now consider the q-difference equation

$$(q^2x^2+1)D_q^2y_n(x;q) - q^2\,[2p-3]_q\,xD_qy_n(x;q) + \lambda_{n,q}y_n(qx;q) = 0, \tag{25}$$

with

$$\lambda_{n,q} = q\,[n]_q\,[2p-n-2]_q,$$

for $n = 0, 1, 2, \ldots$ and $q \in \mathbb{R} \setminus \{-1, 0, 1\}$. The following limit relation holds true

$$\lim_{q\to 1} \lambda_{n,q} = -n(n + 2 - 2p),$$

which is the same value as in the continuous case (5).

The equivalent form of equation (25) is as

$$(x^2 + 1)y_n(qx; q) - \left((q^{2p-2-n} + q^n)x^2 + (q + 1)\right) y_n(x; q) + (q^{2p-2}x^2 + q)\, y_n(q^{-1}x; q) = 0.$$

Theorem 3.7. *Let $\{I_n^{(p)}(x; q)\}_n$ be a sequence of symmetric polynomials that satisfies the q-difference equation (25). For $n = 0, 1, \ldots, N$ we have*

$$\int_{-\infty}^{\infty} w_2(x; q) I_n^{(p)}(x; q) I_m^{(p)}(x; q) d_q x = \left(\int_{-\infty}^{\infty} w_2(x; q) \left(I_n^{(p)}(x; q)\right)^2 d_q x\right) \delta_{n,m},$$

where $q > 1$, $N = max\{m, n\}$, $N < p - 1$, $(-1)^{2p} = -1$, and the symmetric function $w_2(x; q)$ is the solution of the Pearson-type q-difference equation

$$D_q \left(w_2(x; q)(x^2 + 1)\right) = -q^2\, [2p - 3]_q\, w_2(qx; q),$$

which is equivalent to

$$\frac{w_2(x; q)}{w_2(qx; q)} = \frac{q^{2p-1}x^2 + 1}{x^2 + 1}. \tag{26}$$

Proof. First, it can be verified that

$$w_2(x; q) = \frac{x^{1-2p}}{(-x^{-2}; q^{-2})_{p-\frac{1}{2}}} \quad \left(0 < |q^{-2}| < 1\right), \tag{27}$$

is a solution of the Pearson-type q-difference equation (26), provided that $(-1)^{2p} = -1$. Note that

$$\lim_{q\to 1} w_2(x; q) = (1 + x^2)^{-(p-\frac{1}{2})},$$

which gives the same as weight function of orthogonal polynomials $\{I_n^{(p)}(x)\}_n$ [17].

Now change equation (25) in the self-adjoint form

$$D_q[w_2(x; q)(x^2 + 1)D_q I_n^{(p)}(x; q)] + \lambda_{n,q} w_2(qx; q) I_n^{(p)}(qx; q) = 0, \tag{28}$$

and for m as

$$D_q[w_2(x; q)(x^2 + 1)D_q I_m^{(p)}(x; q)] + \lambda_{m,q} w_2(qx; q) I_m^{(p)}(qx; q) = 0. \tag{29}$$

By multiplying (28) by $I_m^{(p)}(qx; q)$ and (29) by $I_n^{(p)}(qx; q)$ and subtracting each other we get

$$\begin{aligned}(\lambda_{m,q} - \lambda_{n,q}) w_2(x; q) I_m^{(p)}(x; q) I_n^{(p)}(x; q) \\ = q^2 D_q[w_2(q^{-1}x; q)(q^{-2}x^2 + 1) D_q I_n^{(p)}(q^{-1}x; q)] I_m^{(p)}(x; q) \\ - q^2 D_q[w_2(q^{-1}x; q)(q^{-2}x^2 + 1) D_q I_m^{(p)}(q^{-1}x; q)] I_n^{(p)}(x; q).\end{aligned} \tag{30}$$

Hence, q-integration by parts on both sides of (30) over $(-\infty,\infty)$ yields

$$(\lambda_{m,q}-\lambda_{n,q})\int_{-\infty}^{\infty} w_2(x;q)I_m^{(p)}(x;q)I_n^{(p)}(x;q)d_qx$$
$$=\int_{-\infty}^{\infty} q^2\Big\{D_q[w_2(q^{-1}x;q)(q^{-2}x^2+1)D_qI_n^{(p)}(q^{-1}x;q)]$$
$$\times I_m^{(p)}(x;q)-D_q[w_2(q^{-1}x;q)(q^{-2}x^2+1)D_qI_m^{(p)}(q^{-1}x;q)]I_n^{(p)}(x;q)\Big\}d_qx$$
$$=q^2\left[w_2(q^{-1}x;q)(q^{-2}x^2+1)\Big(D_qI_n^{(p)}(q^{-1}x;q)I_m^{(p)}(x;q)\ -D_qI_m^{(p)}(q^{-1}x;q)I_n^{(p)}(x;q)\Big)\right]_{-\infty}^{\infty}. \tag{31}$$

But since

$$\max\ \deg\{D_qI_n^{(p)}(q^{-1}x;q)I_m^{(p)}(x;q)-D_qI_m^{(p)}(q^{-1}x;q)I_n^{(p)}(x;q)\}=m+n-1,$$

if $N<p-1$ for $N=\max\{m,n\}$, the following boundary condition holds

$$\lim_{x\to\infty} w_2(q^{-1}x;q)(q^{-2}x^2+1)x^{2N-1}=0.$$

Therefore, the right hand side of (31) tends to zero and consequently

$$\int_{-\infty}^{\infty} w_2(x;q)I_m^{(p)}(x;q)I_n^{(p)}(x;q)d_qx=0,$$

if and only if $m\neq n$, $N<p-1$, $(-1)^{2p}=-1$, and $N=\max\{m,n\}$. □

Corollary 3.8. *The finite polynomial set $\{I_n^{(p)}(x;q)\}_{n=0}^{N<p-1}$ is orthogonal with respect to the even weight fuction $w_2(x;q)=\frac{x^{1-2p}}{(-x^{-2};q^{-2})_{p-\frac{1}{2}}}$ on $(-\infty,\infty)$. Especially, when $q\to 1$ in this set, a q-analogue of [17] is obtained.*

Proposition 3.9. *The monic polynomial solution of equation (25) has the following basic hypergeometric representation*

$$\bar{I}_n^{(p)}(x;q)=\frac{i^n q^{\frac{1}{2}n(n+3)-n(p+\frac{1}{2})}(q^{\frac{3}{2}-p},-q^{\frac{3}{2}-p};q)_n}{(q^{n-2p+2};q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, iq^{\frac{3}{2}-p}x^{-1}, q^{n-2p+2}\\ q^{\frac{3}{2}-p}, -q^{\frac{3}{2}-p}\end{matrix}\ \Big|\, q; iq^{p-\frac{1}{2}}x\right). \tag{32}$$

Proof. To obtain (32), it is enough to expand the polynomial solution of equation (25) as follows

$$I_n^{(p)}(x;q)=\sum_{k=0}^{n} a_{n,k}\frac{(-cx^{-1}q;q)_k}{(q;q)_k}(1-q)^k x^k,$$

where $c\in\mathbb{C}$, and $a_{n,n}\neq 0$, $n=0,1,2,\ldots$. If c satisfies the relation

$$q^{2p-1}c^2+q^2=0,$$

it can be verified that the coefficients $\{a_{n,k}\}_{k=0}^{n}$ satisfy the two-term recurrence relation

$$[n-k]_q(q^{2p-1}-q^{n+k+1})c\,a_{n,k}=(q^{2k}c^2+1)q^{n-k+1}a_{n,k+1},$$

for $k=0,1,\ldots,n-1$, and they are therefore determined uniquely up to the normalizing constant $a_{n,n}\neq 0$ as

$$a_{n,k}=c^{k-n}\left(\prod_{i=1}^{n-k}\frac{c^2q^{2n-i+1}+q^{i+1}}{[i]_q(q^{2p-1}-q^{2n-i+1})}\right)a_{n,n},$$

for $k = 0, 1, \ldots, n-1$. For $a_{n,n} = (1-q)^{-n}\,(q;\,q)_n$, the monic solution is finally derived as

$$\bar{I}_n^{(p)}(x;q) = i^n q^{\left(\frac{n(n+3)}{2}+n(p-\frac{3}{2})\right)} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (iq^{\frac{3}{2}-p}x^{-1};q)_k}{(q;q)_k} \left(\prod_{i=1}^{n-k} \frac{(1-q^{n-i-p+\frac{1}{2}})(1+q^{n-i-p+\frac{1}{2}})}{(q^{2p-1}-q^{2n-i+1})}\right)(iq^{\frac{1}{2}-p}x)^k,$$

which is equivalent to

$$\bar{I}_n^{(p)}(x;q) = \frac{i^n q^{\frac{1}{2}n(n+3)-n(p+\frac{1}{2})}(q^{\frac{3}{2}-p}, -q^{\frac{3}{2}-p};q)_n}{(q^{n-2p+2};q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (iq^{\frac{3}{2}-p}x^{-1};q)_k (q^{n-2p+2};q)_k}{(q^{\frac{3}{2}-p}, -q^{\frac{3}{2}-p})_k (q;q)_k} (iq^{p-\frac{1}{2}}x)^k.$$

□

Remark 3.10. *From (11), we can directly conclude that*

$$\lim_{q\to 1} \frac{(q^{n-2p+2};q)_n}{i^n q^{\frac{1}{2}n(n+3)-n(p+\frac{1}{2})}(q^{\frac{3}{2}-p}, -q^{\frac{3}{2}-p};q)_n} \bar{I}_n^{(p)}(x;q) = I_n^{(p)}(x),$$

where $I_n^{(p)}(x)$ is the polynomial solution of equation (5) and

$$I_n^{(p)}(x) = \frac{(-4i)^n (p-n)_n (\frac{3}{2}-p)_n}{(n+2-2p)_n}\, {}_2F_1\left(\begin{array}{c} -n\,, n+2-2p \\ \dfrac{3}{2}-p \end{array} \middle| \, \frac{1-ix}{2}\right).$$

Moreover, the q-difference equation (25) converges formally to equation (5) when $q \to 1$.

Proposition 3.11. *The following relation holds true*

$$D_q \bar{I}_n^{(p)}(x;q) = q^{1-n}[n]_q \bar{I}_{n-1}^{(p-1)}(qx;q).$$

Proof. From (32) we have

$$\bar{I}_n^{(p)}(x;q) - \bar{I}_n^{(p)}(qx;q) = \frac{i^n q^{\frac{1}{2}n(n+3)-n(p+\frac{1}{2})}(q^{\frac{5}{2}-p}, -q^{\frac{5}{2}-p};q)_{n-1}(1-q^{-n})}{(q^{n-2p+3};q)_{n-1}}$$
$$\times \sum_{k=1}^{n} \frac{(q^{1-n};q)_{k-1} (iq^{\frac{3}{2}-p}x^{-1};q)_{k-1} (q^{n-2p+3};q)_{k-1}}{(q^{\frac{5}{2}-p}, -q^{\frac{5}{2}-p};q)_{k-1}(q;q)_{k-1}} (iq^{p-\frac{1}{2}})^k\, x^k,$$

and

$$x\bar{I}_{n-1}^{(p-1)}(qx;q) = \frac{i^{n-1} q^{\frac{1}{2}(n-1)(n+2)-(n-1)(p-\frac{1}{2})}(q^{\frac{5}{2}-p}, -q^{\frac{5}{2}-p};q)_{n-1}}{(q^{n-2p+3};q)_{n-1}}$$
$$\times \sum_{k=1}^{n} \frac{(q^{1-n};q)_{k-1} (iq^{\frac{3}{2}-p}x^{-1};q)_{k-1} (q^{n-2p+3};q)_{k-1}}{(q^{\frac{5}{2}-p}, -q^{\frac{5}{2}-p};q)_{k-1}(q;q)_{k-1}} (iq^{p-\frac{1}{2}}x)^{k-1}\, x^k.$$

Therefore

$$\bar{I}_n^{(p)}(x;q) - \bar{I}_n^{(p)}(qx;q) = q^{1-n}(1-q^n)x\bar{I}_{n-1}^{(p-1)}(qx;q),$$

which is equivalent to

$$D_q \bar{I}_n^{(p)}(x;q) = q^{1-n}[n]_q \bar{I}_{n-1}^{(p-1)}(qx;q).$$

3.4. Computing the Norm Square Value

It is straightforward to find that the monic polynomial solution of equation (25) satisfies the recurrence relation

$$\bar{I}_{n+1}^{(p)}(x;q) = x\,\bar{I}_n^{(p)}(x;q) - d_n^* \bar{I}_{n-1}^{(p)}(x;q),$$

with the initial terms $\bar{I}_0^{(p)}(x;q) = 1$, $\bar{I}_1^{(p)}(x;q) = x$,

$$d_n^* = -q^{n-2p+2}\,\frac{[n]_q\,[n-2p+1]_q}{[2n-2p]_q\,[2n-2p+2]_q}.$$

Since $d_n^* > 0$ for $n = 1, 2, \ldots, N < p-1$, applying the Favard theorem [7] it yields

$$\int_{-\infty}^{\infty} w_2(x;q)\bar{I}_m^{(p)}(x;q)\bar{I}_n^{(p)}(x;q)d_qx = \left(\prod_{k=1}^{n} d_k^* \int_{-\infty}^{\infty} w_2(x;q)d_qx\right)\delta_{n,m},$$

where $w_2(x;q)$ is given by (27). In order to compute

$$\int_{-\infty}^{\infty} w_2(x;q)d_qx = \int_{-\infty}^{\infty} \frac{x^{1-2p}}{(-x^{-2};q^{-2})_{p-\frac{1}{2}}}d_qx,$$

we use the q-analogue of the change of variable (14) for $u(x) = x^{-\frac{1}{2}}$ to obtain

$$\int_{-\infty}^{\infty} \frac{x^{1-2p}}{(-x^{-2};q^{-2})_{p-\frac{1}{2}}}d_qx = \frac{2q^2}{(q+1)} \int_{0}^{\infty} \frac{x^{p-2}}{(-x;\,q^{-2})_{p-\frac{1}{2}}}\, d_{q^{-2}}x. \tag{33}$$

As the right hand side of equality (33) can be directly computed [8], so we have

$$\int_{-\infty}^{\infty} \frac{x^{1-2p}}{(-x^{-2};q^{-2})_{p-\frac{1}{2}}}d_qx = \frac{4q^2}{(q+1)(-1;q^{-2})_{p-1}(-1;q^{-2})_{2-p}}\frac{\Gamma_{q^{-2}}(p-1)\Gamma_{q^{-2}}(\frac{1}{2})}{\Gamma_{q^{-2}}(p-\frac{1}{2})}. \tag{34}$$

Hence, the norm square value of the monic q-polynomials (32) is computed as

$$\int_{-\infty}^{\infty} \frac{x^{1-2p}}{(-x^{-2};q^{-2})_{p-\frac{1}{2}}}\left(\bar{I}_n^{(p)}(x;q)\right)^2 d_qx = \frac{4(-1)^n q^{\frac{1}{2}(n^2+5n-4np+4)}(q;q)_n(q^{2-2p};q)_n}{(q+1)(-1;q^{-2})_{p-1}(-1;q^{-2})_{2-p}(q^{2-2p};q^2)_n(q^{4-2p};q^2)_n}\frac{\Gamma_{q^{-2}}(p-1)\Gamma_{q^{-2}}(\frac{1}{2})}{\Gamma_{q^{-2}}(p-\frac{1}{2})} \tag{35}$$

if and only if $n = 0, 1, ..., N < p-1$.

Remark 3.12. *Relation (34) helps us compute the moments corresponding to the weight function (27) as*

$$\mu_k = \int_{-\infty}^{\infty} \frac{x^{k+1-2p}}{(-x^{-2};\,q^{-2})_{p-\frac{1}{2}}}d_qx = \frac{4q^2\,\Gamma_{q^{-2}}(k+p-1)\Gamma_{q^{-2}}(\frac{1}{2}-k)}{(q+1)(-1;q^{-2})_{k+p-1}(-1;q^{-2})_{2-p-k}\,\Gamma_{q^{-2}}(p-\frac{1}{2})}.$$

Note that if $n > p-1$ in (35), the above moments would be divergent and the norm square value will be divergent too.

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