



Multiplicatively Weighted Harary Index of Some Composite Graphs

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Abstract. Recently, Alizadeh et al. [Discrete Math., 313 (2013): 26-34] proposed a modification of the Harary index in which the contributions of vertex pairs are weighted by the product of their degrees. It is named multiplicatively weighted Harary index and defined as: $H_M(G) = \sum_{u \neq v} \frac{\delta_G(u) \delta_G(v)}{d_G(u,v)}$, where $\delta_G(u)$ denotes the degree of the vertex u in the graph G and $d_G(u, v)$ denotes the distance between two vertices u and v in the graph G . In this paper, after establishing basic mathematical properties of this new index, we proceed by finding the extremal graphs and presenting explicit formulae for computing the multiplicatively weighted Harary index of the most important graph operations such as the join, composition, disjunction and symmetric difference of graphs.

1. Introduction

All graphs considered in this paper are finite undirected simple connected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $\delta_G(v)$ be the degree of a vertex v in G and $d_G(u, v)$ the distance between two vertices u and v in G . When the graph is clear from the context, we will omit the subscript G from the notation. For other undefined terminology and notations from graph theory, the readers are referred to [3].

A topological index is a number related to a graph invariant under graph isomorphisms. Obviously, the number of vertices and edges of a given graph G are topological indices of G . One of the oldest and well-studied distance-based topological index is the Wiener number $W(G)$, also termed as *Wiener index* in chemical or mathematical chemistry literature, which is defined [26] as the sum of distances over all unordered vertex pairs in G , namely,

$$W(G) = \sum_{u \neq v} d_G(u, v).$$

This equation was introduced by Hosoya [13], although the concept has been introduced by late Wiener. However, the approach by Wiener is applicable only to acyclic structures, whilst Hosoya matrix definition allowed the Wiener index to be used for any structure.

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Another distance-based graph invariant, defined [16, 22] in a fully analogous manner to Wiener index, is the *Harary index*, which is equal to the sum of reciprocal distances over all unordered vertex pairs in G , that is,

$$H(G) = \sum_{u \neq v} \frac{1}{d_G(u, v)}.$$

In 1994, Dobrynin and Kochetova [7] and Gutman [11] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance* or *Schultz molecular topological index*, which is defined for a graph G as

$$DD_A(G) = \sum_{u \neq v} (\delta_G(u) + \delta_G(v))d_G(u, v).$$

The Gutman index is put forward in [11] and called there the *Schultz index of the second kind*, but for which the name *Gutman index* has also sometimes been used [24]. It is defined as

$$DD_M(G) = \sum_{u \neq v} \delta_G(u)\delta_G(v)d_G(u, v).$$

The interested readers may consult [6, 10, 12] for Wiener index, [22] for Harary index, [4, 5, 14, 25] for degree distance and [9, 20] for Gutman index.

Although Harary index is not well known in the mathematical chemistry community, it arises in the study of complex networks. Let n denotes the order of G . By dividing $H(G)$ by $n(n - 1)$, we obtain a normalization of $H(G)$, which is called the efficiency of G [18]. The reciprocal value of the efficiency is called the performance of G [19]. For a given network, both efficiency and performance afford a uniform way to express and quantify the small-world property. Since the strength of interactions between nodes in a network is seldom properly described by their topological distances, one need to consider also the weighted versions of efficiency and performance.

In order to close the gap between the two research communities by drawing their attention to a generalization of a concept, which gives more weight to the contributions of pairs of vertices of high degrees, recently, Alizadeh et al. [1] introduced an invariant, named *additively weighted Harary index*, which is defined as

$$H_A(G) = \sum_{u \neq v} \frac{\delta_G(u) + \delta_G(v)}{d_G(u, v)}.$$

Some basic mathematical properties of this index were established and its behavior under several standard graph products were investigated there.

It is known that the intuitive idea of pairs of close atoms contributing more than the distant ones has been difficult to capture in topological indices. A possibly useful approach could be to replace the additive weighting of pairs by the multiplicative one, thus giving rise to a new invariant, named *multiplicatively weighted Harary index* [1]:

$$H_M(G) = \sum_{u \neq v} \frac{\delta_G(u) \cdot \delta_G(v)}{d_G(u, v)}.$$

Evidently, the additively (multiplicatively, respectively) weighted Harary index is related to the Harary index in the same way as the degree distance (Gutman index, respectively) is related to the Wiener index.

In [1], Alizadeh et al. also proposed an open problem: It would be interesting to explore mathematical properties of multiplicatively weighted Harary index and would be useful to investigate the behavior of $H_M(G)$ under graph operations.

In this paper, we successfully solve this problem. That is, we establish basic mathematical properties of $H_M(G)$, and give the explicit formulae for multiplicatively weighted Harary index of the join, composition,

disjunction and symmetric difference of graphs.

The paper is organized as follows. In Section 2, we give the necessary definitions and some auxiliary results. In Section 3, we present our main results. Some examples will be given in the last section.

2. Preliminaries

2.1 Some definitions

Let K_n, C_n, P_n and S_n denote the n -vertex complete graph, cycle, path and star graph, respectively. For a given graph G , its first and second Zagreb indices are defined as follows:

$$M_1(G) = \sum_{u \in V(G)} \delta^2(u), \quad M_2(G) = \sum_{e=uv \in E(G)} \delta(u)\delta(v),$$

where $\delta^2(u) = \delta(u)^2$. The first Zagreb index can be also expressed as a sum over edges of G ,

$$M_1(G) = \sum_{e=uv \in E(G)} (\delta(u) + \delta(v)).$$

For the proof of this fact and more information on Zagreb indices we encourage the interested reader to [21].

The first and the second Zagreb coindices of a graph G are defined as follows:

$$\overline{M}_1(G) = \sum_{e=uv \notin E(G)} (\delta(u) + \delta(v)), \quad \overline{M}_2(G) = \sum_{e=uv \notin E(G)} \delta(u)\delta(v).$$

The Zagreb indices and Zagreb coindices, in particular for the following result, which was proved in [2], will be helpful in presenting our main results in a more compact form.

Lemma 2.1. Let G be a graph on n vertices and e edges. Then

$$\begin{aligned} \text{(i)} \quad \overline{M}_1(G) &= 2e(n-1) - M_1(G); \\ \text{(ii)} \quad \overline{M}_2(G) &= 2e^2 - M_2(G) - \frac{1}{2}M_1(G). \end{aligned}$$

2.2 Extremal graphs

It is obvious that adding an edge to G will increase its multiplicatively weighted Harary index. So we obtain the following result immediately.

Theorem 2.1. Let G be any graph on n vertices. Then $H_M(G) \leq H_M(K_n)$.

By the same remark mentioned above, the tree has the smallest $H_M(G)$ among all graphs on the same number of vertices. It is known that the extremal tree for the ordinary Harary index is the path [10]. We can prove, following the method of [1], that this is also the case for the multiplicatively weighted version.

Theorem 2.2. Let G be any graph on n vertices. Then $H_M(G) \geq H_M(P_n)$.

Proof. By the above argument, we only need to consider trees on n vertices. Let T_n be such a tree, and let v be any vertex of T_n of degree at least 3 such that at least two of the components of $T_n - v$ are paths. Let those paths be of lengths s and l , with $s \leq l$. We denote the tree induced by the vertices not in the above two paths by R . Let us call such a tree $T_{s,l}$. We transform $T_{s,l}$ by transplanting the end-vertex of the shorter path to the end-vertex of the longer path, obtaining a tree we denote by $T_{s-1,l+1}$. Evidently, R is not affected by such a transformation. The transformation is illustrated in Fig. 1. We proceed by comparing the contributions of various pairs of vertices to the values of $T_{s,l}$ and $T_{s-1,l+1}$. We consider the following two cases.

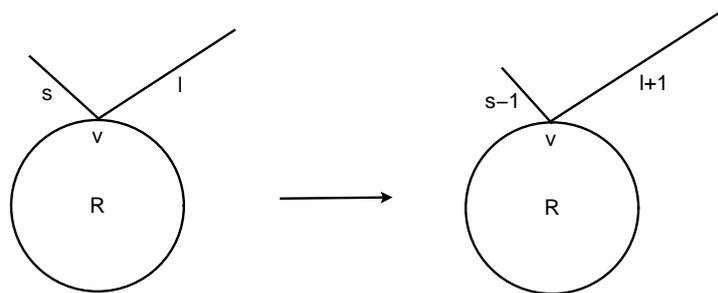


Fig. 1. Transformation for general graphs.

Case 1. $s > 1$.

Obviously, the contributions of all pairs not including the transplanted vertex and its neighbors remain unaffected by our transformation. Moreover, it is obvious that the contributions involving the transplanted vertex are smaller in $T_{s-1,l+1}$ than in $T_{s,l}$, since the distances involved are greater. The only contributions that are greater in $T_{s-1,l+1}$ than in $T_{s,l}$ are those involving the former end-vertex of l -path. For a vertex x at distance d from v such contributions are $\frac{2\delta(x)}{d+l}$ and $\frac{\delta(x)}{d+l}$, respectively. Hence, the net change per vertex u of R is $\frac{\delta(x)}{d+l}$ in surplus for $T_{s-1,l+1}$. That surplus is, however, at least offset by the change in the contributions of pairs containing the new end-vertex of the shorter path. Previous contributions $\frac{2\delta(x)}{d+s-1}$ become $\frac{\delta(x)}{d+s-1}$, resulting in a net loss of $\frac{\delta(x)}{d+s-1}$ per vertex x at distance d from v . Since $s - 1 < l$, such loss more than offsets the gain on the longer side, and hence $H_M(T_{s-1,l+1}) \leq H_M(T_{s,l})$.

Case 2. $s = 1$.

We still follow the same pattern discussed above. In this case, our transformation also changes the degree of v by decreasing it by 1. The only contributions that are greater in $H_M(T_{s-1,l+1})$ than the corresponding contributions in $H_M(T_{s,l})$ are those involving the former end-vertex on the longer side. The net surplus per vertex is again $\frac{\delta(x)}{d+l}$ per vertex x of R at distance d from v . Once more, this is compensated by the loss of $\frac{\delta(x)}{d}$ per each such vertex coming from the decrease in the degree of v . It remains to consider the change in the contributions of pairs (v, y) where y is on the remaining path of length $l + 1$. All such contributions in $H_M(T_{s-1,l+1})$ are smaller than the corresponding contributions in $H_M(T_{s,l})$, except from the last two vertices. Their combined contributions are $\frac{2[\delta(v)-1]}{l} + \frac{\delta(v)-1}{l+1}$. This quantity, however, cannot exceed the value of $\delta(v)$, representing the loss from the transplanted vertex, since $\delta(v) > \frac{2[\delta(v)-1]}{l} + \frac{\delta(v)-1}{l+1}$ for all $l \geq 2$. Again, $H_M(T_{s-1,l+1}) \leq H_M(T_{s,l})$. ■

2.3 Composite graphs

Now we introduce the four standard types of composite graphs that we consider in this paper. Let G_1 and G_2 be two graphs. The sum of these graphs is defined as a graph $G_1 + G_2$ with the vertex set

$V(G_1 + G_2) = V(G_1) \cup V(G_2)$, and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1), u_2 \in V(G_2)\}.$$

In other words, we join every vertex from G_1 to every vertex in G_2 . The sum of two graphs is sometimes also called *join*, and denoted by $G_1 \nabla G_2$ ([23]). Obviously, any two vertices of $G_1 + G_2$ are either at the distance 1 or at the distance 2. When one of the components is K_1 , we have a special case called *suspension* of G , $K_1 + G$.

The next binary operation we consider is the composition of two graphs. For given graphs G_1 and G_2 , their *composition* is the graph $G_1[G_2]$ with the vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$, and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1[G_2]$ are adjacent if and only if $[u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)]$ or $[u_1v_1 \in E(G_1)]$. In other words, $G_1[G_2]$ is obtained by expanding each vertex of G_1 into a copy of G_2 , with each edge of G_1 replaced by the edge set of the corresponding $G_2 + G_2$. Unlike the sum, the composition is not a symmetric operation.

The *disjunction* $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2)\}.$$

The *symmetric difference* $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both}\}.$$

The disjunction and symmetric difference are both symmetric operations that share a number of common properties. The most remarkable is that their diameter never exceeds 2.

For more information about graph products, please see monograph [15]. There is a growing corpus of literature concerned with the study of graph invariants of composite graphs [e.g., 8, 27]. The following result, which was proved in that literature, will be used in our proofs. We refer the reader to [17] for the details.

Lemma 2.2. Let G_1 and G_2 be two graphs. The number of vertices and edges of graph G_i is denoted by n_i and e_i respectively for $i = 1, 2$. Then we have

(a)

$$d_{G_1+G_2}(u, v) = \begin{cases} 1 & uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } (u \in V(G_1) \text{ and } v \in V(G_2)) \\ 2 & \text{otherwise} \end{cases}$$

For a vertex u of G_1 , $\delta_{G_1+G_2}(u) = \delta_{G_1}(u) + n_2$, and for a vertex v of G_2 , $\delta_{G_1+G_2}(v) = \delta_{G_2}(v) + n_1$.

(b)

$$d_{G_1[G_2]}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_{G_1}(u_1, u_2) & v_1 = v_2 \\ 1 & u_1 = u_2, v_1v_2 \in E(G_2) \\ 2 & \text{otherwise} \end{cases}$$

$\delta_{G_1[G_2]}((u, v)) = n_2\delta_{G_1}(u) + \delta_{G_2}(v)$.

(c)

$$d_{G_1 \vee G_2}((u_1, v_1), (u_2, v_2)) = \begin{cases} 1 & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \\ 2 & \text{otherwise} \end{cases}$$

$\delta_{G_1 \vee G_2}((u, v)) = n_2\delta_{G_1}(u) + n_1\delta_{G_2}(v) - \delta_{G_1}(u)\delta_{G_2}(v)$.

(d)

$$d_{G_1 \oplus G_2}((u_1, v_1), (u_2, v_2)) = \begin{cases} 1 & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \text{ but not both} \\ 2 & \text{otherwise} \end{cases}$$

$\delta_{G_1 \oplus G_2}((u, v)) = n_2\delta_{G_1}(u) + n_1\delta_{G_2}(v) - 2\delta_{G_1}(u)\delta_{G_2}(v)$.

3. Main Results

In this section we state and prove our main results, by giving explicit formulae for multiplicatively weighted Harary indices of composite graphs in terms of Harary indices and multiplicatively (additively) weighted Harary indices and some simple graphic invariants of underlying components. We begin with an example. Let $H_n = \sum_{k=1}^n \frac{1}{k}$.

Example 3.1. Evidently, if G is a k -regular graph, then $H_M(G) = k^2H(G)$. The multiplicatively weighted Harary indices of K_n, C_n, P_n and S_n are computed as follows:

$$H_M(K_n) = \frac{n(n-1)^3}{2}$$

$$H_M(C_n) = \begin{cases} 4(nH_{n/2} - 1), & n \text{ is even} \\ 4nH_{(n-1)/2}, & n \text{ is odd} \end{cases}$$

$$H_M(P_n) = n(H_n - 1) + 2H_{n-1} - \frac{2}{n-1}$$

$$H_M(S_n) = \frac{1}{4}(n-1)(5n-6).$$

It is well known [1] that $H(P_n) = n(H_n - 1)$, so $H_M(P_n) = H(P_n) + 2H_{n-1} - \frac{2}{n-1}$.

3.1 Sum

Theorem 3.1. Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} H_M(G_1 + G_2) = & \frac{1}{2}(M_2(G_1) + M_2(G_2)) + \frac{1}{4}(2n_2 - 1)M_1(G_1) + \frac{1}{4}(2n_1 - 1)M_1(G_2) - \frac{1}{4}n_1n_2(n_1 + n_2) \\ & + \frac{1}{2}n_2e_1(6n_1 + n_2 - 2) + \frac{1}{2}n_1e_2(6n_2 + n_1 - 2) + 2e_1e_2 + (e_1 + e_2)^2 + \frac{3}{2}n_1^2n_2^2. \end{aligned}$$

Proof. By definition of the sum of two graphs, one can see that, for any $u, v \in V(G_1 + G_2)$, the distance between them $d_{G_1+G_2}(u, v)$ is either 1 or 2. In the formula for $H_M(G_1 + G_2)$, we partition the set of pairs of vertices of $G_1 + G_2$ into three cases, denoted by A_0, A_1 , and A_2 . In A_0 , we collect all pairs of vertices u and v that u is in G_1 and v is in G_2 . Hence, they are adjacent in $G_1 + G_2$. The set $A_i, i = 1, 2$, is the set of pairs of vertices u and v such that they are in G_i . Also, we partition the sum in the formula of $H_M(G_1 + G_2)$ into three sums S_i so that S_i is over A_i for $i = 0, 1, 2$. For S_0 , we have

$$\begin{aligned} S_0 &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (\delta_{G_1}(u) + n_2)(\delta_{G_2}(v) + n_1) \\ &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (\delta_{G_1}(u)\delta_{G_2}(v) + n_1\delta_{G_1}(u) + n_2\delta_{G_2}(v) + n_1n_2) \\ &= 4e_1e_2 + 2n_1n_2(e_1 + e_2) + n_1^2n_2^2. \end{aligned}$$

Since S_1 and S_2 have the same structure, it is enough to calculate one of them. By Lemma 2.1 (i), we have

$$\begin{aligned} S_1 &= \sum_{\{u,v\} \subseteq V(G_1)} \frac{(\delta_{G_1}(u) + n_2)(\delta_{G_1}(v) + n_2)}{d_{G_1+G_2}(u, v)} \\ &= \sum_{uv \in E(G_1)} \left[\delta_{G_1}(u)\delta_{G_1}(v) + n_2(\delta_{G_1}(u) + \delta_{G_1}(v)) + n_2^2 \right] \\ &\quad + \sum_{uv \notin E(G_1)} \frac{\delta_{G_1}(u)\delta_{G_1}(v) + n_2(\delta_{G_1}(u) + \delta_{G_1}(v)) + n_2^2}{2} \\ &= \frac{1}{2}n_2^2 \binom{n_1}{2} + \frac{1}{2}M_2(G_1) + \frac{1}{4}(2n_2 - 1)M_1(G_1) + e_1^2 + \frac{1}{2}n_2e_1(2n_1 + n_2 - 2). \end{aligned}$$

Similarly,

$$S_2 = \frac{1}{2}n_1^2 \binom{n_2}{2} + \frac{1}{2}M_2(G_2) + \frac{1}{4}(2n_1 - 1)M_1(G_2) + e_2^2 + \frac{1}{2}n_1e_2(2n_2 + n_1 - 2).$$

Thus, addition of the three sums and simplification of the resulting expression, we obtain the desired result. ■

3.2 Composition

The components of composition enter the operation in a markedly asymmetric manner. That fact is reflected in the formula for the $H_M(G_1[G_2])$.

Theorem 3.2. Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} H_M(G_1[G_2]) &= n_2^4 H_M(G_1) + 2n_2^2 e_2 H_A(G_1) + 4e_2^2 H(G_1) + \frac{1}{2}n_2^2 \left(\binom{n_2}{2} + e_2 \right) M_1(G_1) \\ &\quad + \frac{1}{4}(4n_2e_1 - n_1) M_1(G_2) + \frac{1}{2}n_1 M_2(G_2) + e_2 (2e_1 n_2^2 - 2e_1 n_2 + n_1 e_2). \end{aligned}$$

Proof. Suppose $G = G_1[G_2]$. For each vertex x of G_1 , we label the corresponding copy of G_2 $G_{2,x}$. If two vertices x, y of G_1 are adjacent, then every pair of vertices of $G_{2,x}$ and $G_{2,y}$ are adjacent too. We have

$$\begin{aligned} H_M(G) &= \sum_{x,y \in V(G_1)} \sum \left\{ \frac{\delta_G(u) \cdot \delta_G(v)}{d_G(u, v)} \mid u \in G_{2,x}, v \in G_{2,y} \right\} \\ &= \sum_{x \in V(G_1)} \sum \left\{ \frac{\delta_G(u) \cdot \delta_G(v)}{d_G(u, v)} \mid u, v \in G_{2,x} \right\} \\ &\quad + \sum_{x,y \in V(G_1)} \sum \left\{ \frac{\delta_G(u) \cdot \delta_G(v)}{d_G(u, v)} \mid u \in G_{2,x}, v \in G_{2,y}, x \neq y \right\}. \end{aligned}$$

We partition the sum into two sums, S_1 and S_2 . The first one, S_1 , runs over all pairs of vertices u and v in $G_{2,x}$ for each vertex x in G_1 . The second one, S_2 , is over all pairs of vertices u and v such that u is in $G_{2,x}$ and

v is in $G_{2,y}$ for x, y in $G_1, x \neq y$.

$$\begin{aligned} S_1 &= \sum_{x \in V(G_1)} \sum \left\{ \frac{n_2^2 \delta_{G_1}^2(x) + n_2 \delta_{G_1}(x)(\delta_{G_{2,x}}(u) + \delta_{G_{2,x}}(v)) + \delta_{G_{2,x}}(u)\delta_{G_{2,x}}(v)}{d_G(u, v)} \mid u, v \in G_{2,x} \right\} \\ &= \sum_{x \in V(G_1)} \sum_{uv \in E(G_2)} \left\{ n_2^2 \delta_{G_1}^2(x) + n_2 \delta_{G_1}(x)(\delta_{G_{2,x}}(u) + \delta_{G_{2,x}}(v)) + \delta_{G_{2,x}}(u)\delta_{G_{2,x}}(v) \mid u, v \in G_{2,x} \right\} \\ &\quad + \sum_{x \in V(G_1)} \sum_{uv \notin E(G_2)} \left\{ \frac{n_2^2 \delta_{G_1}^2(x) + n_2 \delta_{G_1}(x)(\delta_{G_{2,x}}(u) + \delta_{G_{2,x}}(v)) + \delta_{G_{2,x}}(u)\delta_{G_{2,x}}(v)}{2} \mid u, v \in G_{2,x} \right\} \\ &= \frac{1}{2} n_2^2 \left(\binom{n_2}{2} + e_2 \right) M_1(G_1) + \frac{1}{4} (4n_2 e_1 - n_1) M_1(G_2) + \frac{1}{2} n_1 M_2(G_2) + e_2 (2e_1 n_2^2 - 2e_1 n_2 + n_1 e_2). \end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{x,y \in V(G_1)} \sum \left\{ \frac{\delta_G(u) \cdot \delta_G(v)}{d_G(u, v)} \mid u \in G_{2,x}, v \in G_{2,y}, x \neq y \right\} \quad (d_G(u, v) = d_{G_1}(x, y)) \\ &= \sum_{x,y \in V(G_1)} \sum \left\{ \frac{(n_2 \delta_{G_1}(x) + \delta_{G_2}(u))(n_2 \delta_{G_1}(y) + \delta_{G_2}(v))}{d_{G_1}(x, y)} \mid u \in G_{2,x}, v \in G_{2,y}, x \neq y \right\} \\ &= n_2^4 H_M(G_1) + 4e_2^2 H(G_1) + 2n_2^2 e_2 H_A(G_1). \end{aligned}$$

Hence we have

$$\begin{aligned} H_M(G_1[G_2]) &= n_2^4 H_M(G_1) + 2n_2^2 e_2 H_A(G_1) + 4e_2^2 H(G_1) + \frac{1}{2} n_2^2 \left(\binom{n_2}{2} + e_2 \right) M_1(G_1) \\ &\quad + \frac{1}{4} (4n_2 e_1 - n_1) M_1(G_2) + \frac{1}{2} n_1 M_2(G_2) + e_2 (2e_1 n_2^2 - 2e_1 n_2 + n_1 e_2). \blacksquare \end{aligned}$$

In what follows, for convenience, we assume that \bar{e}_i is equal to $\binom{n_i}{2} - e_i$.

3.3 Disjunction

Theorem 3.3. Let G_1 and G_2 be two graphs. Then

$$\begin{aligned} H_M(G_1 \vee G_2) &= \frac{1}{2} (n_2^3 - 4n_2 e_2 + 2n_2^2 \bar{e}_2) \bar{M}_2(G_1) + \frac{1}{2} (n_1^3 - 4n_1 e_1 + 2n_1^2 \bar{e}_1) \bar{M}_2(G_2) + \frac{1}{2} M_1(G_1) \bar{M}_2(G_2) \\ &\quad + \frac{1}{2} n_1 n_2 \bar{M}_1(G_2) \bar{M}_1(G_1) + (n_1^4 + 4e_1^2 - 6n_1^2 e_1) M_2(G_2) + (n_2^4 + 4e_2^2 - 6n_2^2 e_2) M_2(G_1) \\ &\quad - \frac{1}{2} n_2 M_1(G_1) \bar{M}_1(G_2) - \frac{1}{2} n_1 M_1(G_2) \bar{M}_1(G_1) + \frac{1}{2} M_1(G_2) \bar{M}_2(G_1) + \bar{M}_2(G_2) \bar{M}_2(G_1) \\ &\quad + 4e_1 e_2 (e_1 n_2^2 + e_2 n_1^2) + 2n_2 M_1(G_2) M_2(G_1) - n_1 \bar{M}_1(G_1) \bar{M}_2(G_2) - 2M_2(G_1) M_2(G_2) \\ &\quad + \frac{1}{2} (4n_1^2 n_2 e_1 - 8n_2 e_1^2 + n_1^2 \bar{e}_1) M_1(G_2) + \frac{1}{2} (4n_1 n_2^2 e_2 - 8n_1 e_2^2 + n_2^2 \bar{e}_2) M_1(G_1) \\ &\quad + n_1 n_2 (e_1 \bar{M}_1(G_2) + e_2 \bar{M}_1(G_1)) + 2n_1 M_1(G_1) M_2(G_2) - n_2 \bar{M}_1(G_2) \bar{M}_2(G_1) \\ &\quad - n_1 n_2 M_1(G_1) M_1(G_2). \end{aligned}$$

Proof. Suppose $G = G_1 \vee G_2$. By Lemma 2.2 (c), the degree of vertex (x, u) in G is $n_2 \delta_{G_1}(x) + n_1 \delta_{G_2}(u) -$

$\delta_{G_1}(x)\delta_{G_2}(u)$. Thus

$$\begin{aligned} H_M(G_1 \vee G_2) &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \left\{ \frac{\delta_G((x,u)) \cdot \delta_G((y,v))}{d_G((x,u), (y,v))} \right\} \\ &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \frac{n_2^2 \delta_{G_1}(x) \delta_{G_1}(y) + n_1^2 \delta_{G_2}(u) \delta_{G_2}(v) + \delta_{G_1}(x) \delta_{G_1}(y) \delta_{G_2}(u) \delta_{G_2}(v)}{d_G((x,u), (y,v))} \\ &\quad - \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \frac{n_1(\delta_{G_1}(x) + \delta_{G_1}(y)) \delta_{G_2}(u) \delta_{G_2}(v)}{d_G((x,u), (y,v))} \\ &\quad - \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \frac{n_2(\delta_{G_2}(u) + \delta_{G_2}(v)) \delta_{G_1}(x) \delta_{G_1}(y)}{d_G((x,u), (y,v))} \\ &\quad + \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \frac{n_1 n_2 (\delta_{G_1}(x) \delta_{G_2}(v) + \delta_{G_1}(y) \delta_{G_2}(u))}{d_G((x,u), (y,v))}. \end{aligned}$$

We consider four sums S_1, \dots, S_4 as follows:

$$\begin{aligned} S_1 &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \{ \delta_G((x,u)) \cdot \delta_G((y,v)) | uv \in E(G_2) \} \\ &= 4e_1^2 e_2 n_2^2 + (n_1^4 + 4e_1^2 - 4n_1^2 e_1) M_2(G_2) + (2n_1^2 n_2 e_1 - 4n_2 e_1^2) M_1(G_2). \end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \{ \delta_G((x,u)) \cdot \delta_G((y,v)) | xy \in E(G_1) \} \\ &= 4e_1 e_2^2 n_1^2 + (n_2^4 + 4e_2^2 - 4n_2^2 e_2) M_2(G_1) + (2n_1 n_2^2 e_2 - 4n_1 e_2^2) M_1(G_1). \end{aligned}$$

$$\begin{aligned} S_3 &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \{ \delta_G((x,u)) \cdot \delta_G((y,v)) | xy \in E(G_1), uv \in E(G_2) \} \\ &= 2n_2^2 e_2 M_2(G_1) + 2(n_2^2 e_1 + M_2(G_1) - n_1 M_1(G_1)) M_2(G_2) - 2n_2 M_2(G_1) M_1(G_2) + n_1 n_2 M_1(G_1) M_1(G_2). \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \left\{ \frac{\delta_G((x,u)) \cdot \delta_G((y,v))}{2} \middle| xy \notin E(G_1), uv \notin E(G_2) \right\} \\ &= \frac{1}{2} \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} \{ \delta_G((x,u)) \cdot \delta_G((y,v)) | xy \notin E(G_1), uv \notin E(G_2), x \neq y, u \neq v \} \\ &\quad + \frac{1}{2} \sum_{xy \notin E(G_1)} \sum_{u \in V(G_2)} \left[(n_2^2 - 2n_2 \delta(u) + \delta^2(u)) \delta(x) \delta(y) + (n_1 n_2 \delta(u) - n_1 \delta^2(u)) (\delta(x) + \delta(y)) + n_1^2 \delta^2(u) \right] \\ &\quad + \frac{1}{2} \sum_{x \in V(G_1)} \sum_{uv \notin E(G_2)} \left[(n_1^2 - 2n_1 \delta(x) + \delta^2(x)) \delta(u) \delta(v) + (n_1 n_2 \delta(x) - n_2 \delta^2(x)) (\delta(u) + \delta(v)) + n_2^2 \delta^2(x) \right] \\ &= \frac{1}{2} \left[(n_2^3 - 4n_2 e_2 + M_1(G_2) + 2n_2^2 \bar{e}_2 + 2\bar{M}_2(G_2) - 2n_2 \bar{M}_1(G_2)) \bar{M}_2(G_1) + (n_1^3 - 4n_1 e_1 + M_1(G_1) \right. \\ &\quad \left. + 2n_1^2 \bar{e}_1 - 2n_1 \bar{M}_1(G_1)) \bar{M}_2(G_2) + (2n_1 n_2 e_2 - n_1 M_1(G_2) + n_1 n_2 \bar{M}_1(G_2)) \bar{M}_1(G_1) \right. \\ &\quad \left. + (2n_1 n_2 e_1 - n_2 M_1(G_1)) \bar{M}_1(G_2) + n_1^2 \bar{e}_1 M_1(G_2) + n_2^2 \bar{e}_2 M_1(G_1) \right]. \end{aligned}$$

Hence we have

$$\begin{aligned}
 H_M(G_1 \vee G_2) &= S_1 + S_2 + S_4 - S_3 \\
 &= \frac{1}{2}(n_2^3 - 4n_2e_2 + 2n_2^2\bar{e}_2)\bar{M}_2(G_1) + \frac{1}{2}(n_1^3 - 4n_1e_1 + 2n_1^2\bar{e}_1)\bar{M}_2(G_2) + \frac{1}{2}M_1(G_1)\bar{M}_2(G_2) \\
 &\quad + \frac{1}{2}n_1n_2\bar{M}_1(G_2)\bar{M}_1(G_1) + (n_1^4 + 4e_1^2 - 6n_1^2e_1)M_2(G_2) + (n_2^4 + 4e_2^2 - 6n_2^2e_2)M_2(G_1) \\
 &\quad - \frac{1}{2}n_2M_1(G_1)\bar{M}_1(G_2) - \frac{1}{2}n_1M_1(G_2)\bar{M}_1(G_1) + \frac{1}{2}M_1(G_2)\bar{M}_2(G_1) + \bar{M}_2(G_2)\bar{M}_2(G_1) \\
 &\quad + 4e_1e_2(e_1n_2^2 + e_2n_1^2) + 2n_2M_1(G_2)M_2(G_1) - n_1\bar{M}_1(G_1)\bar{M}_2(G_2) - 2M_2(G_1)M_2(G_2) \\
 &\quad + \frac{1}{2}(4n_1^2n_2e_1 - 8n_2e_1^2 + n_1^2\bar{e}_1)M_1(G_2) + \frac{1}{2}(4n_1n_2^2e_2 - 8n_1e_2^2 + n_2^2\bar{e}_2)M_1(G_1) \\
 &\quad + n_1n_2(e_1\bar{M}_1(G_2) + e_2\bar{M}_1(G_1)) + 2n_1M_1(G_1)M_2(G_2) - n_2\bar{M}_1(G_2)\bar{M}_2(G_1) \\
 &\quad - n_1n_2M_1(G_1)M_1(G_2).
 \end{aligned}$$

This completes the proof. ■

3.4 Symmetric difference

As mentioned in Section 2, the disjunction and symmetric difference are very much alike. So we present the following result similar to Theorem 3.3.

Theorem 3.4. Let G_1 and G_2 be two graphs. Then

$$\begin{aligned}
 H_M(G_1 \oplus G_2) &= \frac{1}{2}n_2(4n_1e_2 + n_2\bar{e}_2 + 2n_2e_2)M_1(G_1) + \frac{1}{2}n_1(4n_2e_1 + n_1\bar{e}_1 + 2n_1e_1)M_1(G_2) + \frac{1}{2}n_2[n_2^2 \\
 &\quad - 8e_2 + n_2^2(n_2 - 1)]\bar{M}_2(G_1) + \frac{1}{2}n_1[n_1^2 - 8e_1 + n_1^2(n_1 - 1)]\bar{M}_2(G_2) + (n_2^3 + n_2^2\binom{n_2}{2} \\
 &\quad - 8n_2e_2 + 8e_2^2)M_2(G_1) + n_1(n_1^2 + n_1\binom{n_1}{2} - 8e_1)M_2(G_2) + 2[\bar{M}_2(G_2) + 2M_2(G_2)] \\
 &\quad \cdot M_1(G_1) - n_2[\bar{M}_1(G_2) + 2M_1(G_2)]M_1(G_1) - n_1[\bar{M}_1(G_1) + 2M_1(G_1)]M_1(G_2) \\
 &\quad + n_1n_2(e_2\bar{M}_1(G_1) + e_1\bar{M}_1(G_2)) - 2n_1[\bar{M}_1(G_1) + M_1(G_1)][\bar{M}_2(G_2) + M_2(G_2)] \\
 &\quad + \frac{1}{2}n_1n_2[\bar{M}_1(G_1) + M_1(G_1)][\bar{M}_1(G_2) + M_1(G_2)] + 2[\bar{M}_2(G_1) + M_2(G_1)]M_1(G_2) \\
 &\quad - 2n_2[\bar{M}_2(G_1) + M_2(G_1)][\bar{M}_1(G_2) + M_1(G_2)] + 4[\bar{M}_2(G_2) + M_2(G_2)]\bar{M}_2(G_1).
 \end{aligned}$$

4. Examples

In this section our theorems for multiplicatively weighted Harary indices are illustrated for several more particular composite graphs. We first give the expressions for suspensions.

Corollary 4.1.

$$H_M(K_1 + G) = \frac{1}{2}M_2(G) + \frac{1}{4}M_1(G) + e^2 + \left(3n - \frac{1}{2}\right)e.$$

Next, the formulae for the *fan graph* $K_1 + P_n$ and the *wheel graph* $W_n = K_1 + C_n$ are presented as follows:

Corollary 4.2.

$$H_M(K_1 + P_n) = 4n^2 - \frac{5}{2}n - 4,$$

$$H_M(K_1 + C_n) = 4n^2 + \frac{5}{2}n.$$

By composing paths and cycles with various small graphs, we can obtain different classes of polymer-like graphs. Thus, we finally state the formulae of the H_M index for the *fence graph* $P_n[K_2]$ and the *closed fence* $C_n[K_2]$.

Corollary 4.3.

$$H_M(P_n[K_2]) = 16H_M(P_n) + 8H_A(P_n) + 4H(P_n) + 25n - 32,$$

$$H_M(C_n[K_2]) = 100H(C_n) + 25n.$$

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