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Graded Diextremities

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Abstract. In this paper, the concept of graded diextremity is defined on textures as a generalization of diextremities on textures and some properties of graded diextremity are obtained. It is shown that each graded diuniformity generates a graded diextremity and each graded diextremity genarates a graded ditopology. Moreover, the relations between graded diextremities (resp. graded diuniformities, graded ditopologies) and diextremities (resp. diuniformities, ditopologies) are investigated in basic categorical aspects.

1. Introduction

The concept of fuzzy topological space was defined in 1968 by C. Chang as ordinary subset of the family of all fuzzy subsets of a given set [8]. As a more suitable approach to the idea of fuzzyness, in 1985, Šostak and Kubiak independently redefined fuzzy topology where a fuzzy subset has a degree of openness rather than being open or not [12, 17].

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different. Indeed, a bitopology consists of two separate topological structures whose interrelations are studied, whereas a ditopology represents a single topological structure.

Ditopological texture spaces were introduced by L.M. Brown as a natural extension of the work on the representation of lattice-valued topologies by bitopologies in [11]. Ditopology is more general than general topology, bitopology and fuzzy topology in Chang's sense. An adequate introduction to the theory of textures and ditopological texture spaces may be obtained from [2–6, 18]. G. Yıldız and R. Ertürk have introduced diextremity as an extension of proximity in the sense of [13] to the texture spaces and investigated interrelations between these two structures in [20].

Recently, L.M. Brown and A. Šostak have presented "graded ditopology" on textures as an extension of ditopology to the case where openness and closedness are given in terms of a priori unrelated grading functions [7]. Graded ditopology is more general than ditopology and fuzzy topology in Šostak's sense. Two sorts of neighborhood structure on graded ditopological texture spaces are presented and investigated in [9].

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The main aim of this work is to generalize the structure of diextremity in ditopological texture spaces defined in [20] to the graded ditopological texture spaces and to obtain fundamental properties of interrelations of these two topological structures; an other aim is to investigate graded ditopologies generated by graded diextremities and graded diextremities generated by graded diuniformities. In addition, the final intention is to study basic categorical perspective of this new structure.

2. Preliminaries

Ditopological texture spaces: ([4]) Let S be a set. A texturing $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point separating (i.e. for all $s, t \in S,\ s \neq t$ there exists a set $A \in \mathcal{S}$ such that $s \in A,\ t \notin A$ or $s \notin A,\ t \in A$), complete, completely distributive lattice with respect to inclusion which contains S, $\emptyset$ and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S, \mathcal{S})$ is then called a texture or a texture space.

In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is said to be a complemented texture. A complementation σ on a texture $(S, \mathcal{S})$ is called "grounded" [16] if there is an involution $s \mapsto s'$ on S such that $\sigma(P_s) = Q_{s'}$ and $\sigma(Q_s) = P_{s'}$ (s' will be denoted by $\sigma(s)$) for all $s \in S$ and in this case the complemented texture space $(S, \mathcal{S}, \sigma)$ is called "complemented grounded texture space".

For any texture $(S, \mathcal{S})$, many properties are conveniently defined in terms of the $p-sets$

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the $q-sets$

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. $\overline{P}_{(s,v)}$, $\overline{Q}_{(s,v)}$ will denote the p-sets and q-sets for the product texture $(S \times V, \mathcal{P}(S) \otimes \mathcal{V})$ and $\overline{P}_{(v,s)}$, $\overline{Q}_{(v,s)}$ will denote the p-sets and q-sets for the product texture $(V \times S, \mathcal{P}(V) \otimes \mathcal{S})$.

Theorem 2.1. ([4]) *In any texture $(S, \mathcal{S})$, the following statements hold:*

1. $s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.
2. $A^\flat = \{s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
3. *For* $A_j \in \mathcal{S}$, $j \in J$ *we have* $(\bigvee_{j \in J} A_j)^\flat = \bigcup_{j \in J} A_j^\flat$.
4. *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.*
5. *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
6. $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.
7. $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

Definition 2.2. ([4]) Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. Then

(1) $r \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
 R1 $r \not\subseteq \overline{Q}_{(s,v)}, P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}_{(s',v)}$.
 R2 $r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',v)}$.

(2) $R \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
 CR1 $\overline{P}_{(s,v)} \not\subseteq R, P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}_{(s',v)} \not\subseteq R$.
 CR2 $\overline{P}_{(s,v)} \not\subseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',v)} \not\subseteq R$.

(3) A pair (r, R), where r is a relation and R a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ is called a direlation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$.

The direlations can be ordered as follows: for direlations (p, P), (q, Q) on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ it is written $(p, P) \sqsubseteq (q, Q)$ if and only if $p \subseteq q$ and $Q \subseteq P$. Moreover, it is defined in [14] that

$$p \sqcap q = \bigvee\{\overline{P}_{(s,v)} \mid \exists t \in S \text{ with } P_s \not\subseteq Q_t \text{ and } p, q \not\subseteq \overline{Q}_{(t,v)}\},$$

$$P \sqcup Q = \bigcap\{\overline{Q}_{(s,v)} \mid \exists t \in S \text{ with } P_t \not\subseteq Q_s \text{ and } \overline{P}_{(t,v)} \not\subseteq P, Q\},$$

$$(p, P) \sqcap (q, Q) = (p \sqcap q, P \sqcup Q).$$

For a texture $(S, \mathcal{S})$, $i = i_S = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\}$ is a relation and $I = I_S = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S\}$ is a co-relation on $(S, \mathcal{S})$ to $(S, \mathcal{S})$. That is, (i, I) is a direlation and we call it the identity direlation on $(S, \mathcal{S})$.

Let (r, R) be a direlation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. The inverses of r and R are defined respectively by $r^{\leftarrow} = \bigcap\{\overline{Q}_{(v,s)} \mid r \not\subseteq \overline{Q}_{(s,v)}\}$ and $R^{\leftarrow} = \bigvee\{\overline{P}_{(v,s)} \mid \overline{P}_{(s,v)} \not\subseteq R\}$ where $R^{\leftarrow}$ is a relation and $r^{\leftarrow}$ is a co-relation on $(V, \mathcal{V})$ to $(S, \mathcal{S})$. The direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ is called the inverse of (r, R).

For $A \in \mathcal{S}$, $r^{\rightarrow}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the A-section of r and $R^{\rightarrow}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the A-section of R.

For $B \in \mathcal{V}$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the B-presection of R.

The family of direlations on a texture space $(S, \mathcal{S})$ will be denoted by $\mathfrak{DR}_S$ or if there is no confusion just by $\mathfrak{DR}$.

For a direlation (d, D), $d^{\rightarrow}P_t$ and $D^{\rightarrow}Q_t$ will be denoted by $d[t]$ and $D[t]$ respectively.

Lemma 2.3. ([4, 19]) *Let r, r_1, r_2 be relations, R, R_1, R_2 co-relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ with $r_1 \subseteq r_2$, $R_1 \subseteq R_2$ and take $A, A_1, A_2 \in \mathcal{S}$ with $A_1 \subseteq A_2$, take $B, B_1, B_2 \in \mathcal{V}$ with $B_1 \subseteq B_2$.*

(1) $r \not\subseteq \overline{Q}_{(s,v)} \Leftrightarrow \overline{P}_{(v,s)} \not\subseteq r^{\leftarrow}$ and $\overline{P}_{(s,v)} \not\subseteq R \Leftrightarrow R^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ for all $s \in S$, $v \in V$.
(2) $(r^{\leftarrow})^{\leftarrow} = r$ and $(R^{\leftarrow})^{\leftarrow} = R$
(3) For a second direlation (m, M) from $(S, \mathcal{S})$ to $(V, \mathcal{V})$, $(r, R) \sqsubseteq (m, M) \Leftrightarrow (r, R)^{\leftarrow} \sqsubseteq (m, M)^{\leftarrow}$
(4) $r^{\rightarrow}\emptyset = \emptyset$, $A \subseteq r^{\leftarrow}(r^{\rightarrow}A)$, $r^{\rightarrow}(r^{\leftarrow}B) \subseteq B$
(5) $R^{\rightarrow}S = V$, $R^{\leftarrow}(R^{\rightarrow}A) \subseteq A$, $B \subseteq R^{\rightarrow}(R^{\leftarrow}B)$
(6) $r_1^{\rightarrow}A_1 \subseteq r_2^{\rightarrow}A_2$, $R_1^{\rightarrow}A_1 \subseteq R_2^{\rightarrow}A_2$, $r_2^{\leftarrow}B_1 \subseteq r_1^{\leftarrow}B_2$, $R_2^{\leftarrow}B_1 \subseteq R_1^{\leftarrow}B_2$.

Proposition 2.4. ([4]) *For a direlation (r, R) on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ we have $r^{\rightarrow}(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} r^{\rightarrow}A_i$, $R^{\rightarrow}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} R^{\rightarrow}A_i$, $r^{\leftarrow}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j \in J} B_j) = \bigvee_{j \in J} R^{\leftarrow}B_j$ for any $A_i \in \mathcal{S}$, $B_j \in \mathcal{V}$, $i \in I$, $j \in J$.*

Definition 2.5. ([4]) Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ and $(Y, \mathcal{Y})$ be textures.

(1) If p is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and q is a relation on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$ then their composition is the relation $q \circ p$ on $(S, \mathcal{S})$ to $(Y, \mathcal{Y})$ defined by

$$q \circ p = \bigvee\{\overline{P}_{(s,y)} \mid \exists v \in V \text{ with } p \not\subseteq \overline{Q}_{(s,v)} \text{ and } q \not\subseteq \overline{Q}_{(v,y)}\}.$$

(2) If P is a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and Q is a co-relation on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$ then their composition is the co-relation $Q \circ P$ on $(S, \mathcal{S})$ to $(Y, \mathcal{Y})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(s,y)} \mid \exists v \in V \text{ with } \overline{P}_{(s,v)} \not\subseteq P \text{ and } \overline{P}_{(v,y)} \not\subseteq Q\}.$$

(3) The composition of direlations (p, P) and (q, Q) is the direlation $(q, Q) \circ (p, P)$ defined by $(q, Q) \circ (p, P) = (q \circ p, Q \circ P)$.

Also it is shown in [4] that the composition of direlations is associative and $[(q, Q) \circ (p, P)]^{\leftarrow} = (p, P)^{\leftarrow} \circ (q, Q)^{\leftarrow}$.

Definition 2.6. ([4]) Let (f,F) be a direlation from $(S,\mathcal{S})$ to $(V,\mathcal{V})$. Then (f,F) is called a difunction from $(S,\mathcal{S})$ to $(V,\mathcal{V})$ if it satisfies the following two conditions:

(DF1) For $s,s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$.

(DF2) For $v,v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.

It is clear that (i_S, I_S) is a difunction on $(S,\mathcal{S})$ and we call it the identity difunction on $(S,\mathcal{S})$. Texture spaces and difunctions form a category denoted by **dfTex** [4].

Proposition 2.7. ([4]) *For a difunction (f,F) on $(S,\mathcal{S})$ to $(V,\mathcal{V})$ we have $f^{\leftarrow}B = F^{\leftarrow}B$ for each $B \in \mathcal{V}$.*

Definition 2.8. ([5]) A dichotomous topology, or ditopology for short, on a texture $(S,\mathcal{S})$ is a pair (τ,κ) of subsets of $\mathcal{S}$, where the set τ of open sets satisfies

(T_1) $S, \emptyset \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set κ of closed sets satisfies

(CT_1) $S, \emptyset \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Thus a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets. When a complementation σ on $(S,\mathcal{S})$ is given, (τ,κ) is called complemented if $\kappa = \sigma(\tau)$.

Definition 2.9. ([5]) Let $(S_k,\mathcal{S}_k,\tau_k,\kappa_k)$, $k = 1,2$ be ditopological texture spaces and $(f,F) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$ a difunction. (f,F) is called continuous if

$$F^{\leftarrow}A \in \tau_1, \quad \text{for all } A \in \tau_2$$

and cocontinuous if

$$f^{\leftarrow}A \in \kappa_1, \quad \text{for all } A \in \kappa_2.$$

The difunction (f,F) is called bicontinuous if it is both continuous and cocontinuous.

Theorem 2.10. ([5]) *Ditopological texture spaces and bicontinuous difunctions form a category denoted by* **dfDiTop**.

Diuniform texture spaces: ([15]) Let $(S,\mathcal{S})$ be a texture and $\mathcal{U}$ a nonempty family of direlations on $(S,\mathcal{S})$, i.e. $\emptyset \neq \mathcal{U} \subseteq \mathfrak{DR}_S$. If $\mathcal{U}$ satisfies the conditions

(U_1) $(i,I) \sqsubseteq (d,D)$ for all $(d,D) \in \mathcal{U}$,
(U_2) $(d,D) \in \mathcal{U}$, $(e,E) \in \mathfrak{DR}$ *and* $(d,D) \sqsubseteq (e,E)$ *implies* $(e,E) \in \mathcal{U}$,
(U_3) (d,D), $(e,E) \in \mathcal{U}$ *implies* $(d,D) \sqcap (e,E) \in \mathcal{U}$,
(U_4) Given for all $(d,D) \in \mathcal{U}$ there exists $(e,E) \in \mathcal{U}$ satisfying $(e,E) \circ (e,E) \sqsubseteq (d,D)$,
(U_5) Given for all $(d,D) \in \mathcal{U}$ there exists $(c,C) \in \mathcal{U}$ satisfying $(c,C)^{\leftarrow} \sqsubseteq (d,D)$,

then $\mathcal{U}$ is called a direlational uniformity on $(S,\mathcal{S})$ and the triple $(S,\mathcal{S},\mathcal{U})$ is known as a direlational uniform texture space. We will use "diuniformity" and "diuniform texture space" instead of the terms "direlational uniformity" and "direlational uniform texture space" respectively.

Proposition 2.11. ([15]) *Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be texture spaces, (d,D) a direlation on $(V,\mathcal{V})$ and $(f,F) : (S,\mathcal{S}) \to (V,\mathcal{V})$ a difunction.*

1. *For the sets*

$$(f,F)^{-1}(d) = \bigvee\{\overline{P}_{(s_1,s_2)} \mid \exists P_{s_1} \not\subseteq Q_{s_1'} : \overline{P}_{(s_1',v_1)} \not\subseteq F,\ f \not\subseteq \overline{Q}_{(s_2,v_2)} \Rightarrow \overline{P}_{(v_1,v_2)} \subseteq d\}$$

and

$$(f,F)^{-1}(D) = \bigcap\{\overline{Q}_{(s_1,s_2)} \mid \exists P_{s_1'} \not\subseteq Q_{s_1} : f \not\subseteq \overline{Q}_{(s_1',v_1)},\ \overline{P}_{(s_2,v_2)} \not\subseteq F \Rightarrow D \subseteq \overline{Q}_{(v_1,v_2)}\},$$

$$(f,F)^{-1}(d,D) = ((f,F)^{-1}(d),(f,F)^{-1}(D))$$

is a direlation on $(S,\mathcal{S})$.

2. $(f,F)^{-1}(i_V,I_V) = (i_S,I_S)$
3. $(i_S,I_S)^{-1}(d,D) = (d,D)$ *for all* $(d,D) \in \mathfrak{DR}_S$.

Let $(S_k,\mathcal{S}_k,\mathcal{U}_k)$, $k = 1,2$ be diuniform texture spaces and $(f,F) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$ a difunction. (f,F) is called $\mathcal{U}_1 - \mathcal{U}_2$ uniformly bicontinuous if $(f,F)^{-1}(d,D) \in \mathcal{U}_1$ for each $(d,D) \in \mathcal{U}_2$. The identity difunction and the composition of uniformly bicontinuous difunctions are uniformly bicontinuous. So, the class of diuniform texture spaces and uniformly bicontinuous difunctions between them form a category denoted by **dfDiU**.

Diextremities: ([20]) Let $(S,\mathcal{S})$ be a texture, δ^a, δ^b two binary relations on $\mathcal{S}$. Then $\delta = (\delta^a,\delta^b)$ is called a diextremity on $(S,\mathcal{S})$ if

(E1) $A\delta^a B$ implies $A \neq \emptyset$, $B \neq S$.

(E2) $(A \cup B)\delta^a C$ iff $A\delta^a C$ or $B\delta^a C$.

(E3) $A\delta^a(B \cap C)$ iff $A\delta^a B$ or $A\delta^a C$.

(E4) If $A \not\delta^a B$, there exists $E \in \mathcal{S}$ such that $A \not\delta^a E$ and $E \not\delta^a B$.

(E5) $A \not\delta^a B$ implies $A \subseteq B$.

(DE) $A\delta^b B \Leftrightarrow B\delta^a A$.

(CE1) $A\delta^b B$ implies $A \neq S$, $B \neq \emptyset$.

(CE2) $A\delta^b(B \cup C)$ iff $A\delta^b B$ or $A\delta^b C$.

(CE3) $(A \cap B)\delta^b C$ iff $A\delta^b C$ or $B\delta^b C$.

(CE4) If $A \not\delta^b B$, there exists $E \in \mathcal{S}$ such that $A \not\delta^b E$ and $E \not\delta^b B$.

(CE5) $A \not\delta^b B$ implies $B \subseteq A$.

In this case it is said that δ^a is an extremity and δ^b a co-extremity. Also, $(S,\mathcal{S},\delta)$ is known as a diextremial texture space.

Let $\delta = (\delta^a,\delta^b)$ be a diextremity on a complemented texture $(S,\mathcal{S},\sigma)$. Define $\dot{\delta} = (\dot{\delta}^a,\dot{\delta}^b)$ by

$$A\dot{\delta}^a B \Leftrightarrow \sigma(A)\delta^b\sigma(B) \text{ and } A\dot{\delta}^b B \Leftrightarrow \sigma(A)\delta^a\sigma(B)$$

where $A,B \in \mathcal{S}$. Then $\dot{\delta}$ is a diextremity on $(S,\mathcal{S},\sigma)$. The diextremity δ is said to be complemented if $\delta = \dot{\delta}$.

Let $(S_1,\mathcal{S}_1,\delta_1)$ and $(S_2,\mathcal{S}_2,\delta_2)$ be diextremial texture spaces and $(f,F) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$ a difunction. Then (f,F) is called extremial bicontinuous if it satisfies one, and hence both, of the following equivalent conditions:

(1) $C \not\delta_2^a D$ implies $f^{\leftarrow}C \not\delta_1^a f^{\leftarrow}D$ for all $C,D \in \mathcal{S}_2$.
(2) $C \not\delta_2^b D$ implies $f^{\leftarrow}C \not\delta_1^b f^{\leftarrow}D$ for all $C,D \in \mathcal{S}_2$.

The identity difunction and the composition of extremial bicontinuous difunctions are extremial bicontinuous. So, the class of diextremial texture spaces and extremial bicontinuous difunctions between them form a category that we will denote by **dfDiE**.

For a diextremial texture space $(S, \mathcal{S}, \delta)$ and for any $A \in \mathcal{S}$ define

$$int(A) = \bigcap \{Q_s \mid P_s \delta^a A\} \text{ and } cl(A) = \bigvee \{P_s \mid Q_s \delta^b A\}.$$

Lemma 2.12. ([20]) *The functions $int : \mathcal{S} \to \mathcal{S}$ and $cl : \mathcal{S} \to \mathcal{S}$ have the following properties:*

(1) $A \not\subseteq int(B)$ implies $\exists s \in S$ such that $P_s \delta^a B$ and $A \not\subseteq Q_s$.
(2) $P_s \delta^a B$ implies $int(B) \subseteq Q_s$.
(3) $A \not\delta^a B$ implies $A \subseteq int(B)$.
(4) $cl(A) \not\subseteq B$ implies $\exists s \in S$ such that $Q_s \delta^b A$ and $P_s \not\subseteq B$.
(5) $Q_s \delta^b B$ implies $P_s \subseteq cl(B)$.
(6) $A \not\delta^b B$ implies $cl(B) \subseteq A$.
(7) $int(A) = \bigvee \{P_s \mid P_s \not\delta^a A\}$.
(8) $cl(A) = \bigcap \{Q_s \mid Q_s \not\delta^b A\}$.
(9) $P_s \not\delta^a B$ implies $P_s \subseteq int(B)$.
(10) $Q_s \not\delta^b B$ implies $cl(B) \subseteq Q_s$.

Theorem 2.13. ([20]) *Let $\delta = (\delta^a, \delta^b)$ be a diextremity on $(S, \mathcal{S})$. The function $int : \mathcal{S} \to \mathcal{S}$ with $int(A) = \bigcap \{Q_s \mid P_s \delta^a A,\ s \in S\}$ satisfies the axioms of interior operation and the function $cl : \mathcal{S} \to \mathcal{S}$ with $cl(A) = \bigvee \{P_s \mid Q_s \delta^b A,\ s \in S\}$ satisfies the axioms of closure operation.*

Each diextremity induces a ditopology: if we set the families $\tau(\delta) = \{A \in \mathcal{S} \mid A = int(A)\}$ and $\kappa(\delta) = \{A \in \mathcal{S} \mid A = cl(A)\}$ then $(\tau(\delta), \kappa(\delta))$ is a ditopology on $(S, \mathcal{S})$. An extremial bicontinuous difunction is also bicontinuous with respect to induced ditopologies.

If a complemented diextremity δ on $(S, \mathcal{S}, \sigma)$ is given then the ditopology induced by δ is also complemented.

Graded Ditopological Texture Spaces: ([7]) Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and consider $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ satisfying

(GT_1) $\mathcal{T}(S) = \mathcal{T}(\emptyset) = V$
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)\ \forall A_1, A_2 \in \mathcal{S}$
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)\ \forall A_j \in \mathcal{S}, j \in J$

and

(GCT_1) $\mathcal{K}(S) = \mathcal{K}(\emptyset) = V$
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)\ \forall A_1, A_2 \in \mathcal{S}$
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)\ \forall A_j \in \mathcal{S}, j \in J$

Then $\mathcal{T}$ is called a $(V, \mathcal{V})$-graded topology, $\mathcal{K}$ a $(V, \mathcal{V})$-graded cotopology and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. For any ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ and for each $v \in V$ let's define the families:

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\},\ \mathcal{K}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{K}(A)\}.$$

Then $(\mathcal{T}^v, \mathcal{K}^v)$ is a ditopology on $(S, \mathcal{S})$ for each $v \in V$. That is, if $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is any graded ditopological texture space, then there exists a ditopology $(\mathcal{T}^v, \mathcal{K}^v)$ on the texture space $(S, \mathcal{S})$ for each $v \in V$.

If $(S, \mathcal{S}, \sigma)$ is a complemented texture and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, then $(\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ is also a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. $(\mathcal{T}, \mathcal{K})$ is called complemented if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$.

Example 2.14. ([7]) Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ the discrete texture on a singleton. Take $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g : \mathcal{S} \to \mathcal{P}(1)$ by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$. Then τ^g is a $(V, \mathcal{V})$-graded topology on $(S, \mathcal{S})$. Likewise, κ^g defined by $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ is a $(V, \mathcal{V})$-graded cotopology on $(S, \mathcal{S})$ and (τ^g, κ^g) is called the graded ditopology on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ).

Definition 2.15. ([7]) Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded ditopological texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ be difunctions. For the pair $((f, F), (h, H))$, (f, F) is called continuous with respect to (h, H) if

$$H^{\leftarrow}\mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow}A), \ \text{ for all } A \in \mathcal{S}_2$$

and cocontinuous with respect to (h, H) if

$$h^{\leftarrow}\mathcal{K}_2(A) \subseteq \mathcal{K}_1(f^{\leftarrow}A), \ \text{ for all } A \in \mathcal{S}_2.$$

The difunction (f, F) is called bicontinuous with respect to (h, H) if it is both continuous and cocontinuous with respect to (h, H).

Theorem 2.16. ([7]) *The class of graded ditopological texture spaces and relatively bicontinuous difunction pairs (in the sense of Definition 2.15) between them form a category denoted by* **dfGDiTop**.

The graded dineighborhood systems of the graded ditopological texture spaces were defined in [9]. From now on, we will use dinhd, shortly instead of dineighborhood. To avoid a long part of preliminaries we will give the following equivalent proposition instead of the definition.

Proposition 2.17. ([9]) *Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$ and $N : S^\flat \to \mathcal{V}^\mathcal{S}$, $M : S \to \mathcal{V}^\mathcal{S}$ be mappings where $N(s) = N_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S^\flat$ and $M(s) = M_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S$. Then (N, M) is a graded dinhd system of the graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ iff*

$$N_s(A) = \begin{cases} sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}, & A \not\subseteq Q_s \\ \emptyset, & A \subseteq Q_s \end{cases} \tag{1}$$

for each $s \in S^\flat$, $A \in \mathcal{S}$ and

$$M_s(A) = \begin{cases} sup\{\mathcal{K}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}, & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases} \tag{2}$$

for each $s \in S$, $A \in \mathcal{S}$.

Theorem 2.18. ([9]) *Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$. If (N, M) is the graded dinhd system of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$, then the following properties hold for all $A, A_1, A_2 \in \mathcal{S}$:*
(**1**) *For each $s \in S^\flat$;*
(**N1**) $N_s(A) \neq \emptyset \Rightarrow A \not\subseteq Q_s$
(**N2**) $N_s(\emptyset) = \emptyset$ *and* $N_s(S) = V$
(**N3**) $A_1 \subseteq A_2 \Rightarrow N_s(A_1) \subseteq N_s(A_2)$
(**N4**) $A_1 \cap A_2 \not\subseteq Q_s \Rightarrow N_s(A_1) \wedge N_s(A_2) \subseteq N_s(A_1 \cap A_2)$
(**N5**) $N_s(A) \subseteq sup\{\bigwedge_{s' \in B^\flat} N_{s'}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$
(**2**) *For each $s \in S$;*
(**M1**) $M_s(A) \neq \emptyset \Rightarrow P_s \not\subseteq A$
(**M2**) $M_s(S) = \emptyset$ *and* $M_s(\emptyset) = V$
(**M3**) $A_1 \subseteq A_2 \Rightarrow M_s(A_2) \subseteq M_s(A_1)$
(**M4**) $M_s(A_1) \wedge M_s(A_2) \subseteq M_s(A_1 \cup A_2)$
(**M5**) $M_s(A) \subseteq sup\{\bigwedge_{s' \in (S \setminus B)} M_{s'}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}$

Theorem 2.19. ([9]) *If the mappings $N : S^\flat \to \mathcal{V}^\mathcal{S}$, $M : S \to \mathcal{V}^\mathcal{S}$ satisfy the conditions $N1 - N4$ and $M1 - M4$ in Theorem 2.18, respectively, then the mappings $\mathcal{T}_N, \mathcal{K}_M : \mathcal{S} \to \mathcal{V}$, defined by*

$$\mathcal{T}_N(A) = \bigcap_{s \in A^\flat} N_s(A) \tag{3}$$

$$\mathcal{K}_M(A) = \bigcap_{s \in S \setminus A} M_s(A) \tag{4}$$

where $A \in \mathcal{S}$, form a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$.

Definition 2.20. ([10]) Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be textures and $\mathfrak{DR} = \mathfrak{DR}_S$ denote the family of all direlations on $(S,\mathcal{S})$. A mapping $\mathfrak{U} : \mathfrak{DR} \to \mathcal{V}$ is called a $(V,\mathcal{V})$-graded diuniformity on $(S,\mathcal{S})$ if it satisfies:

(GU1) $\mathfrak{U}(d,D) \neq \emptyset \Rightarrow (i,I) \sqsubseteq (d,D)$ for all $(d,D) \in \mathfrak{DR}$

(GU2) $(d,D) \sqsubseteq (e,E) \Rightarrow \mathfrak{U}(d,D) \subseteq \mathfrak{U}(e,E)$ for all $(d,D),(e,E) \in \mathfrak{DR}$

(GU3) $\mathfrak{U}(d,D) \wedge \mathfrak{U}(e,E) \subseteq \mathfrak{U}((d,D) \sqcap (e,E))$ for all $(d,D),(e,E) \in \mathfrak{DR}$

(GU4) $\forall (d,D) \in \mathfrak{DR}\ \exists (e,E) \in \mathfrak{DR} : \mathfrak{U}(d,D) \subseteq \mathfrak{U}(e,E)$ and $(e,E) \circ (e,E) \sqsubseteq (d,D)$

(GU5) $\forall (d,D) \in \mathfrak{DR}\ \exists (c,C) \in \mathfrak{DR} : \mathfrak{U}(d,D) \subseteq \mathfrak{U}(c,C)$ and $(c,C)^{\leftarrow} \sqsubseteq (d,D)$

(GU6) $\bigvee \{\mathfrak{U}(d,D) \mid (d,D) \in \mathfrak{DR}\} = V$.

In this case, $(S,\mathcal{S},\mathfrak{U},V,\mathcal{V})$ is called a graded diuniform texture space.

Example 2.21. ([10]) (1) Let $(S,\mathcal{S},\mathfrak{U},V,\mathcal{V})$ be a graded diuniform texture space. Then the set $\mathfrak{U}^v = \{(d,D) \in \mathfrak{DR} \mid P_v \subseteq \mathfrak{U}(d,D)\} \neq \emptyset$ is a diuniformity on $(S,\mathcal{S})$ for each $v \in V^\flat$.

(2) If $\mathcal{U}$ is a diuniformity on $(S,\mathcal{S})$ then the mapping $\mathfrak{U}_{\mathcal{U}} : \mathfrak{DR} \to \mathcal{P}(1)$ defined by

$$\mathfrak{U}_{\mathcal{U}}(d,D) = \begin{cases} 1, & (d,D) \in \mathcal{U} \\ \emptyset, & (d,D) \notin \mathcal{U} \end{cases}$$

is a $(1,\mathcal{P}(1))$-graded diuniformity on $(S,\mathcal{S})$.

Definition 2.22. ([10]) Let $(S_k,\mathcal{S}_k,\mathfrak{U}_k,V_k,\mathcal{V}_k)$, $k = 1,2$ be graded diuniform texture spaces and $(f,F) : (S_1,\mathcal{S}_1) \to (S_2,\mathcal{S}_2)$, $(h,H) : (V_1,\mathcal{V}_1) \to (V_2,\mathcal{V}_2)$ difunctions. If $H^{\leftarrow}(\mathfrak{U}_2(d,D)) \subseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$ for each $(d,D) \in \mathfrak{DR}_{S_2}$ then (f,F) is called $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniformly bicontinuous with respect to (h,H).

Theorem 2.23. ([10]) *Graded diuniform texture spaces and relatively uniformly bicontinuous difunction pairs between them form a category that we will denote by* **dfGDiU**.

Theorem 2.24. ([10]) *Let $(S,\mathcal{S},\mathfrak{U},V,\mathcal{V})$ be a graded diuniform texture space. Then the mappings $\mathcal{T}_{\mathfrak{U}},\mathcal{K}_{\mathfrak{U}} : \mathcal{S} \to \mathcal{V}$ defined by*

$$\mathcal{T}_{\mathfrak{U}}(A) = \bigcap_{t \in A^\flat} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d,D), \quad \mathcal{K}_{\mathfrak{U}}(A) = \bigcap_{t \in S \setminus A} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d,D) \tag{5}$$

where $A \in \mathcal{S}$, form a $(V,\mathcal{V})$-graded ditopology $(\mathcal{T}_{\mathfrak{U}},\mathcal{K}_{\mathfrak{U}})$ on $(S,\mathcal{S})$.

3. Graded Diextremities

In this chapter, the concept of diextremity on textures will be generalized to the graded case. Moreover, the relations of this new structure with graded ditopologies and graded diuniformities will be investigated.

Definition 3.1. Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be textures and $\mathfrak{e}^a, \mathfrak{e}^b : \mathcal{S} \times \mathcal{S} \to \mathcal{V}$ mappings. Then $\mathfrak{e} = (\mathfrak{e}^a, \mathfrak{e}^b)$ is called a $(V,\mathcal{V})$-graded diextremity on $(S,\mathcal{S})$ if for all $A,B,C \in \mathcal{S}$ it satisfies:

(GE1) $\mathfrak{e}^a(A,B) \neq \emptyset \Rightarrow A \neq \emptyset,\ B \neq S$

(GE2) $\mathfrak{e}^a(A \cup B, C) = \mathfrak{e}^a(A,C) \vee \mathfrak{e}^a(B,C)$

(GE3) $\mathfrak{e}^a(A, B \cap C) = \mathfrak{e}^a(A,B) \vee \mathfrak{e}^a(A,C)$

(GE4) $\forall A,B \in \mathcal{S}\ \exists E \in \mathcal{S}\ :\ \mathfrak{e}^a(A,E) \vee \mathfrak{e}^a(E,B) \subseteq \mathfrak{e}^a(A,B)$

(GE5) $\mathfrak{e}^a(A,B) \neq V \Rightarrow A \subseteq B$

(GDE) $\mathfrak{e}^b(A, B) = \mathfrak{e}^a(B, A)$

(GCE1) $\mathfrak{e}^b(A, B) \neq \emptyset \Rightarrow A \neq S,\ B \neq \emptyset$

(GCE2) $\mathfrak{e}^b(A, B \cup C) = \mathfrak{e}^b(A, B) \vee \mathfrak{e}^b(A, C)$

(GCE3) $\mathfrak{e}^b(A \cap B, C) = \mathfrak{e}^b(A, C) \vee \mathfrak{e}^b(B, C)$

(GCE4) $\forall A, B \in \mathcal{S}\ \exists E \in \mathcal{S}\ :\ \mathfrak{e}^b(A, E) \vee \mathfrak{e}^b(E, B) \subseteq \mathfrak{e}^b(A, B)$

(GCE5) $\mathfrak{e}^b(A, B) \neq V \Rightarrow B \subseteq A$.

In this case $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$ is called a graded diextremial texture space; $\mathfrak{e}^a$ a $(V, \mathcal{V})$-graded extremity and $\mathfrak{e}^b$ a $(V, \mathcal{V})$-graded co-extremity.

Let $\mathfrak{e} = (\mathfrak{e}^a, \mathfrak{e}^b)$ be a $(V, \mathcal{V})$-graded diextremity on a complemented texture $(S, \mathcal{S}, \sigma)$. Define $\dot{\mathfrak{e}} = (\dot{\mathfrak{e}}^a, \dot{\mathfrak{e}}^b)$ by

$$\dot{\mathfrak{e}}^a(A, B) = \mathfrak{e}^b(\sigma(A), \sigma(B)) \text{ and } \dot{\mathfrak{e}}^b(A, B) = \mathfrak{e}^a(\sigma(A), \sigma(B))$$

where $A, B \in \mathcal{S}$. Then $\dot{\mathfrak{e}}$ is a $(V, \mathcal{V})$-graded diextremity on $(S, \mathcal{S}, \sigma)$. $\mathfrak{e}$ is called complemented if $\mathfrak{e} = \dot{\mathfrak{e}}$.

Corollary 3.2. *Let $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$ be a graded diextremial texture space. For all $A, B, C, D \in \mathcal{S}$ we have*

$$A \subseteq C \Rightarrow \mathfrak{e}^a(A, B) \subseteq \mathfrak{e}^a(C, B),\ B \subseteq D \Rightarrow \mathfrak{e}^a(A, D) \subseteq \mathfrak{e}^a(A, B) \tag{6}$$

and

$$A \subseteq C \Rightarrow \mathfrak{e}^b(C, B) \subseteq \mathfrak{e}^b(A, B),\ B \subseteq D \Rightarrow \mathfrak{e}^b(A, B) \subseteq \mathfrak{e}^b(A, D). \tag{7}$$

Example 3.3. (1) If $\delta = (\delta^a, \delta^b)$ is a diextremity on a texture $(S, \mathcal{S})$ then the mappings $\mathfrak{e}^a_\delta, \mathfrak{e}^b_\delta : \mathcal{S} \times \mathcal{S} \to \mathcal{P}(1)$ (The notation 1 denotes the set $\{0\}$) defined by

$$\mathfrak{e}^a_\delta(A, B) = \begin{cases} 1, & A\delta^a B \\ \emptyset, & A \not\delta^a B \end{cases} \tag{8}$$

and

$$\mathfrak{e}^b_\delta(A, B) = \begin{cases} 1, & A\delta^b B \\ \emptyset, & A \not\delta^b B \end{cases} \tag{9}$$

form a $(1, \mathcal{P}(1))$-graded diextremity $\mathfrak{e}_\delta = (\mathfrak{e}^a_\delta, \mathfrak{e}^b_\delta)$ on $(S, \mathcal{S})$.

(2) If $\mathfrak{e} = (\mathfrak{e}^a, \mathfrak{e}^b)$ is a $(V, \mathcal{V})$-graded diextremity on $(S, \mathcal{S})$ then for each $v \in V$ the relations defined by

$$A\delta^a_{\mathfrak{e}^v} B \Leftrightarrow P_v \subseteq \mathfrak{e}^a(A, B),\ A\delta^b_{\mathfrak{e}^v} B \Leftrightarrow P_v \subseteq \mathfrak{e}^b(A, B),\ \forall A, B \in \mathcal{S}$$

describe a diextremity $\delta_{\mathfrak{e}^v} = (\delta^a_{\mathfrak{e}^v}, \delta^b_{\mathfrak{e}^v})$ on $(S, \mathcal{S})$.

Definition 3.4. Let $(S_k, \mathcal{S}_k, \mathfrak{e}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded diextremial texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. (f, F) is called extremial bicontinuous with respect to (h, H) if for all $A, B \in \mathcal{S}_2$; one of the following equivalent conditions is satisfied:

(i) $\mathfrak{e}^a_1(f^{\leftarrow} A, f^{\leftarrow} B) \subseteq H^{\leftarrow} \mathfrak{e}^a_2(A, B)$
(ii) $\mathfrak{e}^b_1(f^{\leftarrow} A, f^{\leftarrow} B) \subseteq H^{\leftarrow} \mathfrak{e}^b_2(A, B)$.

Example 3.5. For a graded diextremial texture space $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$; the identity difunction (i_S, I_S) on $(S, \mathcal{S})$ is extremial bicontinuous with respect to the identity difunction (i_V, I_V) on $(V, \mathcal{V})$. Indeed, $\mathfrak{e}^a(i_S^{\leftarrow} A, i_S^{\leftarrow} B) = \mathfrak{e}^a(A, B) = I_V^{\leftarrow} \mathfrak{e}^a(A, B)$ for all $A, B \in \mathcal{S}$.

Proposition 3.6. *Relatively extremial bicontinuity is preserved under composition of difunctions.*

Proof. Let $(S_j, \mathcal{S}_j, \mathfrak{e}_j, V_j, \mathcal{V}_j)$, $j = 1, 2, 3$ be graded diextremial texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$, $(g, G) : (S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$, $(k, K) : (V_2, \mathcal{V}_2) \to (V_3, \mathcal{V}_3)$ be difunctions where (f, F) is extremial bicontinuous with respect to (h, H) and (g, G) is extremial bicontinuous with respect to (k, K). For all $A, B \in \mathcal{S}_3$ we have;

$$\begin{aligned}\mathfrak{e}_1^a((g \circ f)^{\leftarrow} A, (g \circ f)^{\leftarrow} B) = \mathfrak{e}_1^a(f^{\leftarrow}(g^{\leftarrow} A), f^{\leftarrow}(g^{\leftarrow} B)) &\subseteq H^{\leftarrow} \mathfrak{e}_2^a(g^{\leftarrow} A, g^{\leftarrow} B)\\ &\subseteq H^{\leftarrow}(K^{\leftarrow} \mathfrak{e}_3^a(A, B)) = (K \circ H)^{\leftarrow} \mathfrak{e}_3^a(A, B).\end{aligned}$$

Hence $(g, G) \circ (f, F)$ is extremial bicontinuous with respect to $(k, K) \circ (h, H)$. □

Corollary 3.7. *Graded diextremial texture spaces and relatively extremial bicontinuous difunction pairs between them form a category that we will denote by* **dfGDiE**.

Proposition 3.8. *Let $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$ be a graded diextremial texture space and define the mappings $N^{\mathfrak{e}} : S^{\flat} \to \mathcal{V}^{\mathcal{S}}$, $M^{\mathfrak{e}} : S \to \mathcal{V}^{\mathcal{S}}$ where $N^{\mathfrak{e}}(s) = N_s^{\mathfrak{e}} : \mathcal{S} \to \mathcal{V}$ for each $s \in S^{\flat}$ and $M^{\mathfrak{e}}(s) = M_s^{\mathfrak{e}} : \mathcal{S} \to \mathcal{V}$ for each $s \in S$ by*

$$N_s^{\mathfrak{e}}(A) = \begin{cases} sup\{P_v \ : \ P_v \cap \mathfrak{e}^a(P_s, A) = \emptyset\}, & A \not\subseteq Q_s \\ \emptyset, & A \subseteq Q_s \end{cases} \tag{10}$$

and

$$M_s^{\mathfrak{e}}(A) = \begin{cases} sup\{P_v \ : \ P_v \cap \mathfrak{e}^b(Q_s, A) = \emptyset\}, & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases} \tag{11}$$

for each $A \in \mathcal{S}$. Then the mappings $N^{\mathfrak{e}}, M^{\mathfrak{e}}$ satisfy the properties $N1 - N4$ and $M1 - M4$.

Proof. (N1) is clear.
(N2): Since $\emptyset \subseteq Q_s$, $S \not\subseteq Q_s$ and $\mathfrak{e}^a(P_s, S) = \emptyset$ by $(GE1)$ for all $s \in S^{\flat}$ we have $N_s^{\mathfrak{e}}(\emptyset) = \emptyset$ and $N_s^{\mathfrak{e}}(S) = sup\{P_v \ : \ P_v \cap \mathfrak{e}^a(P_s, S) = \emptyset\} = sup\{P_v \ : \ P_v \cap \emptyset = \emptyset\} = V$.
(N3): Let $A_1, A_2 \in \mathcal{S}$ and $A_1 \subseteq A_2$. If $A_1 \subseteq Q_s$ then we have $N_s^{\mathfrak{e}}(A_1) = \emptyset \subseteq N_s^{\mathfrak{e}}(A_2)$. If $A_1 \not\subseteq Q_s$ then we get $A_2 \not\subseteq Q_s$ and $\mathfrak{e}^a(P_s, A_2) \subseteq \mathfrak{e}^a(P_s, A_1)$ by Corollary 3.2. Thus $N_s^{\mathfrak{e}}(A_1) = sup\{P_v \ : \ P_v \cap \mathfrak{e}^a(P_s, A_1) = \emptyset\} \subseteq sup\{P_v \ : \ P_v \cap \mathfrak{e}^a(P_s, A_2) = \emptyset\} = N_s^{\mathfrak{e}}(A_2)$ is obtained.
(N4): Let $A_1 \cap A_2 \not\subseteq Q_s$. Then $A_1, A_2 \not\subseteq Q_s$. Since every texture is a completely distributive lattice and thus satisfies join infinite distributivity and also by using (GE3) we obtain $N_s^{\mathfrak{e}}(A_1) \wedge N_s^{\mathfrak{e}}(A_2) = sup\{P_v \ : \ P_v \cap \mathfrak{e}^a(P_s, A_1) = \emptyset\} \wedge sup\{P_t \ : \ P_t \cap \mathfrak{e}^a(P_s, A_2) = \emptyset\} = sup\{P_v \cap P_t \ : \ P_v \cap \mathfrak{e}^a(P_s, A_1) = \emptyset, \ P_t \cap \mathfrak{e}^a(P_s, A_2) = \emptyset\} = sup\{P_r \ : \ P_r \cap (\mathfrak{e}^a(P_s, A_1) \vee \mathfrak{e}^a(P_s, A_2)) = \emptyset\} = sup\{P_r \ : \ P_r \cap \mathfrak{e}^a(P_s, A_1 \cap A_2) = \emptyset\} = N_s^{\mathfrak{e}}(A_1 \cap A_2)$.
The proof of $M1 - M4$ is similar. □

Corollary 3.9. *Let $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$ be a graded diextremial texture space. Then the mappings $\mathcal{T}_{\mathfrak{e}}, \mathcal{K}_{\mathfrak{e}} : \mathcal{S} \to \mathcal{V}$ defined by*

$$\mathcal{T}_{\mathfrak{e}}(A) = \bigcap_{s \in A^{\flat}} N_s^{\mathfrak{e}}(A) = \bigcap_{s \in A^{\flat}} \bigvee_{P_v \cap \mathfrak{e}^a(P_s, A) = \emptyset} P_v, \tag{12}$$

$$\mathcal{K}_{\mathfrak{e}}(A) = \bigcap_{s \in S \setminus A} M_s^{\mathfrak{e}}(A) = \bigcap_{s \in S \setminus A} \bigvee_{P_v \cap \mathfrak{e}^b(Q_s, A) = \emptyset} P_v \tag{13}$$

where $A \in \mathcal{S}$, form a $(V, \mathcal{V})$-graded ditopology (induced by $\mathfrak{e}$) $(\mathcal{T}_{\mathfrak{e}}, \mathcal{K}_{\mathfrak{e}})$ on $(S, \mathcal{S})$.

Proof. It is clear from Theorem 2.19. □

Theorem 3.10. *Let $(S, \mathcal{S}, \mathfrak{e}, V, \mathcal{V})$ be a graded diextremial texture space and σ be a grounded complementation on $(S, \mathcal{S})$. If $\mathfrak{e}$ is complemented then the graded ditopology induced by $\mathfrak{e}$ is also complemented.*

Proof. Since $\mathfrak{e}$ is complemented, for any set $A \in \mathcal{S}$ we have

$$\mathcal{K}_{\mathfrak{e}}(\sigma(A)) = \bigcap_{s\in S\setminus\sigma(A)} \bigvee_{P_v\cap \mathfrak{e}^b(Q_s,\sigma(A))=\emptyset} P_v = \bigcap_{P_s\not\subseteq\sigma(A)} \bigvee_{P_v\cap \mathfrak{e}^a(\sigma(Q_s),\sigma(\sigma(A)))=\emptyset} P_v = \bigcap_{A\not\subseteq Q_{\sigma(s)}} \bigvee_{P_v\cap \mathfrak{e}^a(P_{\sigma(s)},A)=\emptyset} P_v$$
$$= \bigcap_{\sigma(s)\in A^b} \bigvee_{P_v\cap \mathfrak{e}^a(P_{\sigma(s)},A)=\emptyset} P_v = \mathcal{T}_{\mathfrak{e}}(A)$$

and so $\mathcal{K}_{\mathfrak{e}} \circ \sigma = \mathcal{T}_{\mathfrak{e}}$. Similarly it can be shown that $\mathcal{T}_{\mathfrak{e}} \circ \sigma = \mathcal{K}_{\mathfrak{e}}$. Therefore we obtain that $(\mathcal{T}_{\mathfrak{e}}, \mathcal{K}_{\mathfrak{e}})$ is complemented. □

Theorem 3.11. *Let $(S_k, \mathcal{S}_k, \mathfrak{e}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded diextremial texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. If (f, F) is $\mathfrak{e}_1$ - $\mathfrak{e}_2$ extremial bicontinuous with respect to (h, H) then it is $(\mathcal{T}_{\mathfrak{e}_1}, \mathcal{K}_{\mathfrak{e}_1})$ - $(\mathcal{T}_{\mathfrak{e}_2}, \mathcal{K}_{\mathfrak{e}_2})$ bicontinuous with respect to (h, H) with the notations given in Corollary 3.9.*

Proof. Let (f, F) be $\mathfrak{e}_1$ - $\mathfrak{e}_2$ extremial bicontinuous with respect to (h, H). Suppose that (f, F) is not $(\mathcal{T}_{\mathfrak{e}_1}, \mathcal{K}_{\mathfrak{e}_1})$ - $(\mathcal{T}_{\mathfrak{e}_2}, \mathcal{K}_{\mathfrak{e}_2})$ continuous with respect to (h, H). Then $H^{\leftarrow}\mathcal{T}_{\mathfrak{e}_2}A \not\subseteq \mathcal{T}_{\mathfrak{e}_1}F^{\leftarrow}A$ for some $A \in \mathcal{S}_2$. So, using Theorem 2.1(5), there exists $v_0 \in V_1$ such that $H^{\leftarrow}\mathcal{T}_{\mathfrak{e}_2}A \not\subseteq Q_{v_0}$ and $P_{v_0} \not\subseteq \mathcal{T}_{\mathfrak{e}_1}(F^{\leftarrow}A)$. Thus, using Propositions 2.4 and 2.7 we have

$$H^{\leftarrow}\mathcal{T}_{\mathfrak{e}_2}A \not\subseteq Q_{v_0} \Rightarrow H^{\leftarrow}\Big(\bigcap_{t\in A^b} \bigvee_{P_v\cap \mathfrak{e}_2^a(P_t,A)=\emptyset} P_v\Big) \not\subseteq Q_{v_0}$$
$$\Rightarrow \bigcap_{t\in A^b} \bigvee_{P_v\cap \mathfrak{e}_2^a(P_t,A)=\emptyset} H^{\leftarrow}P_v \not\subseteq Q_{v_0}$$

and so there exists $v_t \in V_2$,

$$\text{"}P_{v_t} \cap \mathfrak{e}_2^a(P_t, A) = \emptyset \text{ and } H^{\leftarrow}P_{v_t} \not\subseteq Q_{v_0}\text{" for all } t \in A^b. \tag{14}$$

On the other hand, we have

$$P_{v_0} \not\subseteq \mathcal{T}_{\mathfrak{e}_1}(F^{\leftarrow}A) \Rightarrow P_{v_0} \not\subseteq \bigcap_{s\in(F^{\leftarrow}A)^b} \bigvee_{P_v\cap \mathfrak{e}_1^a(P_s,F^{\leftarrow}A)=\emptyset} P_v$$

and so, there exists $s_1 \in (F^{\leftarrow}A)^b$ such that "$P_v \cap \mathfrak{e}_1^a(P_{s_1}, F^{\leftarrow}A) = \emptyset \Rightarrow P_v \subseteq Q_{v_0}$". Thus, we have

$$\text{"}P_v \not\subseteq Q_{v_0} \Rightarrow P_v \cap \mathfrak{e}_1^a(P_{s_1}, F^{\leftarrow}A) \neq \emptyset\text{" for some } s_1 \in (F^{\leftarrow}A)^b. \tag{15}$$

Since $s_1 \in (F^{\leftarrow}A)^b = F^{\leftarrow}(\bigvee_{t\in A^b} P_t)^b = F^{\leftarrow}(\bigcup_{t\in A^b} P_t^b) = \bigcup_{t\in A^b} F^{\leftarrow}(P_t^b) \subseteq \bigcup_{t\in A^b} F^{\leftarrow}P_t$ there exists $t_0 \in A^b$ such that $P_{s_1} \subseteq F^{\leftarrow}P_{t_0}$.

On the other hand, because of $t_0 \in A^b$, using (14) there exists $v_{t_0} \in V_2$ such that "$P_{v_{t_0}} \cap \mathfrak{e}_2^a(P_{t_0}, A) = \emptyset$ and $H^{\leftarrow}P_{v_{t_0}} \not\subseteq Q_{v_0}$". Moreover $H^{\leftarrow}P_{v_{t_0}} \not\subseteq Q_{v_0}$ implies that there exists $v_1 \in V_1$ such that $P_{v_1} \subseteq H^{\leftarrow}P_{v_{t_0}}$ and $P_{v_1} \not\subseteq Q_{v_0}$. From (15) we get $P_{v_1} \cap \mathfrak{e}_1^a(P_{s_1}, F^{\leftarrow}A) \neq \emptyset$ and so, there exists $v_2 \in P_{v_1}$ such that $P_{v_2} \subseteq \mathfrak{e}_1^a(P_{s_1}, F^{\leftarrow}A)$. Since (f, F) is $\mathfrak{e}_1$ - $\mathfrak{e}_2$ extremial bicontinuous with respect to (h, H), using Corollary 3.2 we obtain that

$$P_{v_2} \subseteq \mathfrak{e}_1^a(P_{s_1}, F^{\leftarrow}A) \subseteq \mathfrak{e}_1^a(F^{\leftarrow}P_{t_0}, F^{\leftarrow}A) \subseteq H^{\leftarrow}\mathfrak{e}_2^a(P_{t_0}, A). \tag{16}$$

Recall that $P_{v_1} \subseteq H^{\leftarrow}P_{v_{t_0}}$ and $v_2 \in P_{v_1}$ so we get $P_{v_2} \subseteq H^{\leftarrow}P_{v_{t_0}}$. Using (16), Lemma 2.3. and recalling the fact that $P_{v_{t_0}} \cap \mathfrak{e}_2^a(P_{t_0}, A) = \emptyset$, we have $P_{v_2} \subseteq H^{\leftarrow}P_{v_{t_0}} \cap H^{\leftarrow}\mathfrak{e}_2^a(P_{t_0}, A) = H^{\leftarrow}(P_{v_{t_0}} \cap \mathfrak{e}_2^a(P_{t_0}, A)) = H^{\leftarrow}(\emptyset) = \emptyset$. However, this result leads the contradiction $P_{v_2} \subseteq \emptyset$. Thus, (f, F) is $(\mathcal{T}_{\mathfrak{e}_1}, \mathcal{K}_{\mathfrak{e}_1})$ - $(\mathcal{T}_{\mathfrak{e}_2}, \mathcal{K}_{\mathfrak{e}_2})$ continuous with respect to (h, H).

Similarly, it can be shown the cocontinuity part of the proof. □

Proposition 3.12. *Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be a graded diuniform texture space and define the mappings $\mathrm{e}^a_{\mathfrak{U}}, \mathrm{e}^b_{\mathfrak{U}} : \mathcal{S} \times \mathcal{S} \to \mathcal{V}$ by*

$$\mathrm{e}^a_{\mathfrak{U}}(A, B) = \bigvee_{P_v \not\subseteq \varphi^a_{\mathfrak{U}}(A,B)} P_v \qquad \text{and} \qquad \mathrm{e}^b_{\mathfrak{U}}(A, B) = \bigvee_{P_v \not\subseteq \varphi^b_{\mathfrak{U}}(A,B)} P_v$$

where

$$\varphi^a_{\mathfrak{U}}(A, B) = \bigvee \{\mathfrak{U}(d, D) \ : \ d^{\to}A \subseteq B\} \qquad \text{and} \qquad \varphi^b_{\mathfrak{U}}(A, B) = \bigvee \{\mathfrak{U}(d, D) \ : \ B \subseteq D^{\to}A\}$$

for all $A, B \in \mathcal{S}$. Then the mapping $\mathrm{e}_{\mathfrak{U}} = (\mathrm{e}^a_{\mathfrak{U}}, \mathrm{e}^b_{\mathfrak{U}})$ is a $(V, \mathcal{V})$-graded diextremity (induced by the graded diuniformity $\mathfrak{U}$) on $(S, \mathcal{S})$.

Proof. (GE1) If $A = \emptyset$ or $B = S$ then we have $\varphi^a_{\mathfrak{U}}(A, B) = V$ by $(GU6)$ and so $\mathrm{e}^a_{\mathfrak{U}}(A, B) = \emptyset$.
(GE2) Using the fact that $d^{\to}(A \cup B) = d^{\to}A \cup d^{\to}B$ we have $\varphi^a_{\mathfrak{U}}(A \cup B, C) = \varphi^a_{\mathfrak{U}}(A, C) \wedge \varphi^a_{\mathfrak{U}}(B, C)$ and so $\mathrm{e}^a_{\mathfrak{U}}(A \cup B, C) = \mathrm{e}^a_{\mathfrak{U}}(A, C) \vee \mathrm{e}^a_{\mathfrak{U}}(B, C)$.
(GE3) Considering $d^{\to}A \subseteq (B \cap C) \Leftrightarrow d^{\to}A \subseteq B$ and $d^{\to}A \subseteq C$ we have $\varphi^a_{\mathfrak{U}}(A, B \cap C) = \varphi^a_{\mathfrak{U}}(A, B) \wedge \varphi^a_{\mathfrak{U}}(A, C)$ and so $\mathrm{e}^a_{\mathfrak{U}}(A, B \cap C) = \mathrm{e}^a_{\mathfrak{U}}(A, B) \vee \mathrm{e}^a_{\mathfrak{U}}(A, C)$.
(GE4) For each $(d, D) \in \mathfrak{DR}$ there exists $(r, R) \in \mathfrak{DR}$ such that $\mathfrak{U}(d, D) \subseteq \mathfrak{U}(r, R)$ and $(r, R) \circ (r, R) \sqsubseteq (d, D)$ by $(GU4)$. So, if $d^{\to}A \subseteq B$ then we have $(r, R) \in \mathfrak{DR}$ such that $\mathfrak{U}(d, D) \subseteq \mathfrak{U}(r, R)$ and $r^{\to}(r^{\to}A) \subseteq d^{\to}A \subseteq B$. Now, applying $r^{\leftarrow}$ we get $r^{\leftarrow}r^{\to}(r^{\to}A) \subseteq r^{\leftarrow}B$ and $r^{\to}A \subseteq r^{\leftarrow}B$. If we denote $r^{\leftarrow}B$ by E then we have $r^{\to}A \subseteq E$ and $r^{\to}E \subseteq B$ by the fact that $r^{\to}r^{\leftarrow}B \subseteq B$. Therefore, considering $\mathfrak{U}(d, D) \subseteq \mathfrak{U}(r, R)$ we obtain $\varphi^a_{\mathfrak{U}}(A, B) \subseteq \varphi^a_{\mathfrak{U}}(A, E)$ and $\varphi^a_{\mathfrak{U}}(A, B) \subseteq \varphi^a_{\mathfrak{U}}(E, B)$. Thus $\mathrm{e}^a_{\mathfrak{U}}(A, E) \vee \mathrm{e}^a_{\mathfrak{U}}(E, B) \subseteq \mathrm{e}^a_{\mathfrak{U}}(A, B)$.
(GE5) If $\mathrm{e}^a_{\mathfrak{U}}(A, B) \neq V$ then we have $\varphi^a_{\mathfrak{U}}(A, B) \neq \emptyset$ and so, there exist a direlation (d, D) such that $\mathfrak{U}(d, D) \neq \emptyset$ and $d^{\to}A \subseteq B$. On the other hand, since $\mathfrak{U}(d, D) \neq \emptyset$ we have $(i, I) \sqsubseteq (d, D)$ by $(GU1)$. Thus we get $A = i^{\to}A \subseteq d^{\to}A \subseteq B$ and $A \subseteq B$.
(GDE) It is sufficient to show that $\varphi^a_{\mathfrak{U}}(A, B) = \varphi^b_{\mathfrak{U}}(B, A)$. Let $d^{\to}A \subseteq B$ for a direlation (d, D). Then there exists a direlation (c, C) such that $\mathfrak{U}(d, D) \subseteq \mathfrak{U}(c, C)$ and $(c, C)^{\leftarrow} \sqsubseteq (d, D)$ by (GU5). Since $d^{\to}A \subseteq B$, using Lemma 2.3. (4) we have $A \subseteq d^{\leftarrow}(d^{\to}A) \subseteq d^{\leftarrow}B$ and so $A \subseteq d^{\leftarrow}B$. Since $(c, C)^{\leftarrow} \sqsubseteq (d, D)$ we get $C^{\leftarrow} \subseteq d$ and so using Lemma 2.3. (3), $d^{\leftarrow} \subseteq C$. Thus we obtain $A \subseteq C^{\to}B$ by $A \subseteq d^{\leftarrow}B$. Since $\mathfrak{U}(d, D) \subseteq \mathfrak{U}(c, C)$ we get $\varphi^a_{\mathfrak{U}}(A, B) \subseteq \varphi^b_{\mathfrak{U}}(B, A)$. Similarly, it can be shown that $\varphi^b_{\mathfrak{U}}(B, A) \subseteq \varphi^a_{\mathfrak{U}}(A, B)$.
The proof of (GCE1)-(GCE5) is similar and so omitted. □

Lemma 3.13. ([14, Proposition 6.13]) *Let $(S_k, \mathcal{S}_k)$, $k = 1, 2$ be texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ a difunction and $(d, D) \in \mathfrak{DR}_{S_2}$. If $\overline{P}_{(s_1,s_2)} \not\subseteq F$ and $d[s_2] \subseteq A$ for $s_1 \in S_1$, $s_2 \in S_2$, $A \in \mathcal{S}_2$ then $(f, F)^{-1}(d)[s_1] \subseteq F^{\leftarrow}A$.*

Lemma 3.14. *Let $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ be a difunction and (r, R) a direlation on $(S_2, \mathcal{S}_2)$. Then*

(i) $r^{\to}A \subseteq B \Rightarrow (f, F)^{-1}(r)(f^{\leftarrow}A) \subseteq f^{\leftarrow}B$
(ii) $B \subseteq R^{\to}A \Rightarrow F^{\leftarrow}B \subseteq (f, F)^{-1}(R)(F^{\leftarrow}A)$

for all $A, B \in \mathcal{S}_2$.

Proof. (i) Let $r^{\to}A \subseteq B$ and $s \in (f^{\leftarrow}A)^{\flat}$. Then we have $f^{\leftarrow}A = F^{\leftarrow}A \not\subseteq Q_s$. Recall that $F^{\leftarrow}A = \bigcap\{Q_s \mid \forall t, \overline{P}_{(s,t)} \not\subseteq F \Rightarrow A \subseteq Q_t\}$ so, there exists $t_0 \in S_2$ such that $\overline{P}_{(s,t_0)} \not\subseteq F$ and $A \not\subseteq Q_{t_0}$. Since $A \not\subseteq Q_{t_0}$ we get $P_{t_0} \subseteq A$ and so $r[t_0] \subseteq r^{\to}A \subseteq B$. Thus we have $\overline{P}_{(s,t_0)} \not\subseteq F$ and $r[t_0] \subseteq B$. Now considering Lemma 3.13. we have $(f, F)^{-1}(r)[s] \subseteq f^{\leftarrow}B$ for each $s \in (f^{\leftarrow}A)^{\flat}$. Therefore we obtain that $(f, F)^{-1}(r)(f^{\leftarrow}A) = (f, F)^{-1}(r)(\bigvee_{s \in (f^{\leftarrow}A)^{\flat}} P_s) = \bigvee_{s \in (f^{\leftarrow}A)^{\flat}} (f, F)^{-1}(r)P_s = \bigvee_{s \in (f^{\leftarrow}A)^{\flat}} (f, F)^{-1}(r)[s] \subseteq f^{\leftarrow}B$.

(ii) Similar to (i). □

Theorem 3.15. *Let $(S_k, \mathcal{S}_k, \mathfrak{U}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded diuniform texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. If (f, F) is $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniformly bicontinuous with respect to (h, H) then it is $\mathrm{e}_{\mathfrak{U}_1}$ - $\mathrm{e}_{\mathfrak{U}_2}$ extremial bicontinuous with respect to (h, H).*

Proof. Let (f, F) be $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniformly bicontinuous and suppose that (f, F) is not $\mathfrak{e}_{\mathfrak{U}_1}$ - $\mathfrak{e}_{\mathfrak{U}_2}$ extremial bicontinuous with respect to (h, H). Then $\mathfrak{e}^a_{\mathfrak{U}_1}(f^{\leftarrow}A, f^{\leftarrow}B) \not\subseteq H^{\leftarrow}(\mathfrak{e}^a_{\mathfrak{U}_2}(A, B))$ for some $A, B \in \mathcal{S}_2$ and it follows that $\bigvee_{P_v \not\subseteq \varphi^a_{\mathfrak{U}_1}(f^{\leftarrow}A, f^{\leftarrow}B)} P_v \not\subseteq H^{\leftarrow}(\bigvee_{P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A,B)} P_t)$ for some $A, B \in \mathcal{S}_2$. So there exists $v \in V_1$ such that $P_v \not\subseteq \varphi^a_{\mathfrak{U}_1}(f^{\leftarrow}A, f^{\leftarrow}B)$ and $P_v \not\subseteq H^{\leftarrow}(\bigvee_{P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A,B)} P_t)$.

Since $P_v \not\subseteq \varphi^a_{\mathfrak{U}_1}(f^{\leftarrow}A, f^{\leftarrow}B)$ we have "$d^{\rightarrow}(f^{\leftarrow}A) \subseteq f^{\leftarrow}B \Rightarrow P_v \not\subseteq \mathfrak{U}_1(d, D)$" and so "$d^{\rightarrow}(f^{\leftarrow}A) \subseteq f^{\leftarrow}B \Rightarrow \mathfrak{U}_1(d, D) \subseteq Q_v$" for all $(d, D) \in \mathfrak{DR}_{S_1}$. Thus we get:

$$\bigvee_{d^{\rightarrow}(f^{\leftarrow}A) \subseteq f^{\leftarrow}B} \mathfrak{U}_1(d, D) \subseteq Q_v. \tag{17}$$

On the other hand, $P_v \not\subseteq H^{\leftarrow}(\bigvee_{P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A,B)} P_t)$ implies that $P_v \not\subseteq (\bigvee_{P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A,B)} H^{\leftarrow}P_t)$ and so $\bigvee_{P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A,B)} H^{\leftarrow}P_t \subseteq Q_v$. This implies that "$P_t \not\subseteq \varphi^a_{\mathfrak{U}_2}(A, B) \Rightarrow H^{\leftarrow}P_t \subseteq Q_v$". Hence we have

$$P_t \not\subseteq \bigvee_{r^{\rightarrow}A \subseteq B,\ (r,R) \in \mathfrak{DR}_{S_2}} \mathfrak{U}_2(r, R) \Rightarrow H^{\leftarrow}P_t \subseteq Q_v$$

and it follows that "$H^{\leftarrow}P_t \not\subseteq Q_v \Rightarrow P_t \subseteq \bigvee_{r^{\rightarrow}A \subseteq B,\ (r,R) \in \mathfrak{DR}_{S_2}} \mathfrak{U}_2(r, R)$". Since (f, F) is $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniformly bicontinuous with respect to (h, H), by considering Lemma 3.14. we obtain that

$$\begin{aligned} H^{\leftarrow}P_t \not\subseteq Q_v \Rightarrow H^{\leftarrow}P_t \subseteq H^{\leftarrow} \bigvee_{r^{\rightarrow}A \subseteq B} \mathfrak{U}_2(r, R) &= \bigvee_{r^{\rightarrow}A \subseteq B} H^{\leftarrow}\mathfrak{U}_2(r, R) \\ &\subseteq \bigvee_{r^{\rightarrow}A \subseteq B} \mathfrak{U}_1((f, F)^{-1}(r, R)) \\ &\subseteq \bigvee_{d^{\rightarrow}f^{\leftarrow}A \subseteq f^{\leftarrow}B} \mathfrak{U}_1(d, D). \end{aligned}$$

Therefore we have $H^{\leftarrow}P_t \not\subseteq Q_v \Rightarrow H^{\leftarrow}P_t \subseteq \bigvee_{d^{\rightarrow}f^{\leftarrow}A \subseteq f^{\leftarrow}B} \mathfrak{U}_1(d, D)$. So, by recalling (17) we get the contradiction "$H^{\leftarrow}P_t \not\subseteq Q_v \Rightarrow H^{\leftarrow}P_t \subseteq Q_v$." Thus, (f, F) is $\mathfrak{e}_{\mathfrak{U}_1}$ - $\mathfrak{e}_{\mathfrak{U}_2}$ extremial bicontinuous with respect to (h, H). □

Theorem 3.16. *Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be a graded diuniform texture space. Then we have*

$$(\mathcal{T}_{\mathfrak{e}_{\mathfrak{U}}}, \mathcal{K}_{\mathfrak{e}_{\mathfrak{U}}}) \subseteq (\mathcal{T}_{\mathfrak{U}}, \mathcal{K}_{\mathfrak{U}}).$$

Proof. Let $A \in \mathcal{S}$. By recalling Corollary 3.9 and Theorem 2.24, we have $\mathcal{T}_{\mathfrak{e}_{\mathfrak{U}}}(A) = \bigcap_{s \in A^b} \bigvee_{P_v \cap \mathfrak{e}^a_{\mathfrak{U}}(P_s, A) = \emptyset} P_v$ and $\mathcal{T}_{\mathfrak{U}}(A) = \bigcap_{s \in A^b} \bigvee_{d[s] \subseteq A} \mathfrak{U}(d, D)$. So it is sufficient to show that $\bigvee_{P_v \cap \mathfrak{e}^a_{\mathfrak{U}}(P_s, A) = \emptyset} P_v \subseteq \bigvee_{d[s] \subseteq A} \mathfrak{U}(d, D)$ for each $s \in A^b$.

Let $s \in A^b$ and $P_v \cap \mathfrak{e}^a_{\mathfrak{U}}(P_s, A) = \emptyset$. Then by using Proposition 3.12 we get $P_v \subseteq \varphi^a_{\mathfrak{U}}(P_s, A) = \bigvee_{d[s] \subseteq A} \mathfrak{U}(d, D)$. So, we obtain that $\mathcal{T}_{\mathfrak{e}_{\mathfrak{U}}} \subseteq \mathcal{T}_{\mathfrak{U}}$.

Similarly, it can be shown that $\mathcal{K}_{\mathfrak{e}_{\mathfrak{U}}} \subseteq \mathcal{K}_{\mathfrak{U}}$. □

4. The Relations of the Category dfGDiE with Some Other Categories

In this section we investigate the relations of the category **dfGDiE** with the categories **dfGDiU**, **dfGDiTop**, **dfDiE**, **dfDiU**, **dfDiTop**. Our reference for category theory is [1].

Proposition 4.1. ([20]) *Let $\mathcal{U}$ be a diuniformity on the texture $(S, \mathcal{S})$. Define*

$$A\delta^a B \Leftrightarrow d^{\rightarrow}A \not\subseteq B\ \forall (d, D) \in \mathcal{U} \text{ and } A\delta^b B \Leftrightarrow B \not\subseteq D^{\rightarrow}A\ \forall (d, D) \in \mathcal{U}.$$

Then $\delta = (\delta^a, \delta^b)$ is a diextremity on $(S, \mathcal{S})$.

Theorem 4.2. ([20]) *The diextremity defined in Proposition 4.1 is called the diextremity induced on* $(S, \mathcal{S})$ *by* $\mathcal{U}$*, or the induced diextremity for short, and is denoted by* $\delta_{\mathcal{U}} = (\delta^a_{\mathcal{U}}, \delta^b_{\mathcal{U}})$*. A uniformly bicontinuous difunction is also extremial bicontinuous with respect to the induced diextremities.*

Corollary 4.3. *With the above notations, the mapping* $\mathfrak{F}_1 : \mathbf{dfDiU} \to \mathbf{dfDiE}$ *defined by*

$$\mathfrak{F}_1((f,F) : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2)) = ((f,F) : (S_1, \mathcal{S}_1, \delta_{\mathcal{U}_1}) \to (S_2, \mathcal{S}_2, \delta_{\mathcal{U}_2}))$$

is a faithful and full functor.

Proof. By recalling Proposition 4.1 and Theorem 4.2 we get that $\mathfrak{F}_1$ is a functor. Because of the definition of $\mathfrak{F}_1$, it is a faithful and full functor. □

Corollary 4.4. *With the above notations, the mapping* $\mathfrak{F}_2 : \mathbf{dfDiE} \to \mathbf{dfDiTop}$ *defined by*

$$\mathfrak{F}_2((f,F) : (S_1, \mathcal{S}_1, \delta_1) \to (S_2, \mathcal{S}_2, \delta_2)) = ((f,F) : (S_1, \mathcal{S}_1, \tau_{\delta_1}, \kappa_{\delta_1}) \to (S_2, \mathcal{S}_2, \tau_{\delta_2}, \kappa_{\delta_2}))$$

is a faithful and full functor.

Proof. By Theorem 2.13, we get that $\mathfrak{F}_2$ is a functor. Because of the definition of $\mathfrak{F}_2$, it is a faithful and full functor. □

Theorem 4.5. ([7]) *The functor* $\mathfrak{H}_1 : \mathbf{dfDiTop} \to \mathbf{dfGDiTop}$ *defined by*

$$\begin{aligned}&\mathfrak{H}_1((f,F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2))\\ &= ((f,F),(i_1,I_1)) : (S_1, \mathcal{S}_1, \tau^g_1, \kappa^g_1, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \tau^g_2, \kappa^g_2, 1, \mathcal{P}(1))\end{aligned}$$

is an embedding of the category **dfDiTop** *as a full subcategory* $\mathbf{dfGDiTop}_{(1,\mathcal{P}(1))}$ *of the category* **dfGDiTop**.

Theorem 4.6. ([10]) *The functor* $\mathfrak{H}_2 : \mathbf{dfDiU} \to \mathbf{dfGDiU}$ *defined by*

$$\begin{aligned}&\mathfrak{H}_2((f,F) : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2))\\ &= ((f,F),(i_1,I_1)) : (S_1, \mathcal{S}_1, \mathfrak{U}_{\mathcal{U}_1}, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \mathfrak{U}_{\mathcal{U}_2}, 1, \mathcal{P}(1))\end{aligned}$$

is an embedding of the category **dfDiU** *as a full subcategory* $\mathbf{dfGDiU}_{(1,\mathcal{P}(1))}$ *of the category* **dfGDiU**.

Theorem 4.7. *The mapping* $\mathfrak{H}_3 : \mathbf{dfDiE} \to \mathbf{dfGDiE}$ *defined by*

$$\begin{aligned}&\mathfrak{H}_3((f,F) : (S_1, \mathcal{S}_1, \delta_1) \to (S_2, \mathcal{S}_2, \delta_2))\\ &= ((f,F),(i_1,I_1)) : (S_1, \mathcal{S}_1, \mathfrak{e}_{\delta_1}, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \mathfrak{e}_{\delta_2}, 1, \mathcal{P}(1))\end{aligned}$$

is an embedding of the category **dfDiE** *as a full subcategory* $\mathbf{dfGDiE}_{(1,\mathcal{P}(1))}$ *of the category* **dfGDiE**.

Proof. Since an extremial bicontinuous difunction $(f,F) : (S_1, \mathcal{S}_1, \delta_1) \to (S_2, \mathcal{S}_2, \delta_2)$ is $\mathfrak{e}_{\delta_1} - \mathfrak{e}_{\delta_2}$ extremial bicontinuous with respect to (i_1, I_1), $\mathfrak{H}_3$ is a functor. $\mathfrak{H}_3$ is also a full embedding from Example 3.3 (1), Definition 3.4 and the definition of extremial bicontinuity. □

Theorem 4.8. *With the above notations,* $\mathfrak{G}_1 : \mathbf{dfGDiU} \to \mathbf{dfGDiE}$ *defined by*

$$\begin{aligned}&\mathfrak{G}_1(((f,F),(h,H)) : (S_1, \mathcal{S}_1, \mathfrak{U}_1, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathfrak{U}_2, V_2, \mathcal{V}_2))\\ &= ((f,F),(h,H)) : (S_1, \mathcal{S}_1, \mathfrak{e}_{\mathfrak{U}_1}, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathfrak{e}_{\mathfrak{U}_2}, V_2, \mathcal{V}_2)\end{aligned}$$

is a faithful and full functor.

Proof. By Proposition 3.12 and Theorem 3.15 we have the fact that $\mathfrak{G}_1$ is a functor. Because of the definition of $\mathfrak{G}_1$, it is a faithful and full functor. □

Theorem 4.9. *With the above notations,* $\mathfrak{G}_2 : \mathbf{dfGDiE} \to \mathbf{dfGDiTop}$ *defined by*

$$\mathfrak{G}_2(((f,F),(h,H)) : (S_1,\mathcal{S}_1,\mathfrak{e}_1,V_1,\mathcal{V}_1) \to (S_2,\mathcal{S}_2,\mathfrak{e}_2,V_2,\mathcal{V}_2))$$
$$= ((f,F),(h,H)) : (S_1,\mathcal{S}_1,\mathcal{T}_{\mathfrak{e}_1},\mathcal{K}_{\mathfrak{e}_1},V_1,\mathcal{V}_1) \to (S_2,\mathcal{S}_2,\mathcal{T}_{\mathfrak{e}_2},\mathcal{K}_{\mathfrak{e}_2},V_2,\mathcal{V}_2)$$

is a faithful and full functor.

Proof. By Corollary 3.9 and Theorem 3.11 we have the fact that $\mathfrak{G}_2$ is a functor. Besides, from the definition of $\mathfrak{G}_2$, it is a faithful and full functor. □

Consequently, we have the diagram

$$\begin{tikzcd}
\mathbf{dfDiU} \arrow[r, "\mathfrak{F}_1"] \arrow[d, "\mathfrak{H}_2"] & \mathbf{dfDiE} \arrow[r, "\mathfrak{F}_2"] \arrow[d, "\mathfrak{H}_3"] & \mathbf{dfDiTop} \arrow[d, "\mathfrak{H}_1"] \\
\mathbf{dfGDiU} \arrow[r, "\mathfrak{G}_1"] & \mathbf{dfGDiE} \arrow[r, "\mathfrak{G}_2"] & \mathbf{dfGDiTop}
\end{tikzcd}$$

where $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1, \mathfrak{G}_2$ are faithful and full functors; also, $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ are embeddings.

5. Conclusion

The concept of proximity as a kind of ”nearness relation” provides an extensive perspective to the theory of topology; for instance, there is a one to one correspondence between the proximities and the totally bounded uniformities on a set.

Since the textures are complement free structures; Yıldız and Ertürk introduced the concept of diextremity, as an alternative suitable ”nearness relation” to proximities on textures in [20]. The relationship of diextremities with dimetrics and diuniformities is also investigated in [20].

In this study, graded diextremity is introduced as a generalization of diextremities on textures to the graded case. As expected, each graded diuniformity induces a graded diextremity and each graded diextremity induces a graded ditopology (see Proposition 3.12 and Corollary 3.9, resp.). In Section 4, this new structure is investigated with some categorical aspects; the relations of the category **dfGDiE** with the categories **dfGDiU**, **dfGDiTop**, **dfDiE**, **dfDiU**, **dfDiTop** are studied.

Clearly, graded diextremities can be useful to discover new properties of graded ditopological texture spaces and for deeper investigation of the theory of graded ditopology.

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