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Geodesic Mappings of Manifolds with Affine Connection onto the Ricci Symmetric Manifolds

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Abstract. In the present paper we investigate geodesic mappings of manifolds with affine connection onto Ricci symmetric manifolds which are characterized by the covariantly constant Ricci tensor. We obtained a fundamental system for this problem in a form of a system of Cauchy type equations in covariant derivatives depending on no more than $n(n+1)$ real parameters. Analogous results are obtained for geodesic mappings of manifolds with afine connection onto symmetric manifolds.

1. Introduction

The paper is devoted to a further development of Theory of geodesic mappings of manifolds with affine connection. This theory is inspired by T. Levi-Civita [13] where he set and solved in a special system of coordinates the problem of finding Riemannian metrics with common geodesics. It should be noted that it was linked with the study of equations of dynamical mechanical systems.

Later the theory of geodesic mappings was developed in the works of Thomas, Weyl, Cartan, Shirokov, Solodovnikov, Petrov, Sinyukov, Prvanović, Mikeš, etc. [4, 5, 20, 25, 30, 33, 35, 36, 40].

Geodesic mappings play a fundamental role in theoretical mechanics and physics, especially in the general theory of relativity [12, 13, 32, 33], and also in fundamental geometry of special lines structures [29].

Geodesic mappings of more general classes of manifolds (Finsler and generaly Einstein) have been studied in the works [22, 31, 43–45].

Geodesic mappings and projective transformations of symmetric, recurrent, Ricci symmetric, Einstein spaces and their generalizations were studied in [7, 14–20, 25–27, 30, 33, 35, 36, 38, 40].

In the fifties of the 20th century Sinyukov [38] studied the geodesic mappings of equiaffine symmetric manifolds with affine connection onto (pseudo-) Riemannian spaces. He proved that these (pseudo-) Riemannian spaces have a constant curvature. This is also valid for recurrent manifolds. These results have been many times repeated, the detailed description see [20, 25, 30, 40].

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Mikeš [14, 17–20] has followed up with the study based on the results obtained by Sinyukov. The above mentioned results are also valid for general recurrent spaces. In the paper [7] Hinterleitner and Mikeš proved that the targed spaces may be the Weyl manifolds.

Similar results are also true for geodesic mappings of non-Einstenian Ricci symmetric spaces onto (pseudo-) Riemannian spaces [15, 18]. Let us note that geodesic mappings of Einstein spaces have been studied by Mikes [16] and others, see [10, 25, 30].

Sinyukov [39, 40], while studying geodesic mappings between (pseudo-) Riemannian spaces, found out that general solution depends on a finite number of real parameters. This result holds also for geodesic mappings of manifolds with affine connection onto (pseudo-) Riemannian spaces [21], and for the case of geodesic mappings of generalized Finsler spaces onto (pseudo-) Riemannian spaces [22].

In the given study the main equations of geodesic mappings of manifolds A_n with affine connections onto Ricci symmetric manifolds $\bar{A}_n$ were obtained in the form of a system of partial differential equations of Cauchy type. The conditions of their integrability have been found. The numbers of independent real parameters that influence the general solutions of this system have been obtained.

Let us assume that the studied manifolds are simply connected and the dimension of the manifolds is $n \geq 2$. We supposed that geometrical objects are continuous and sufficient differentiable.

2. The Main Concepts of Geodesic Mappings on Manifolds with Affine Connection Theory

Among diffeomorphisms of manifolds geodesic mappings play an important role.

Let us suppose that the manifold A_n with affine connection ∇ admits a diffeomorphism f on the manifold $\bar{A}_n$ with affine connection $\bar{\nabla}$, and these manifolds are related to the common coordinate system $(x^1, x^2, \ldots, x^n)$ with respect to f.

We assume that

$$P_{ij}^h(x) = \bar{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x) \tag{1}$$

where $\Gamma_{ij}^h(x)$ and $\bar{\Gamma}_{ij}^h(x)$ are components of affine connections ∇ and $\bar{\nabla}$, $P_{ij}^h(x)$ is called a *deformation tensor* of connections with respect to diffeomorphism f.

A curve defined in space A_n is called *geodesic* if its tangent vector is parallel along it. A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* if f maps any geodesic in A_n onto a geodesic in $\bar{A}_n$.

It is well know [5, 25, 30, 40] that a mapping A_n onto $\bar{A}_n$ is geodesic if and only if in a common coordinate system $(x^1, x^2, \ldots, x^n)$ the tensor of deformation has the form

$$P_{ij}^h(x) = \psi_i(x)\,\delta_j^h + \psi_j(x)\,\delta_i^h \tag{2}$$

where δ_i^h is the Kronecker symbol and $\psi_i(x)$ are components of a covector.

A geodesic mapping is called *non-trivial* if $\psi_i(x) \neq 0$. It is evident that any arbitrary manifold A_n with affine connection ∇ allows a non-trivial geodesic mapping onto a certain other manifold $\bar{A}_n$ with affine connection $\bar{\nabla}$. Such a suggestion is generally recognized being not true regarding geodesic mapping of (pseudo-) Riemannian spaces onto (pseudo-) Riemannian spaces.

Particularly there were discovered (pseudo-) Riemannian spaces and also spaces with affine connection which do not admit non-trivial geodesic mappings onto (pseudo-) Riemannian spaces [14, 15, 17, 20, 25, 30, 40].

We proved [21] that the main equations of geodesic mappings of A_n onto (pseudo-) Riemannian spaces V_n have a Cauchy type form in covariant derivatives. The main equations of geodesic mappings of equi-affine manifolds on (pseudo-) Riemannian spaces have a linear form. É. Cartan [2] and J.M. Thomas [42] (see [4, p. 105]) demonstrated that any A_n is projectively equivalent with an equi-affine manifold. From this it follows that this system is linear in any case, see [25, p. 282].

3. Geodesic Mappings of Manifolds with Affine Connection onto Ricci Symmetric Manifolds

The manifold A_n with affine connection is called *Ricci symmetric* if the Ricci tensor is covariantly constant. From this it follows that a Ricci symmetric manifold $\bar{A}_n$ is characterized by the condition

$$\bar{R}_{ij|k} \equiv 0 \tag{3}$$

where $\bar{R}_{ij}$ is the Ricci tensor of $\bar{A}_n$, "$|$" denotes the covariant derivative with respect to the connection $\bar{\nabla}$ in $\bar{A}_n$.

Because

$$\bar{R}^h_{ijk|m} = \frac{\partial \bar{R}^h_{ijk}}{\partial x^m} + \bar{\Gamma}^h_{\alpha m}\bar{R}^\alpha_{ijk} - \bar{\Gamma}^\alpha_{im}\bar{R}^h_{\alpha jk} - \bar{\Gamma}^\alpha_{jm}\bar{R}^h_{i\alpha k} - \bar{\Gamma}^\alpha_{km}\bar{R}^h_{ij\alpha}$$

and considering formula (1) we can write

$$\bar{R}^h_{ijk|m} = \bar{R}^h_{ijk,m} + P^h_{\alpha m}\bar{R}^\alpha_{ijk} - P^\alpha_{im}\bar{R}^h_{\alpha jk} - P^\alpha_{jm}\bar{R}^h_{i\alpha k} - P^\alpha_{km}\bar{R}^h_{ij\alpha} \tag{4}$$

where comma denotes the covariant derivative with respect to the connection ∇ in A_n, $\bar{R}^h_{ijk}$ are components of the Riemann tensor of $\bar{A}_n$.

After contracting (4) with respect to the indices h and k we get

$$\bar{R}_{ij|m} = \bar{R}_{ij,m} - P^\alpha_{im}\bar{R}_{\alpha j} - P^\alpha_{jm}\bar{R}_{i\alpha}. \tag{5}$$

Further let us suppose that $\bar{A}_n$ is Ricci symmetric. Basing on formula (3) we obtain

$$\bar{R}_{ij,m} = P^\alpha_{im}\bar{R}_{\alpha j} + P^\alpha_{jm}\bar{R}_{i\alpha}. \tag{6}$$

Let us consider geodesic mappings of manifolds A_n with affine connection onto a Ricci symmetric manifold $\bar{A}_n$. Because the deformation tensor $P^h_{ij}(x)$ of connections has the form (2), than basing on formula (6), we obtain

$$\bar{R}_{ij,m} = 2\psi_m\bar{R}_{ij} + \psi_i\bar{R}_{mj} + \psi_j\bar{R}_{im}. \tag{7}$$

It is known [4, 5, 25, 30, 40] that between the Riemann tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of the manifolds A_n and $\bar{A}_n$, respectively, there is a dependence

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}. \tag{8}$$

Considering that

$$P^h_{ij,k}(x) = \psi_{i,k}(x)\,\delta^h_j + \psi_{j,k}(x)\,\delta^h_i$$

from formula (8) after computation we get

$$\bar{R}^h_{ijk} = R^h_{ijk} - \psi_{i,k}\,\delta^h_j + \psi_{i,j}\,\delta^h_k - (\psi_{j,k} - \psi_{k,j})\,\delta^h_i + \psi_i\psi_k\,\delta^h_j - \psi_i\psi_j\,\delta^h_k. \tag{9}$$

If we contract (9) with respect to the indices h and k we will obtain the following

$$\bar{R}_{ij} = R_{ij} + n\,\psi_{i,j} - \psi_{j,i} - (n-1)\,\psi_i\psi_j. \tag{10}$$

After alternation the equation (10) with respect to the indices i and j we can write

$$\bar{R}_{[ij]} = R_{[ij]} + (n+1)\,(\psi_{i,j} - \psi_{j,i}) \tag{11}$$

where $[ij]$ denotes alternation with respect to the indices i and j.

From equation (11) we find

$$\psi_{i,j} - \psi_{j,i} = \frac{1}{n+1}\,(\bar{R}_{[ij]} - R_{[ij]}) \tag{12}$$

and from equation (10) considering (12) we have the following formula

$$\psi_{i,j} = \frac{1}{n^2-1}\,\Big[n\bar{R}_{ij} + \bar{R}_{ji} - (nR_{ij} + R_{ji})\Big] + \psi_i\psi_j. \tag{13}$$

It is evident that the equations (7) and (13) in the given manifold A_n represent a closed Cauchy type system with respect to the unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.

Finally, we obtain the following theorem.

Theorem 3.1. *A manifold A_n with affine connection admits a geodesic mapping onto a Ricci symmetric manifold $\bar{A}_n$ if and only if in A_n exists a solution of a closed system of Cauchy type equations in covariant derivative* (7) *and* (13) *with respect to the unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.*

General solution of a closed system of equations (7) and (13) depends on no more than $n(n+1)$ independent real parameters.

It is evident that the equations (7) and (13) are non linear with respect to the unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$. We obtain the integrability conditions of them.

We differentiate equation (7) covariantly along x^k in the manifold A_n, and then alternate it with respect to the indices m and k. Considering the Ricci identity and formulas (7) and (13) after transformations we can write

$$\begin{aligned}&(n^2-1)\,(\bar{R}_{\alpha j}R^{\alpha}_{ikm} + \bar{R}_{i\alpha}R^{\alpha}_{jkm}) - 2(n-1)\,(\bar{R}_{km} - \bar{R}_{mk})\,\bar{R}_{ij} -\\ &n\,(\bar{R}_{kj} - \bar{R}_{jk})\,\bar{R}_{im} - (\bar{R}_{mi} - \bar{R}_{im})\,\bar{R}_{kj} - n\,(\bar{R}_{jm} - \bar{R}_{mj})\,\bar{R}_{ki} -\\ &(\bar{R}_{ik} - \bar{R}_{ki})\,\bar{R}_{mj} + (nR_{jm} + R_{mj})\,\bar{R}_{ik} + (nR_{im} + R_{mi})\,\bar{R}_{kj} -\\ &\quad (nR_{ik} + R_{ki})\,\bar{R}_{mj} - (nR_{jk} + R_{kj})\,\bar{R}_{im} = 0.\end{aligned} \tag{14}$$

Differenting equation (13) covariantly along x^k in A_n , then alternating with respect to the indices j and k, considering the Ricci identity and formulas (7) and (13), we obtain after transformations

$$\begin{aligned}&(n^2-1)\,\psi_i R^{\alpha}_{ijk} + n\psi_j R_{ik} + \psi_j R_{ki} - n\psi_k R_{ij} - \psi_k R_{ji} +\\ &(n-1)\,\psi_i\,(R_{jk} - R_{kj}) = nR_{ik,j} - nR_{ij,k} + R_{ki,j} - R_{ji,k}.\end{aligned} \tag{15}$$

It is evident that equation (15) is linear with respect to the ψ_i (from $\bar{R}_{ij}$ it does not depend).

Equation (14) will be linear with respect to $\bar{R}_{ij}$ (from ψ_i does not depend), if the symmetric manifold A_n ($\bar{R}^h_{ijk|m} = 0$) is also equiaffine.

The manifold A_n is called *equaffine*, if the Ricci tensor is symmetric, i.e.

$$\bar{R}_{ij} = \bar{R}_{ji}.$$

Thus, the integrability condition of the equations (7) and (13) under geodesic mappings of the manifolds A_n with affine connection on equiaffine symmetric manifolds $\bar{A}_n$ will be linear with respect to the unknown functions $\bar{R}_{ij}(x)$ and $\psi_i(x)$.

4. Geodesic Mappings of Manifolds with Affine Connection onto Symmetric Manifolds

The manifold $\bar{A}_n$ with affine connection $\bar{\nabla}$ is called (locally) *symmetric* if its Riemann tensor is covariantly constant (P.A. Shirokov [37], E. Cartan [2], S. Helgason [6]).

Thus, symmetric manifolds $\bar{A}_n$ are characterized by the condition

$$\bar{R}^h_{ijk|m} = 0. \tag{16}$$

It is evident that symmetric manifolds are also simultaneously Ricci symmetric manifolds. On the other hand, symmetric manifolds are semisymmetric. In 1950 these spaces were introduced by Sinyukov [38], they are characterized by the following condition

$$\bar{R}^h_{ijk|lm} - \bar{R}^h_{ijk|ml} = 0.$$

Geodesic mappings of semisymmetric spaces were studied by Sinyukov [38, 40] and Mikeš [14, 15, 17–20, 23, 25, 26, 28, 30, 40].

Let us consider a geodesic mapping f of the manifold A_n with affine connection ∇ onto a symmetric manifold $\bar{A}_n$ with affine connection $\bar{\nabla}$.

Equations (7) and (13) are valid for the above mentioned geodesic mapping, because, as we have mentioned, the symmetric manifold $\bar{A}_n$ is also Ricci symmetric.

Due to the condition (16) from equation (4) we get

$$\bar{R}^h_{ijk,m} = P^\alpha_{im}\bar{R}^h_{\alpha jk} + P^\alpha_{jm}\bar{R}^h_{i\alpha k} + P^\alpha_{km}\bar{R}^h_{ij\alpha} - P^h_{\alpha m}\bar{R}^\alpha_{ijk} \tag{17}$$

Taking into consideration that the deformation tensor $P^h_{ij}(x)$ has the structure (2), from (17) we obtain the following equation

$$\bar{R}^h_{ijk,m} = 2\psi_m\bar{R}^h_{ijk} + \psi_i\bar{R}^h_{mjk} + \psi_j\bar{R}^h_{imk} + \psi_k\bar{R}^h_{ijm} - \delta^h_m\psi_\alpha\bar{R}^\alpha_{ijk}. \tag{18}$$

Contracting (18) on the indices h and k, we obtain equation (7). This means that equation (7) is a consequence of equation (18).

It is evident that equations (13) and (18) in the given manifold A_n are a Cauchy type closed system of differential equations regarding the unknown functions $\bar{R}^h_{ijk}(x)$ and $\psi_i(x)$ that, naturally, should satisfy moreover the final conditions of algebraic character

$$\bar{R}^h_{i(jk)}(x) = 0, \qquad \bar{R}^h_{(ijk)}(x) = 0 \tag{19}$$

where brackets mean the symmetrization of the given indices without division.

From the above mentioned we prove the following theorem.

Theorem 4.1. *The manifold A_n with affine connection admits a geodesic mapping onto a symmetric manifold $\bar{A}_n$ if and only if in A_n exists a solution of a mixed system of Cauchy type equations in covariant derivative* (13), (18) *and* (19) *with respect to the unknown functions $\bar{R}^h_{ijk}(x)$ and $\psi_i(x)$.*

It is evident that the general solution of the mixed closed system of Cauchy type equations depends no more that from

$$\frac{1}{2}\, n^3(n-1) + n$$

independent parameters.

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