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# Trees with Equal Total Domination and 2-Rainbow Domination Numbers

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**Abstract.** A 2-rainbow dominating function (2RDF) of a graph $G$ is a function $f : V(G) \to \mathcal{P}(\{1,2\})$ such that for each $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = \{1,2\}$. For a 2RDF $f$ of a graph $G$, the weight $w(f)$ of $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight over all 2RDFs of $G$ is called the 2-rainbow domination number of $G$, which is denoted by $\gamma_{r2}(G)$. A subset $S$ of vertices of a graph $G$ without isolated vertices, is a total dominating set of $G$ if every vertex in $V(G)$ has a neighbor in $S$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of $G$. Chellali, Haynes and Hedetniemi conjectured that $\gamma_t(G) \le \gamma_{r2}(G)$ [M. Chellali, T.W. Haynes and S.T. Hedetniemi, Bounds on weak Roman and 2-rainbow domination numbers, Discrete Appl. Math. 178 (2014), 27–32.], and later Furuya confirmed the conjecture [M. Furuya, A note on total domination and 2-rainbow domination in graphs, Discrete Appl. Math. 184 (2015), 229–230.]. In this paper, we provide a constructive characterization of trees $T$ with $\gamma_{r2}(T) = \gamma_t(T)$.

## 1. Introduction

In this paper, we shall only consider graphs without multiple edges or loops or isolated vertices. Let $G$ be a graph, $S \subseteq V(G)$, $v \in V(G)$, the *open neighborhood* of $v$ in $S$ is denoted by $N_S(v)$. That is to say $N_S(v) = \{u | uv \in E(G), u \in S\}$. The *closed neighborhood* $N_S[v]$ of $v$ in $S$ is defined as $N_S[v] = \{v\} \cup N_S(v)$. If $S = V(G)$, then $N_S(v)$ and $N_S[v]$ are denoted by $N(v)$ and $N[v]$, respectively. The degree of $v$ is the number of neighbors of $v$ and it is denoted by $\deg(v)$, i.e. $\deg(v) = |N(v)|$. A *leaf* of $G$ is a vertex with degree one in $G$ and a vertex that has a leaf neighbor is called a *support vertex*. The set of leaf neighbors of a vertex $v$ is denoted by $L(v)$. A *strong support vertex* is a support vertex adjacent to at least two leaves. An *end support vertex* is a support vertex whose all neighbors with exception at most one are leaves. We denote by $P_n$ the path on $n$ vertices. A *pendant path* $P$ of a graph $G$ is an induced path such that one of end points has degree one in $G$, and its other end point is the only vertex of $P$ adjacent to some vertex in $G - P$. The *distance* $d_G(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $uv$-path in $G$. The *diameter* of a graph $G$, denoted by diam$(G)$, is the greatest distance between two vertices of $G$. A *double star* is a tree

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with exactly two vertices that are not leaves. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denotes the set of children of $v$, $D(v)$ denotes the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of $v$, depth($v$), is the largest distance from $v$ to a vertex in $D(v)$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D(v) \cup \{v\}$, and is denoted by $T_v$.

In a graph $G$, a vertex is said to *dominate* all vertices adjacent to it. A *total dominating set* (TDS) in a graph $G$ is a subset $S \subseteq V(G)$ such that each vertex in $V(G)$ is *dominated* by at least a vertex in $S$, that is $N(S) = V(G)$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of $G$. A TDS with cardinality $\gamma_t(G)$ is called a $\gamma_t$-set of $G$ (or $\gamma_t(G)$-set). The total domination number was introduced by Cockayne, Dawes and Hedetniemi [5] and is now well studied in graph theory. The literatures on this subject has been surveyed and detailed in the book by Henning and Yeo [10].

A 2-*rainbow dominating function* (2RDF) of a graph $G$ is a function $f : V(G) \to \mathcal{P}(\{1, 2\})$ such that for each $v \in V(G)$ with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$. For a 2RDF $f$ of a graph $G$, the weight $w(f)$ of $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight over all 2RDFs of $G$ is called the 2-*rainbow domination number* of $G$, and is denoted by $\gamma_{r2}(G)$. A 2RDF with weight $\gamma_{r2}(G)$ is called a $\gamma_{r2}$-function of $G$ or a $\gamma_{r2}(G)$-*function*. The rainbow domination number was introduced by Brešar, Henning, and Rall [1] and has been studied by several authors (see for example [3, 6, 11, 13, 14]).

Chellali, Haynes and Hedetniemi [4] investigated difference between many domination-like parameters and they conjectured that $\gamma_t(G) \le \gamma_{r2}(G)$ for any graph $G$ without isolated vertices. Later, Furuya [7] confirmed this conjecture. A natural problem that may arise is the characterization of graphs (or trees) $G$ with $\gamma_t(G) = \gamma_{r2}(G)$. In this paper, we provide a constructive characterization of trees $T$ with $\gamma_{r2}(T) = \gamma_t(T)$.

We make use of the following results in this paper.

**Observation 1.1.** *([6]) Let $G$ be a connected graph. If there is a path $v_3v_2v_1$ in $G$ with* $\deg(v_2) = 2$ *and* $\deg(v_1) = 1$, *then $G$ has a $\gamma_{r2}(G)$-function $f$ such that $f(v_1) = \{1\}$ and $2 \in f(v_3)$.*

**Observation 1.2.** *Let $H$ be an induced subgraph of a graph $G$ such that $G$ and $H$ have no isolated vertices. If $\gamma_{r2}(H) = \gamma_t(H)$, $\gamma_t(G) \ge \gamma_t(H) + s$ and $\gamma_{r2}(G) \le \gamma_{r2}(H) + s$ for some positive integer $s$, then $\gamma_{r2}(G) = \gamma_t(G)$.*

*Proof.* It follows from the assumptions and the fact $\gamma_t(G) \le \gamma_{r2}(G)$ that

$$\gamma_t(G) \ge \gamma_t(H) + s = \gamma_{r2}(H) + s \ge \gamma_{r2}(G) \ge \gamma_t(G)$$

and this leads to the result. □

**Observation 1.3.** *Let $H$ be an induced subgraph of a graph $G$ such that $G$ and $H$ have no isolated. If $\gamma_{r2}(G) = \gamma_t(G)$, $\gamma_t(G) \le \gamma_t(H) + s$ and $\gamma_{r2}(G) \ge \gamma_{r2}(H) + s$ for some positive integer $s$, then $\gamma_{r2}(H) = \gamma_t(H)$.*

*Proof.* By assumptions and the fact $\gamma_t(H) \le \gamma_{r2}(H)$, we have

$$\gamma_t(G) \le \gamma_t(H) + s \le \gamma_{r2}(H) + s \le \gamma_{r2}(G) = \gamma_t(G)$$

and this leads to the result. □

## 2. Trees with equal total domination and 2-rainbow domination numbers

In this section, we provide a constructive characterization of all trees with $\gamma_t(T) = \gamma_{r2}(T)$. We begin with three definitions.

**Definition 2.1.** *For a tree $T$ and $v \in V(T)$, let*

$$\gamma_t(T, v) = \min\{|S| : S \subseteq V(T) \text{ and each vertex } w \neq v \text{ has a neighbor in } S\}.$$

*Clearly $\gamma_t(T, v) \le \gamma_t(v)$ for each $v \in V(T)$. We define*

$$W_T^1 = \{v | \gamma_t(T, v) = \gamma_t(T)\}.$$

**Definition 2.2.** *For a tree $T$ and $v \in V(T)$, we say $v$ has property $P$ in $T$ if there exists a $\gamma_{r2}(T)$-function $f$ such that $f(v) \neq \emptyset$. Define*

$$W_T^2 = \{v | v \text{ has property } P \text{ in } T\}.$$

**Definition 2.3.** *An extended spider with $t$ $(t \geq 2)$ feet is a tree obtained from star $K_{1,t}$ by subdividing every edge of $K_{1,t}$ twice. The center of star is called the* head *of spider.*

In order to presenting our constructive characterization, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees $T$ that can be obtained from a sequence $T_1, T_2, \cdots, T_k$ of trees for some $k \geq 1$, where $T_1$ is $P_2$ or $P_3$ and $T = T_k$. If $k \geq 2$, $T_{i+1}$ can be obtained from $T_i$ by one of the following ten operations.

**Operation $O_1$:** If $x \in V(T_i)$ and $x$ is a strong support vertex, then $O_1$ adds a vertex $y$ and an edge $xy$ to obtain $T_{i+1}$.

**Operation $O_2$:** If $x \in W_{T_i}^1$, then $O_2$ adds a star $K_{1,s}$ $(s \geq 3)$ with a leaf $c$ and an edge $xc$ to obtain $T_{i+1}$ (see Fig. 1 (a));

**Operation $O_3$:** If $x \in V(T_i)$ and there is a pendant path $xyz$, then $O_3$ adds a pendant path $xba$ to obtain $T_{i+1}$ (see Fig. 1 (b));

**Operation $O_4$:** If $x \in V(T_i)$ and $x$ is adjacent to the center of a pendant star $K_{1,s}$ $(s \geq 1)$, then $O_4$ adds a pendant path $xcba$ to obtain $T_{i+1}$ (see Fig. 1 (c));

**Operation $O_5$:** If $T_i$ contains a strong support vertex $z$ and a pendant path $zyx$, then $O_5$ adds a pendant edge $xa$ to obtain $T_{i+1}$ (see Fig. 1 (d));

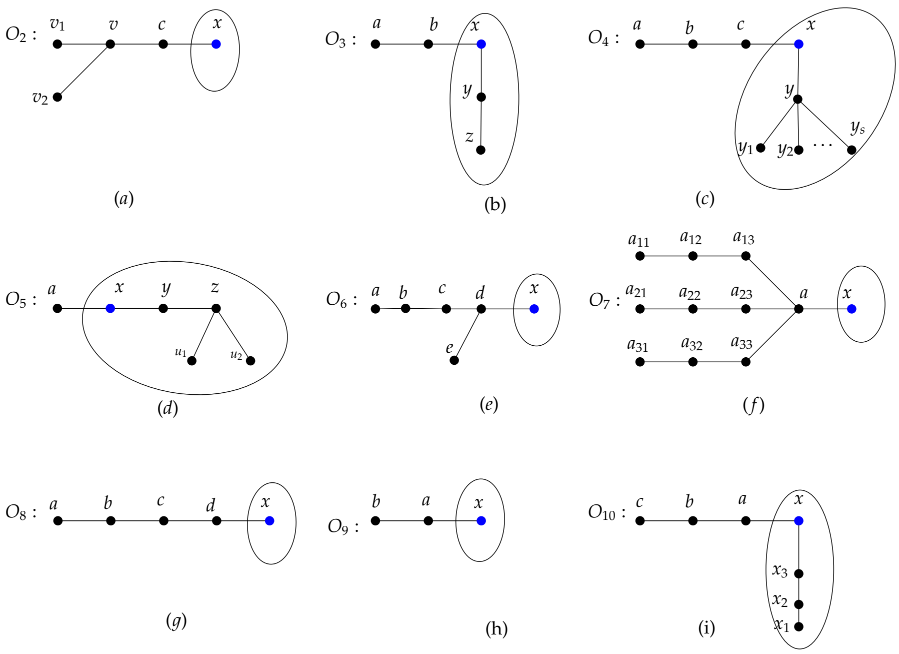
**Operation $O_6$:** If $x \in W_{T_i}^1$, then $O_6$ adds a path $P_5 = abcde$ and an edge $xd$ to obtain $T_{i+1}$ (see Fig. 1 (e));

**Operation $O_7$:** If $x \in V(T_i)$, then $O_7$ adds an extended spider headed at $a$ with $k \geq 2$ feet and joins $x$ to $a$ for obtaining $T_{i+1}$ (see Fig. 1 (f));

**Operation $O_8$:** If $x \in W_{T_i}^2$, then $O_8$ adds a pendant path $xdcba$ to obtain $T_{i+1}$ (see Fig. 1 (g)).

**Operation $O_9$:** If $x \in W_{T_i}^1$ and $x$ is a strong support vertex, then $O_9$ adds a pendant path $xab$ to obtain $T_{i+1}$ (see Fig. 1 (h)).

**Operation $O_{10}$:** If $x \in T_i$ is a support vertex and there is a pendant path $xx_3x_2x_1$, then $O_{10}$ adds a pendant path $xabc$ to obtain $T_{i+1}$ (see Fig. 1 (i)).




The proof of the first lemma is trivial.

**Lemma 2.4.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_1$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

**Lemma 2.5.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_2$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* Let $f$ be a $\gamma_{r2}$-function of $T_i$, we can obtain a 2RDF $f'$ of $T_{i+1}$ by letting $f'(t) = f(t)$ for $t \in V(T_i)$, $f'(v) = \{1, 2\}$, $f'(u) = \emptyset$ for $u \in N(v)$. Hence $\gamma_{r2}(T_{i+1}) \le \gamma_{r2}(T_i) + 2$. On the other hand, let $S$ be a $\gamma_t(T_{i+1})$-set containing no leaves and let $v$ be the central vertex of the star added by Operation $O_2$. Then we have $v, c \in S$ and clearly $S - \{c, v\}$ is a subset of vertices such that each vertex $w \in V(T_i) - \{x\}$ has a neighbor in $S - \{v, c\}$. Since $x \in W^1_{T_i}$, we have $|S - \{v, c\}| \ge \gamma_t(T_i)$ and so $\gamma_t(T_{i+1}) \ge \gamma_t(T_i) + 2$. Now the result follows from Observation 1.2. □

**Lemma 2.6.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_3$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* By Observation 1.1, there exists a $\gamma_{r2}$-function $f$ of $T_i$ such that $f(z) = \{1\}$ and $2 \in f(x)$, now we can extend $f$ to a 2RDF $f'$ of $T_{i+1}$ by letting $f'(a) = \{1\}$ and $f'(b) = \emptyset$. Hence we have $\gamma_{r2}(T_{i+1}) \le \gamma_{r2}(T_i) + 1$. On the other hand, if $S$ is a $\gamma_t(T_{i+1})$-set containing no leaves, then $y, x, b \in S$ and it follows that $S - \{b\}$ is a TDS of $T_i$ yielding $\gamma_t(T_{i+1}) \ge \gamma_t(T_i) + 1$. By Observation 1.2, we have $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$ as desired. □

**Lemma 2.7.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_4$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* By observation 1.2, it is enough to show that $\gamma_{r2}(T_{i+1}) \le \gamma_{r2}(T_i) + 2$ and $\gamma_t(T_{i+1}) \ge \gamma_t(T_i) + 2$. Clearly any $\gamma_{r2}(T_i)$-function $f$ can be extended to a 2RDF of $T_{i+1}$ by assigning $\{1\}$ to $a$, $\emptyset$ to $b$ and $\{2\}$ to $c$ and this implies that $\gamma_{r2}(T_{i+1}) \le \gamma_{r2}(T_i) + 2$. Now let $S$ be a $\gamma_t(T_{i+1})$-set containing no leaves. Then we have $b, c, x, y \in S$ where $y$ is the center of the star $K_{1,s}$. Then obviously $S - \{b, c\}$ is a TDS of $T_i$ of size $\gamma_t(T_{i+1}) - 2$ and so $\gamma_t(T_{i+1}) \ge \gamma_t(T_i) + 2$. This completes the proof. □

**Lemma 2.8.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_5$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* By observation 1.2, we need only to show that $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 1$ and $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 1$. By Observation 1.1, there exists a $\gamma_{r2}(T_i)$-function $f$ such that $f(x) = \{1\}$. Then the function $g : V(T_{i+1}) \to \mathcal{P}(\{1, 2\})$ defined by $g(a) = \{1\}$ and $g(z) = f(z)$ for $z \in V(T_{i+1}) - \{a\}$ is a 2RDF of $T_{i+1}$ of weight $\gamma_{r2}(T_i) + 1$ and so $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 1$.

Now let $S$ be a $\gamma_t(T_{i+1})$-set which contains no leaf. Then we must have $x, y, z \in S$ and clearly $S - \{x\}$ is a TDS of $T_i$. Therefore $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 1$ and the proof is complete. □

**Lemma 2.9.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_6$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* We show that $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 3$ and $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 3$. Let $f$ be an arbitrary $\gamma_{r2}$-function of $T_i$ and define $g : V(T_{i+1}) \to \mathcal{P}(\{1, 2\}$ by $g(a) = g(e) = \{1\}$, $g(c) = \{2\}$, $g(b) = g(d) = \emptyset$ and $g(z) = f(z)$ for $z \in V(T_i)$. Clearly $g$ is a 2RDF of $T_{i+1}$ of weight $\gamma_{r2}(T_i) + 3$ and hence $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 3$.

Now let $S$ be a $\gamma_t(T_{i+1})$-set which contains no leaf. Then we have $b, c, d \in S$. It is not hard to see that $S' = S - \{b, c, d\}$ is a set of vertices of $T_i$ such that any vertex $w \neq x$ has a neighbor in $S'$. Since $x \in W^1_{T_i}$, we have $|S'| \geq \gamma_t(T_i)$ and this implies that $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 3$. Now the result follows by Observation 1.2. □

**Lemma 2.10.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_7$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* Let $k \geq 2$ and $T_1$ be the extended spider with $k$ feet added by Operation $O_7$. Assume $T_1$ headed at $a$ and its feet are $a_{i1}a_{i2}a_{i3}$ with $a_{i3}a \in E(T_1)$ for $1 \leq i \leq k$. Let $f$ be a $\gamma_{r2}$-function of $T_i$, we can obtain a 2RDF $f'$ of $T_{i+1}$ by letting $f'(a) = \emptyset$, $f'(a_{i1}) = \{2\}$, $f'(a_{i2}) = \emptyset$ and $f'(a_{i3}) = \{1\}$ for $i = 1, 2, \ldots, k$. Hence $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 2k$. Now we show that $\gamma_t(T_i) \leq \gamma_t(T_{i+1}) - 2k$. Let $S$ be a $\gamma_t(T_{i+1})$-set containing no leaves. Then we must have $a_{i2}, a_{i3} \in S$ for $i = 1, 2, \ldots, k$. If $a \notin S$, then $S' = S - \{a_{i2}, a_{i3} \mid i = 1, \ldots, k\}$ is a TDS of $T_i$ of weight at most $\gamma_t(T_{i+1}) - 2k$ yielding $\gamma_t(T_i) \leq \gamma_t(T_{i+1}) - 2k$. Suppose that $a \in S$. This implies that $u \notin S$ for each $u \in N(x) \setminus \{a\}$. Then $S' = (S - \{a, a_{i2}, a_{i3} \mid i = 1, \ldots, k\}) \cup \{u\}$ for each $u \in N(x) \setminus \{a\}$ is clearly a TDS of $T_i$ of size at most $\gamma_{r2}(T_{i+1}) - 2k$. Thus $\gamma_t(T_i) \leq \gamma_t(T_{i+1}) - 2k$. We now deduce from Observation 1.2 that $\gamma_{r2}(T_{i+1}) = \gamma_t(T_{i+1})$ and the proof is complete. □

**Lemma 2.11.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_8$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* Let $f$ be a $\gamma_{r2}$-function of $T_i$ such that $f(x) \neq \emptyset$ (since $x \in W^2_T$, so such a function exists). Assume without loss of generality that $1 \in f(x)$. Then $f$ can be extended to a 2RDF $f'$ of $T_{i+1}$ by letting $f'(a) = \{1\}$, $f'(b) = f'(d) = \emptyset$ and $f'(c) = \{2\}$. Then we have $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 2$.

On the other hand, let $S$ be a $\gamma_t(T_{i+1})$-set containing no leaves. Then we have $b, c \in S$. We claim that there exists a TDS $S'$ of $G$ of size at most $|S|$ such that $b, c \in S'$ and $a, d \notin S'$. If $d \notin S$, then let $S' = S$. Assume that $d \in S$. This implies that $u \notin S$ for each $u \in N(x) \setminus \{d\}$. Now let $S' = (S - \{d\}) \cup \{u\}$ for some $u \in N(x) \setminus \{d\}$. Clearly $S'$ is a TDS of $T_{i+1}$ of size at most $|S|$ satisfying our claim. Then $S' - \{b, c\}$ is a TDS of $T_i$ of size at most $\gamma_{r2}(T_{i+1}) - 2$. This yields $\gamma_t(T_i) \leq |S'| - 2 \leq |S| - 2 = \gamma_t(T_{i+1}) - 2$. Therefore, $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 2$ and the result follows by Observation 1.2. □

**Lemma 2.12.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_9$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* Let $f$ be a $\gamma_{r2}$-function of $T_i$ that assigns $\{1, 2\}$ to each strong support vertex. We can obtain a 2RDF $f'$ of $T_{i+1}$ by letting $f'(t) = f(t)$ for $t \in V(T_i)$, $f'(b) = \{1\}$, $f'(a) = \emptyset$. Hence $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 1$. Now let $S$ be a $\gamma_t(T_{i+1})$-set containing no leaves. Then we must have $x, a \in S$ and clearly $S - \{a\}$ is a set of vertices of $T_i$ such that each vertex $w \in V(T_i) - \{x\}$ has a neighbor in $S - \{a\}$. Since $x \in W^1_{T_i}$, we have $|S - \{a\}| \geq \gamma_t(T_i)$ and so $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 1$. Now the result follows from Observation 1.2. □

**Lemma 2.13.** *If $T_i$ is a tree with $\gamma_t(T_i) = \gamma_{r2}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $\mathcal{O}_{10}$, then $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$.*

*Proof.* Assume $S$ is an arbitrary $\gamma_t(T_{i+1})$-set containing no leaves. Then we have $a, b, x_3, x_2, x \in S$ and clearly $S - \{a, b\}$ is a TDS of $T_i$ yielding $\gamma_t(T_{i+1}) \geq \gamma_t(T_i) + 2$. On the other hand, any $\gamma_{r2}(T_i)$-function can be extended to a 2RDF of $T_{i+1}$ by assigning $\{1\}$ to $a$, $\{2\}$ to $c$ and $\emptyset$ to $b$, and hence $\gamma_{r2}(T_{i+1}) \leq \gamma_{r2}(T_i) + 2$. Thus $\gamma_t(T_{i+1}) = \gamma_{r2}(T_{i+1})$ by Observation 1.2. □

**Theorem 2.14.** *If $T \in \mathcal{T}$, then $\gamma_{r2}(T) = \gamma_t(T)$.*

*Proof.* Obviously, if $T$ is $P_2$ or $P_3$, then $\gamma_{r2}(T) = \gamma_t(T)$. Now assume that $T \in \mathcal{T}$, then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ ($k \geq 1$) such that $T_1$ is $P_2$ or $P_3$, and if $k \geq 2$, then $T_{i+1}$ can be obtained recursively from $T_i$ by Operation $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{10}$ for $i = 1, 2, \ldots, k-1$. We apply induction on the number of operations performed to construct $T$. It can be seen that if $k = 1$, the result holds. Suppose that the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{r2}(T') = \gamma_t(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{10}$ from $T'$, we conclude from above Lemmas that $\gamma_{r2}(T) = \gamma_t(T)$. □

**Observation 2.15.** If $T$ is a double star, then $\gamma_{r2}(T) \neq \gamma_t(T)$.

**Theorem 2.16.** Let $T$ be a tree of order $n \geq 2$. Then $\gamma_{r2}(T) = \gamma_t(T)$ if and only if $T \in \mathcal{T}$.

*Proof.* The sufficiency follows from Theorem 2.14. In order to prove the necessity we proceed by induction on $n$. If $n = 2, 3$, then the only trees $T$ of order $2, 3$ and $\gamma_{r2}(T) = \gamma_t(T)$ are $P_2, P_3 \in \mathcal{T}$. Let $n \geq 4$ and let the statement holds for all trees of order less than $n$. Assume that $T$ is a tree of order $n$ with $\gamma_{r2}(T) = \gamma_t(T)$. If $\mathrm{diam}(T) = 2$ then $T$ is a star and $T$ can be obtained from $P_3$ by applying Operation $\mathcal{O}_1$ and so $T \in \mathcal{T}$. Let $\mathrm{diam}(T) \geq 3$. By Observation 2.15, we have $\mathrm{diam}(T) \geq 4$.

Let $v_1 v_2 \ldots v_k$ ($k \geq 5$) be a diametral path in $T$ such that $|L_{v_2}|$ is as large as possible and root $T$ at $v_k$. Also suppose among paths with this property we choose a path such that $|L_{v_3}|$ is as large as possible. We consider two cases.

**Case 1.** $\deg(v_2) \geq 3$.
We claim that $\deg(v_3) = 2$. Assume, to the contrary, that $\deg(v_3) \geq 3$. We distinguish four subcases.

**Subcase 1.1.** $v_3$ is a strong support vertex or is adjacent to a strong support vertex other than $v_2, v_4$.
Let $T' = T - T_{v_2}$. Then any $\gamma_t(T')$-set containing no leaves contains $v_3$ and such a $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2$ and so $\gamma_t(T) \leq \gamma_t(T') + 1$. Suppose now $f$ is a $\gamma_{r2}(T)$-function. We may assume that $f$ assigns $\{1, 2\}$ to each strong support vertex. Hence the function $f$, restricted to $T'$ is a 2RDF and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$. Thus $\gamma_t(T') + 2 \leq \gamma_{r2}(T') + 2 \leq \gamma_{r2}(T) = \gamma_t(T) \leq \gamma_t(T') + 1$ which is a contradiction.

**Subcase 1.2.** $v_3$ is adjacent to a support vertex of degree 2 other than $v_4$.
Let $T' = T - T_{v_2}$. As above we have $\gamma_t(T) \leq \gamma_t(T') + 1$. Let $f$ be a $\gamma_{r2}(T)$-function that assigns $\{1, 2\}$ to each strong support vertex. By Observation 1.1, we may assume that $f(v_3) \neq \emptyset$. Then the function $f$, restricted to $T'$ is a 2RDF and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$. Now we get a contradiction as above.

**Subcase 1.3.** $\deg(v_3) = 3$ and $v_3$ is adjacent to a leaf $u$.
Let $T' = T - T_{v_3}$. Then any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_3, v_2$ and so $\gamma_t(T) \leq \gamma_t(T') + 2$. Let $f$ be a $\gamma_{r2}(T)$-function that assigns $\{1, 2\}$ to each strong support vertex. Clearly $|f(v_3)| + |f(u)| \geq 1$. If $|f(v_3)| + |f(u)| \geq 2$, then the function $g : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g(v_4) = \{1\} \cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T') - \{v_4\}$ is a 2RDF of $T'$ of weight $\omega(f) - 3$ and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$. If $|f(v_3)| + |f(u)| = 1$, then clearly $f(v_3) = \emptyset$ and the function $f$, restricted to $T'$ is a 2RDF of $T'$ of weight $\omega(f) - 3$ and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$. Thus $\gamma_t(T') + 3 \leq \gamma_{r2}(T') + 3 \leq \gamma_{r2}(T) = \gamma_t(T) \leq \gamma_t(T') + 2$ which is a contradiction.

Thus $\deg(v_3) = 2$. Assume that $T' = T - T_{v_3}$. Let $f$ be a $\gamma_{r2}(T)$-function that assigns $\{1, 2\}$ to each strong support vertex. Then the function $g : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g(v_4) = f(v_3) \cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T') - \{v_4\}$ is a 2RDF of $T'$ of weight $\omega(f) - 2$ and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$. On the other hand, any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_3, v_2$ and so $\gamma_t(T) \leq \gamma_t(T') + 2$. It follows from Observation 1.3 that $\gamma_{r2}(T') = \gamma_t(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Now we show that $v_4 \in W^1_{T'}$. Assume, to the contrary, that $\gamma_t(T', v_4) < \gamma_t(T')$. Let $S \subseteq V(T')$ be a set of vertices of size $\gamma_t(T', v_4)$ such that each vertex $w \in V(T') - \{v_4\}$ has a neighbor in $S$. Then $S \cup \{v_2, v_3\}$ is a total dominating set of $T$ of size less than $\gamma_t(T)$ which is a contradiction. Thus $v_4 \in W^1_{T'}$ and so $T \in \mathcal{T}$ since it can be obtained from $T'$ by Operation $\mathcal{O}_2$.

**Case 2.** $\deg(v_2) = 2$.
By the choice of diametral path, we may assume that every end-support vertex on a diametral path has degree 2. In particular, $\deg(v_{k-1}) = 2$. We consider the following subcases.

**Subcase 2.1.** $\deg(v_3) \geq 3$ and there is a pendant path $v_3z_2z_1$ in $T$ where $z_2 \notin \{v_2, v_4\}$.
Then $\deg(z_2) = 2$ and $\deg(z_1) = 1$. Let $T' = T - T_{v_2}$. Clearly any $\gamma_t(T')$-set containing no leaf can be extended to a TDS of $T$ by adding $v_2$ and so $\gamma_t(T) \leq \gamma_t(T') + 1$. Applying Observation 1.1, it is easy to see that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 1$ and so $\gamma_t(T') = \gamma_{r2}(T')$ by Observation 1.3. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Now since $T$ can be obtained from $T'$ by Operation $\mathcal{O}_3$, we deduce that $T \in \mathcal{T}$.

**Subcase 2.2.** $\deg(v_3) \geq 4$ and all neighbors of $v_3$ with exception $v_2, v_4$, are leaves.
Let $T' = T - T_{v_2}$. Suppose $f$ is a $\gamma_{r2}(T)$-function that assigns $\{1,2\}$ to each strong support vertex. By Observation 1.1, we may assume that $f(v_1) = \{1\}$. Then the function $f$, restricted to $T'$ is a 2RDF of $T'$ of weight at most $\omega(f) - 1$ and so $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 1$. On the other hand, any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2$ and so $\gamma_t(T) \leq \gamma_t(T') + 1$. By Observation 1.3, we obtain $\gamma_t(T') = \gamma_{r2}(T')$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. Next we show that $v_3 \in W^1_{T'}$. Assume, to the contrary, that $\gamma_t(T', v_3) < \gamma_t(T')$ and let $S \subseteq V(T')$ be a set of vertices of $T'$ of size $\gamma_t(T', v_3)$ such that each vertex $w \in V(T') - \{v_3\}$ has a neighbor in $S$. We note that $v_3 \in S$. Then $S \cup \{v_2\}$ is a total dominating set of $T$ of size less than $\gamma_t(T)$ which is a contradiction. Thus $v_3 \in W^1_{T'}$ and so $T$ can be obtained from $T'$ by Operation $\mathcal{O}_9$. Therefore, $T \in \mathcal{T}$.

**Subcase 2.3.** $\deg(v_3) = 3$ and $v_3$ is adjacent to a leaf $u$.
Since $\deg(v_{k-1}) = 2$, we have $\mathrm{diam}(T) \geq 5$. We show that this case is impossible. Consider the following.

- $\deg(v_4) = 2$.
  Let $T' = T - T_{v_4}$. Clearly, any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2, v_3$ and hence $\gamma_t(T) \leq \gamma_t(T') + 2$. On the other hand, if $f$ is a $\gamma_{r2}(T)$-function, then obviously $|f(u)| + |f(v_3)| + |f(v_2)| + |f(v_1)| \geq 3$ and the function $g : V(T') \to \mathcal{P}(\{1,2\})$ defined by $g(v_5) = f(v_5) \cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$, is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 3$. Therefore

  $$\gamma_t(T) = \gamma_{r2}(T) \geq \gamma_{r2}(T') + 3 > \gamma_t(T') + 2 \geq \gamma_t(T)$$

  which is a contradiction.

- $v_4$ is a strong support vertex.
  Let $T' = T - T_{v_3}$. Clearly, any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2, v_3$ and the restriction of any $\gamma_{r2}(T)$-function assigning $\{1,2\}$ to each strong support vertex to $T'$, is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 3$. Therefore $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$ and we get a contradiction as above.

- $v_4$ is adjacent to an end support vertex.
  Let $T' = T - T_{v_3}$. It is not hard to see that $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$ and this leads to a contradiction.

- $v_4$ has a neighbor $z_3$ other than $v_3, v_5$ such that $T_{z_3} = T_{v_3}$.
  Let $T' = T - T_{v_3}$. As above we have $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let $f$ be a $\gamma_{r2}(T)$-function. Then obviously $\sum_{z \in V(T_{v_3})} |f(z)| \geq 3$ and $\sum_{z \in V(T_{z_3})} |f(z)| \geq 3$. Define $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g(v_1) = \{1\}, g(v_3) = \{1,2\}, g(v_2) = g(u) = \emptyset$ and $g(z) = f(z)$ for $z \in V(T')$. It is easy to see that $g$ is a $\gamma_{r2}(T)$-function and the restriction of $g$ to $T'$ is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 3$. Thus $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$ and we obtain a contradiction as above.

- $\deg(v_4) = 3$ and $v_4$ is adjacent to a leaf $w$ where $w \neq v_5$.
  Let $T' = T - T_{v_4}$. Clearly any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2, v_3, v_4$ and hence

$\gamma_t(T) \le \gamma_t(T') + 3$. Now let $f$ be a $\gamma_{r2}(T)$-function. It is easy to verify that $\sum_{z\in V(T_{v_4})} |f(z)| \ge 5$ when $f(v_4) \neq \emptyset$. Define $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g = f$ when $f(v_4) = \emptyset$ and by $g(v_5) = f(v_5)\cup\{1\}$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$. It is easy to see that $g$ is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 4$ and this implies that $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 4$. This leads to a contradiction as above.

- $\deg(v_4) = 3$ and there is a pendant path $v_4z_3z_2z_1$ where $z_3 \neq v_5$.
  Let $T' = T - T_{v_4}$. Clearly any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2, v_3, z_3, z_2$ and hence $\gamma_t(T) \le \gamma_t(T') + 4$. Now let $f$ be a $\gamma_{r2}(T)$-function. It is easy to see that $\sum_{z\in V(T_{v_3})} |f(z)| \ge 3$ and $\sum_{z\in V(T_{z_3})} |f(z)| \ge 2$. Define $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g(v_5) = f(v_5)\cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T')-\{v_5\}$. It is easy to see that $g$ is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 5$ and so $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 5$. Again we get a contradiction.

- There are two pendant paths $v_4z_3z_2z_1$ and $v_4y_3y_2y_1$ where $v_5 \notin \{y_3, z_3\}$.
  Let $T' = T - T_{v_3}$. Clearly $\gamma_t(T) \le \gamma_t(T') + 2$. Now let $f$ be a $\gamma_{r2}(T)$-function. It is easy to see that $\sum_{z\in V(T_{v_3})} |f(z)| \ge 3$, $\sum_{z\in V(T_{y_3})} |f(z)| \ge 2$ and $\sum_{z\in V(T_{z_3})} |f(z)| \ge 2$. Define $g : V(T) \to \mathcal{P}(\{1,2\})$ by $g(y_1) = \{1\}, g(y_3) = \{2\}, g(z_1) = \{2\}, g(z_3) = \{1\}$, $g(y_2) = g(z_2) = \emptyset$ and $g(z) = f(z)$ otherwise. It is easy to see that $g$ is a $\gamma_{r2}(T)$-function and the function $g$ restricted to $T'$ is a 2RDF of $T'$ of weight at most $\gamma_{r2}(T) - 3$. Hence $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 3$ and we get a contradiction again.

Considering Subcases 2.1, 2.2 and 2.3, we may assume that $\deg(v_3) = 2$. If there exists a path $v_4z_3z_2z_1$ where $z_4 \notin \{v_3, v_5\}$ in $T$, then by the choice of diametral path, we have $\deg(z_3) = \deg(z_2) = 2$. If $\mathrm{diam}(T) = 4$, then $T = P_5$ and $T \in \mathcal{T}$ since it can be obtained from $P_3$ by Operation $\mathcal{O}_3$. Hence, we assume that $\mathrm{diam}(T) \ge 5$. We proceed with more cases.

**Subcase 2.4.** $\deg(v_4) = 2$.
Let $T' = T - T_{v_4}$. Clearly, every $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding the vertices $v_2, v_3$ and so $\gamma_t(T) \le \gamma_t(T') + 2$. Next we show that $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 2$. Let $f$ be a $\gamma_{r2}(T)$-function. By Observation 1.1, we may assume that $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If $f(v_3) = \{1,2\}$, then define $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g(v_5) = f(v_5) \cup \{1\}$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$, and if $f(v_3) = \{2\}$, then define $g : V(T') \to \mathcal{P}(\{1,2\})$ by $g(v_5) = f(v_5) \cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$. Obviously, $g$ is a 2RDF of $T'$ of weight $\omega(f) - 2$ and so $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 2$. It follows from Observation 1.3 that $\gamma_{r2}(T') = \gamma_t(T')$ and hence $T' \in \mathcal{T}$. Now, we show that $v_5 \in W^2_{T'}$. Let $f$ be a $\gamma_{r2}(T)$-function and assume that $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If $\sum_{i=1}^{4} |f(v_i)| \ge 3$, then the function $g$ defined above, is a $\gamma_{r2}(T')$-function with $g(v_5) \neq \emptyset$. If $\sum_{i=1}^{4} |f(v_i)| = 2$, then we must have $f(v_1) = \{1\}, f(v_3) = \{2\}, f(v_2) = f(v_4) = \emptyset$ and to rainbowly dominate $v_4$, we must have $f(v_5) = \{1\}$. Thus the function $f$, restricted to $T'$ is a $\gamma_{r2}(T')$-function with $f(v_5) \neq \emptyset$. Thus $v_5 \in W^2_{T'}$ and since $T$ can be obtained from $T'$ by Operation $\mathcal{O}_8$, we obtain that $T \in \mathcal{T}$.

**Subcase 2.5.** $v_4$ is a strong support vertex.
Let $T' = T - v_1$. Then any $\gamma_t(T')$-set containing no leaves can be extended to a TDS of $T$ by adding $v_2$ and so $\gamma_t(T) \le \gamma_t(T') + 1$. Now we show that $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 1$. Let $f$ be a $\gamma_{r2}(T)$-function that assigns $\{1,2\}$ to each strong support vertex. By Observation 1.1, we may assume that $f(v_1) = 1$ and $2 \in f(v_3)$. Then the function $g : V(T') \to \mathcal{P}(\{1,2\})$ defined by $g(v_2) = \{1\}, g(v_3) = \emptyset$ and $g(z) = f(z)$ for $z \in V(T') - \{v_2, v_3\}$ is a 2RDF of $T'$ of weight $\omega(f) - 1$ and so $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 1$. By Observation 1.3, $\gamma_t(T') = \gamma_{r2}(T')$ and by the induction hypothesis on $T'$ we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$, since it is obtained from $T'$ by Operation $\mathcal{O}_5$.

**Subcase 2.6.** $v_4$ is adjacent to a support vertex $y$.
Then clearly the depth of $y$ is 1. Let $T' = T - T_{v_3}$. It is not hard to see that $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{r2}(T) = \gamma_{r2}(T') + 2$. This yields $\gamma_{r2}(T') = \gamma_t(T')$ and hence $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $\mathcal{O}_4$.

**Subcase 2.7.** $\deg(v_4) \ge 4$ and $v_4$ is a support vertex.
By Cases 6,7 and 4, we may assume that $v_4$ is adjacent to exactly one leaf, say $u$, and that there exists a pendant path $v_4z_3z_2z_1$ in $T$ where $z_3 \notin \{v_3, v_5\}$. Let $T' = T - T_{v_3}$. Clearly any $\gamma_t(T')$-set containing no leaves can be extended to a TDS of $T$ by adding $v_2, v_3$ and so $\gamma_t(T) \le \gamma_t(T') + 2$. Now we show that $\gamma_{r2}(T) \ge \gamma_{r2}(T') + 2$. Let $f$ be a $\gamma_{r2}(T)$-function. By Observation 1.1, we may assume that $f(v_1) = f(z_1) = \{1\}, 2 \in f(v_2)$ and

$2 \in f(z_2)$. If $f(v_4) \neq \emptyset$, then the function $f$, restricted to $T'$ is a 2RDF of $T$ of weight $\omega(f) - 2$. Assume that $f(v_4) = \emptyset$. Then we may assume without loss of generality that $f(u) = \{1\}$. Again the function $f$, restricted to $T'$ is a 2RDF of $T$ of weight $\omega(f) - 2$. Thus $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2$, and we deduce from Observation 1.3 that $\gamma_{r2}(T') = \gamma_t(T')$. By the induction hypothesis on $T'$, we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$, since it is obtained from $T'$ by Operation $\mathcal{O}_{10}$.

**Subcase 2.8.** $\deg(v_4) \geq 3$ and $v_4$ is not a support vertex.
Considering Case 6, we may assume that $T_{v_4}$ is an extended spider where $v_4$ is the head of spider. Let $T' = T - T_{v_4}$ and let $\deg(v_4) = t + 1$. Clearly any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding all support vertices of $T_{v_4}$ and all neighbors of $v_4$ with exception $v_5$ implying that $\gamma_t(T) \leq \gamma_t(T') + 2t$. Now we show that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2t$. Let $f$ be a $\gamma_{r2}(T)$-function. By Observation 1.1, we may assume that $f$ assigns $\{1\}$ to all leaves of $T_{v_4}$ and $\{2\}$ to all neighbors of $v_4$ in $T_{v_4}$. Then the function $g : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g(v_5) = f(v_5) \cup f(v_4)$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$ is a 2RDF of $T$ of weight at most $\omega(f) - 2t$ and this implies that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 2t$. It follows from Observation 1.3 that $\gamma_{r2}(T') = \gamma_t(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Now $T \in \mathcal{T}$, since it can be obtained from $T'$ by Operation $\mathcal{O}_7$.

**Subcase 2.9.** $\deg(v_4) = 3$ and $v_4$ is adjacent to a leaf, say $w$.
Let $T' = T - T_{v_4}$. First we show that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$. Let $f$ be a $\gamma_{r2}(T)$-function. By Observation 1.1, we may assume that $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If $f(v_4) = \emptyset$, then $|f(w)| \geq 1$ and the function $f$, restricted to $T'$ is a 2RDF of $T'$ of weight $\omega(f) - 3$. Assume that $f(v_4) \neq \emptyset$. Then we have $|f(v_4)| + |f(w)| \geq 2$ and the function $g : V(T') \to \mathcal{P}(\{1, 2\})$ defined by $g(v_5) = f(v_5) \cup \{1\}$ and $g(z) = f(z)$ for $z \in V(T') - \{v_5\}$ is a 2RDF of $T'$ of weight at most $\omega(f) - 3$. This implies that $\gamma_{r2}(T) \geq \gamma_{r2}(T') + 3$. On the other hand, any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $v_2, v_3, v_4$ and so $\gamma_t(T) \leq \gamma_t(T') + 3$. It follows from Observation 1.3 that $\gamma_t(T') = \gamma_{r2}(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Next we show that $v_5 \in W^1_{T'}$. Assume, to the contrary, that $\gamma_t(T', v_5) < \gamma_t(T')$. Let $S \subseteq V(T')$ be a set of vertices of $T'$ of size $\gamma_t(T', v_5)$ such that each vertex $w \in V(T') - \{v_5\}$ has a neighbor in $S$. Then $S \cup \{v_2, v_3, v_4\}$ is a total dominating set of $T$ of weight less than $\gamma_t(T)$ which is a contradiction. Thus $v_5 \in W^1_{T'}$ and so $T \in \mathcal{T}$, since it can be obtained from $T'$ by Operation $\mathcal{O}_6$. This completes the proof. □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## References


[1] B. Brešar, and T. K. Šumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007), 2394-2400.
[2] R.C. Brigham and R.D. Dutton, *Factor domination in graphs*, Discrete Math. **86** (1990), 127–136.
[3] G.J. Chang, J. Wu and X. Zhu, *Rainbow domination on trees*, Discrete Appl. Math. **158** (2010), 8-12.
[4] M. Chellali, T.W. Haynes, S.T. Hedetniemi, *Bounds on weak Roman and 2-rainbow domination numbers*, Discrete Appl. Math. *178* (2014), 27–32.
[5] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), 211–219.
[6] N. Dehgardi, S.M. Sheikholeslami and L. Volkmann, *The rainbow domination subdivision number of a graph*, Mat. Vesnik **67** (2015), 102–114.
[7] M. Furuya, *A note on total domination and 2-rainbow domination in graphs*, Discrete Appl. Math. *184* (2015), 229–230.
[8] M. R. Garey and D. S. Johnson, Computers and Intractability: a Guide to the Theory of NP-Completeness, W. H. Freeman and Co., San Francisco, Calif., 1979.
[9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, NewYork (1998).
[10] M.A. Henning and A. Yeo, *Total Domination in Graphs*, Springer Monographs in Mathematics, 2013.
[11] D. Meierling, S. M. Sheikholeslami and L. Volkmann, *Nordhaus-Gaddum bounds on the k-rainbow domatic number of a graph*, Appl. Math. Lett. **24** (2011), 1758-1761.
[12] O. Ore, Theory of Graphs, American Mathematical Society, Providence, R.I., 1967.
[13] S.M. Sheikholeslami and L. Volkmann, *The k-rainbow domatic number of a graph*, Discuss. Math. Graph Theory **32** (2012), 129–140.
[14] C. Tong, X. Lin, Y. Yang and M. Luo, *2-rainbow domination of generalized Petersen graphs* $P(n, 2)$, Discrete Appl. Math. **157** (2009), 1932-1937.