Filomat 32:1 (2018), 35–44
https://doi.org/10.2298/FIL1801035U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Warped Product Submanifolds of Kaehler Manifolds with Pointwise Slant Fiber

Siraj Uddin[a], Mića S. Stanković[b]

[a]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[b]Department of Mathematics, Faculty of Sciences and Mathematics, University of Nis, Serbia

Abstract. It was shown in [15, 16] that there does not exist any warped product submanifold of a Kaehler manifold such that the spherical manifold of the warped product is proper slant. In this paper, we introduce the notion of warped product submanifolds with a slant function. We show that there exists a class of non-trivial warped product submanifolds of a Kaehler manifold such that the spherical manifold is pointwise slant by giving an example and a characterization theorem. We also prove that if the warped product is mixed totally geodesic then the warping function is constant.

1. Introduction

In [8], B.-Y. Chen and O.J. Garay introduced the notion of pointwise slant submanifolds of an almost Hermitian manifold and they have obtained many interesting result and gave a method how to construct such submanifolds in Euclidean space. They defined these submanifolds as follows: For any non-zero vector $X \in T_pM$, $p \in M$, the angle $\theta(X)$ between JX and the tangent space T_pM is called the *Wirtinger angle* of X. The Wirtinger angle gives rise a real-valued function $\theta : TM - \{0\} \to \mathbb{R}$, called a wirtinger function, defined on the set $T^*M = TM - \{0\}$ consisting of all nonzero vectors on M. A submanifold M of an almost Hermitian manifold $\widetilde{M}$ is called *pointwise slant* if, at each point $p \in M$, the Wirtinger angle $\theta(X)$ is independent of the choice of the nonzero tangent vector $X \in T^*_pM$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold. We note that the poitwise slant submanifolds have been studied in [11] by F. Etayo under the name of quasi-slant submanifolds. We also note that every slant submanifold is pointwise slant but converse may not be true. These submanifolds are also studied in [14].

On the other hand, the geometry of warped product submanifolds became an active field of research after Chen' papers on the geometry of warped product CR-submanifolds [4, 5]. He proved that there do not exist warped product submanifolds of the form $M_\perp \times_f M_T$ in a Kaehler manifold $\widetilde{M}$. Then he introduced the notion of CR-warped products of Kaehler manifolds as follows: A submanifold of a Kaehler manifold is called the CR-warped product if it is the warped product of the form $M_T \times_f M_\perp$, where M_T and $M_\perp$ are holomorphic and totally real submanifolds of $\widetilde{M}$, respectively. He obtained several fundamental results

2010 *Mathematics Subject Classification*. 53C15; 53C40; 53C42; 53B25
Keywords. slant submanifolds; pointwise slant; hemi-slant; pointwise hemi-slant; mixed totally geodesic; warped product; Kaehler manifolds
Received: 21 January 2017; Accepted:09 April 2017
Communicated by Dragan S. Djordjević
Email addresses: siraj.ch@gmail.com (Siraj Uddin), stmica@ptt.rs (Mića S. Stanković)

including, a characterization and a sharp inequality for the squared norm of the second fundamental form $\|h\|^2$. Later on, Sahin [15] proved that there do not exist warped product submanifolds of the form $M_T \times_f M_\theta$ and $M_\theta \times_f M_T$ such that M_T and M_θ are holomorphic and proper slant submanifolds of $\widetilde{M}$, respectively. Using the notion of pointwise slant submanifolds, Sahin introduced pointwise semi-slant submanifolds of Kaehler manifolds and investigated their warped products [17].

Moreover, in [16], Sahin also proved the non-existence of warped product submanifolds $M_\perp \times_f M_\theta$ of a Kaehler manifold $\widetilde{M}$, where $M_\perp$ is a totally real submanifold and M_θ is a proper slant submanifold of $\widetilde{M}$. Then he introduced the notion of hemi-slant warped products $M_\theta \times_f M_\perp$. He provided many examples of such submanifolds and obtained interesting results, including a characterization and an inequality. In this paper, first we define pointwise hemi-slant submanifolds of Kaehler manifolds and then we show that there exists a class of non-trivial warped product submanifolds of the form $M_\perp \times_f M_\theta$ in a Kaehler manifold $\widetilde{M}$ such that $M_\perp$ and M_θ are totally real and proper pointwise slant submanifolds of $\widetilde{M}$, respectively. We note that one of the characterization result of such warped products is given in [18] by using different technique. It is also notice that the warped product hemi-slant submanifolds of almost Hermitian manifolds were studied under the name of warped product pseudo-slant submanifolds in [18–20].

As we know that there exist nontrivial warped product submanifolds of the from $M_\theta \times_f M_\perp$ in a Kaehler manifold $\widetilde{M}$ such that M_θ is proper slant (see [16]) and if we assume that M_θ is poitwise then the warped product poitwise hemi-slant submanifolds of the form $M_\theta \times_f M_\perp$ is a special case of warped product hemi-slant submanifolds $M_\theta \times_f M_\perp$. Thus, we shall leave this case for the repetition purpose i.e., there is no meaning to study warped product pointwise hemi-slant submanifolds of the form $M_\theta \times_f M_\perp$; while M_θ is pointwise slant. For the survey on this topic we refer to Chen's books [6, 9] and his survey article [7]. We also note that, in [21], we studied warped product bi-slant submanifolds of Kaehler manifolds which is a more general case of warped product submanifolds.

The paper is organised as follows: In Section 2 we give basic information needed for this paper. In Section 3, we define and studied pointwise hemi-slant submanifolds of Kaehler manifolds. In Section 4, we study warped product pointwise hemi-slant submanifolds of the form $M_\perp \times_f M_\theta$ in Kaehler manifolds such $M_\perp$ is a totally real submanifold and M_θ is a pointwise submanifold. In this section, we provide an example and present a characterization theorem for such warped products.

2. Preliminaries

Let $(\widetilde{M}, J, g)$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$J^2 = -I, \tag{1}$$

$$g(JX, JY) = g(X, Y) \tag{2}$$

for all $X, Y \in \mathcal{X}(\widetilde{M})$, where I is the identity map.

Let $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}$. If the almost complex structure J satisfies

$$(\widetilde{\nabla}_X J)Y = 0 \tag{3}$$

for $X, Y \in \mathcal{X}(\widetilde{M})$, then $\widetilde{M}$ is called a *Kaehler manifold*.

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$. Then M is called a *complex* submanifold if $J(T_xM) \subseteq T_xM$ for any $x \in M$, where T_xM is the tangent space of M at x. The submanifold M is called *totally real* if $J(T_xM) \subseteq T_x^\perp M$ for any $x \in M$, where $T_x^\perp M$ denotes the normal space of M at x.

Let M be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M, then the Gauss and Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of M with respect to N. Moreover, $h : TM \times TM \to T^{\perp}M$ is the second fundamental form of M in $\widetilde{M}$. Furthermore, A_N and h are related by [22]

$$g(h(X, Y), N) = g(A_N X, Y) \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$.

For any X tanget to M, we write

$$JX = PX + FX, \tag{7}$$

where PX and FX are the tangential and normal components of JX, respectively. Then P is an endomorphism of tangent bundle TM and ω is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$JN = tN + fN, \tag{8}$$

where tN and fN are the tangential and normal components of JN, respectively. Moreover, from (2) and (7), we have $g(PX, Y) = -g(X, PY)$, for any $X, Y \in \Gamma(TM)$.

A submanifold M of a locally product Riemnnian manifold $\widetilde{M}$ is said to be *totally umbilical submanifold* if $h(X, Y) = g(X, Y)H$, for any $X, Y \in \Gamma(TM)$, where $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, the mean curvature vector of M. A submanifold M is said to be totally geodesic if $h(X, Y) = 0$. A totally umbilical submanifold of dimension greater than or equal to 2 with non-vanishing parallel mean curvature vector is called an *extrinsic sphere*.

A (differentiable) distribution $\mathcal{D}$ defined on a submanifold M of $(\widetilde{M}, J, g)$ is called pointwise *θ-slant* if, for each point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and $\mathcal{D}$ is independent of the choice of the nonzero vector $X \in \mathcal{D}$ (cf. [2, 3, 8]). A pointwise θ-slant distribution is called slant if θ is globally constant. Also, it is holomorphic or complex if $\theta = 0$; and it is called totally real if $\theta = \frac{\pi}{2}$, globally. A poitwise θ-slant distribution is called proper pointwise slant whenever $\theta \neq 0, \frac{\pi}{2}$ and θ is not a constant.

From Chen's result (Lemma 2.1) of [8], it is known that M is a pointwise slant submanifold of an almost Hermitian manifold $\widetilde{M}$ if and only if

$$P^2 = -(\cos^2\theta)I, \tag{9}$$

for some real-valued function θ defined on M, where I denotes the identity transformation of the tangent bundle TM of M. The following relations are the consequences of (9) as

$$g(PX, PY) = \cos^2\theta\, g(X, Y), \tag{10}$$

$$g(FX, FY) = \sin^2\theta\, g(X, Y) \tag{11}$$

for any $X, Y \in \Gamma(TM)$. Another important relation for a poitwise slant submanifold of an almost Hermitian manifold is obtained by using (1), (7), (8) and (9) as

$$tFX = -(\sin^2\theta)\, X, \quad fFX = -FPX \tag{12}$$

for any $X \in \Gamma(TM)$.

3. Pointwise Hemi-slant Submanifolds

In this section, we study pointwise hemi-slant submanifolds of Kaehler manifolds. First, we define these submanifolds as follows.

Definition 3.1. *Let $\widetilde{M}$ be a Kaehler manifold and M a real submanifold of $\widetilde{M}$. Then, we say that M is a pointwise hemi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on M such that*

(i) The tangent space TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$.
(ii) The distribution $\mathcal{D}^\perp$ is totally real, i.e. $J(\mathcal{D}^\perp) \subset T^\perp M$.
(iii) The distribution $\mathcal{D}^\theta$ is pointwise slant with slant function θ.

In the above definition, the angle θ is called the slant function of the pointwise slant distribution $\mathcal{D}^\theta$. The totally real distribution $\mathcal{D}^\perp$ of a pointwise hemi-slant submanifold is a pointwise slant distribution with slant function $\theta = \frac{\pi}{2}$. If we denote the dimensions of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by m_1 and m_2, respectively, then we have the following possible cases:

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is a totally real submanifold.
(iii) If $m_1 = 0$ and $\theta = 0$, then M is a holomorphic submanifold.
(iv) If θ is constant on M, then M is a hemi-slant submanifold with slant angle θ.
(v) If $\theta = 0$, then M is a CR-submanifold.

We note that a pointwise hemi-slant submanifold is proper if $m_1 \neq 0$ and θ is not a constant. The normal bundle $T^\perp M$ of a poitwise hemi-slant submanifold M is decomposed by

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus F\mathcal{D}^\theta, \;\; \varphi\mathcal{D}^\perp \perp F\mathcal{D}^\theta.$$

Now, we give the following useful lemma.

Lemma 3.2. *Let M be a pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then the totally real distribution $\mathcal{D}^\perp$ is always integrable.*

The proof of Lemma 3.2 is similar to Theorem 3.5 of [16].

Lemma 3.3. *Let M be a pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

(i) For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$\cos^2\theta\, g(\nabla_X Y, Z) = g(A_{JZ}PY, X) - g(A_{FPY}Z, X). \tag{13}$$

(ii) For any $Z, V \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D}^\theta)$, we have

$$\cos^2\theta\, g(\nabla_Z V, X) = g(A_{FPX}V, Z) - g(A_{JV}PX, Z). \tag{14}$$

Proof. We prove (i) and (ii) in a similar way. For any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(J\widetilde{\nabla}_X Y, JZ).$$

Using (3) and (7), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\widetilde{\nabla}_X PY, JZ) + g(\widetilde{\nabla}_X FY, JZ) \\ &= g(h(X, PY), JZ) - g(\widetilde{\nabla}_X JFY, Z). \end{aligned}$$

Then from (8), we get

$$g(\nabla_X Y, Z) = g(h(A_{JZ}PY, X) - g(\widetilde{\nabla}_X tFY, Z) - g(\widetilde{\nabla}_X fFY, Z).$$

Thus from (12), we derive

$$g(\nabla_X Y, Z) = g(h(A_{JZ}PY, X) + g(\widetilde{\nabla}_X \sin^2\theta\, Y, Z) + g(\widetilde{\nabla}_X FPY, Z).$$
$$= g(h(A_{FZ}PY, X) + \sin^2\theta\, g(\widetilde{\nabla}_X Y, Z) + \sin 2\theta\, X(\theta)\, g(Y, Z)$$
$$- g(A_{FPY}X, Z).$$

Then by the orthogonality of two distributions and the symmetry of the shape operator, we get (i). In a similar way we can prove (ii). □

4. Warped Products $M_\perp \times_f M_\theta$ in Kaehler Manifolds

In [1], Bishop and O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant. Let X be an unit vector field tangent to M_1 and Z be an another unit vector field on M_2, then from Lemma 7.3 of [1], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z \tag{15}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ be a warped product manifold then the base manifold M_1 is totally geodesic in M and the fiberM_2 is totally umbilical in M [1, 4].

Analogous to CR-warped products introduced in [4], we define the notion of warped product pointwise hemi-slant submanifolds as follows.

Definition 4.1. A warped product $M_\perp \times_f M_\theta$ of totally real and pointwise slant submanifolds $M_\perp$ and M_θ of an almost Hermitian manifold $(\widetilde{M}, J, g)$ is called a *warped product pointwise hemi-slant submanifold*.

A warped product pointwise hemi-slant submanifold $M_\perp \times_f M_\theta$ is called *proper* if M_θ is proper pointwise slant and $M_\perp$ is totally real in $\widetilde{M}$. Otherwise, $M_\perp \times_f M_\theta$ is called *non-proper*.

In [16], Sahin proved that there are no warped product hemi-slant submanifolds of the form $M_\perp \times_f M_\theta$ in a Kaehler manifold $\widetilde{M}$ such that M_θ is proper slant. But if we assume that M_θ is a pointwise slant submanifold of $\widetilde{M}$, then there exists a class of nontrivial warped products.

Next, we provide an example of warped product pointwise hemi-slant submanifold of the form $M_\perp \times_f M_\theta$ such M_θ is a pointwise slant submanifold.

Let $\mathbb{E}^{2n}$ be the Euclidean $2n$-space with the standard metric and let $\mathbb{C}^n$ denote the complex Euclidean n-space $(\mathbb{E}^{2n}, J)$ equipped with the canonical complex structure J defined by

$$J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n).$$

Thus we have

$$J\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n. \tag{16}$$

Example 4.2. Consider a submanifold M of $\mathbb{R}^{10}$ defined by

$$\phi(u,v,w) = (u\cos v, u\sin v, u\cos w, u\sin w, -v+w, v+w, -u\cos v, u\sin v, \\ -u\cos w, u\sin w).$$

such that $u \neq 0$ is a real valued function on M. It is easy to see that the tangent bundle TM of M is spanned by the following vectors

$$Z_1 = \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial y_1} + \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - \cos v\frac{\partial}{\partial x_4} \\ + \sin v\frac{\partial}{\partial y_4} - \cos w\frac{\partial}{\partial x_5} + \sin w\frac{\partial}{\partial y_5},$$

$$Z_2 = -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3} + u\sin v\frac{\partial}{\partial x_4} + u\cos v\frac{\partial}{\partial y_4},$$

$$Z_3 = -u\sin w\frac{\partial}{\partial x_2} + u\cos w\frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3} + u\sin w\frac{\partial}{\partial x_5} + u\cos w\frac{\partial}{\partial y_5}.$$

Then, using the canonical complex structure (16) of $\mathbb{R}^{10}$, we have

$$JZ_1 = -\cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial x_1} - \cos w\frac{\partial}{\partial y_2} + \sin w\frac{\partial}{\partial x_2} + \cos v\frac{\partial}{\partial y_4} \\ + \sin v\frac{\partial}{\partial x_4} + \cos w\frac{\partial}{\partial y_5} + \sin w\frac{\partial}{\partial x_5},$$

$$JZ_2 = u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3} - u\sin v\frac{\partial}{\partial y_4} + u\cos v\frac{\partial}{\partial x_4},$$

$$JZ_3 = u\sin w\frac{\partial}{\partial y_2} + u\cos w\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_3} + \frac{\partial}{\partial x_3} - u\sin w\frac{\partial}{\partial y_5} + u\cos w\frac{\partial}{\partial x_5}.$$

It is clear that JZ_1 is orthogonal to TM. Thus $\mathcal{D}^\perp = \mathrm{Span}\{Z_1\}$ is a totally real distribution. Moreover, it is easy to see that $\mathcal{D}^\theta = \mathrm{Span}\{Z_2, Z_3\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{1+u^2}\right)$. It is easy to verify that both distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ are completely integrable. Let $M_\perp$ and M_θ be the integral manifolds of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. Then the metric tensor of M is given by

$$g = 4du^2 + \left(2+2u^2\right)\left(dv^2+dw^2\right) = g_{M_\perp} + \left(\sqrt{2(1+u^2)}\right)^2 g_{M_\theta},$$

where g_{M_θ} and $g_{M^\perp}$ are the metric tensors of M_θ and $M_\perp$, respectively. Consequently, $M = M_\perp \times_f M_\theta$ is a warped product pointwise hemi-slant submanifold of $\mathbb{R}^{10}$ with warping function $f = \sqrt{2(1+u^2)}$ and the slant function $\theta = \cos^{-1}\left(\frac{1}{1+u^2}\right)$.

Now, we investigate the geometry of the warped product pointwise hemi-slant submanifolds of form $M_\perp \times_f M_\theta$. First, we prove the following useful lemma for later use.

Lemma 4.3. *Let $M = M_\perp \times {}_f M_\theta$ be a warped product pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

(i) $g(h(Z,V),FX) = g(h(X,Z),JV)$;
(ii) $g(h(X,Y),JZ) = Z(\ln f)\,g(X,PY) + g(h(X,Z),FY)$

for any $Z,V \in \Gamma(TM_\perp)$ *and* $X,Y \in \Gamma(TM_\theta)$.

Proof. For any $Z,V \in \Gamma(TM_\perp)$ and $X \in \Gamma(TM_\theta)$, we have

$$\begin{aligned} g(h(Z,V),FX) &= g(\widetilde{\nabla}_Z V, FX) \\ &= g(\widetilde{\nabla}_Z V, JX) - g(\widetilde{\nabla}_Z V, PX) \\ &= -g(\widetilde{\nabla}_Z JV, X) - g(\widetilde{\nabla}_Z PX, V). \end{aligned}$$

Then from (4), (5) and (15), we obtain

$$g(h(Z,V),FX) = g(A_{JV}Z,X) + Z(\ln f)\,g(PX,V).$$

From the orthogonality of the vector fields and (5), we find

$$g(h(Z,V),FX) = -g(h(X,Z),JV)$$

which is (i). For the second part of the lemma, we have

$$g(h(X,Y),JZ) = g(\widetilde{\nabla}_X Y, JZ) = -g(\widetilde{\nabla}_X JY, Z)$$

for any $X,Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. Using (7) and (5), we obtain

$$\begin{aligned} g(h(X,Y),JZ) &= -g(\widetilde{\nabla}_X PY, Z) - g(\widetilde{\nabla}_X FY, Z) \\ &= g(\widetilde{\nabla}_X Z, PY) + g(A_{FY}X, Z). \end{aligned}$$

Thus, (ii) follows from the above relation by using (6) and (15), which proves the lemma completely. □

If we interchange X by PX and Y by PY in Lemma 4.3 (ii), for any $X,Y \in \Gamma(TM_\theta)$, then by using (9) and (10), we have the following relations

$$g(h(PX,Y),JZ) = \cos^2\theta\, Z(lnf)\,g(X,Y) + g(h(PX,Z),FY), \tag{17}$$

$$g(h(X,PY),JZ) = -\cos^2\theta\, Z(lnf)\,g(X,Y) + g(h(X,Z),FPY) \tag{18}$$

and

$$g(h(PX,PY),JZ) = \cos^2\theta\, Z(lnf)\,g(X,PY) + g(h(PX,Z),FPY). \tag{19}$$

Lemma 4.4. *Let* $M = M_\perp \times_f M_\theta$ *be a proper warped product pointwise hemi-slant submanifold of a Kaehler manifold* $\widetilde{M}$. *Then*

$$g(A_{FPX}Y - A_{FY}PX, Z) = 2\cos^2\theta\, Z(\ln f)\,g(X,Y)$$

for any $Z \in \Gamma(TM_\perp)$ *and* $X,Y \in \Gamma(TM_\theta)$.

Proof. Interchanging X by Y in Lemma 4.3 (ii), we have

$$\begin{aligned} g(h(X,Y),JZ) &= Z(\ln f)\,g(Y,PX) + g(h(Y,Z),FX) \\ &= -Z(\ln f)\,g(X,PY) + g(h(Y,Z),FX) \end{aligned} \tag{20}$$

Subtracting (20) from Lemma 4.3 (ii), thus we derive

$$g(h(Y,Z),FX) - g(h(X,Z),FY) = 2Z(\ln f)\,g(X,PY). \tag{21}$$

Interchange X by PX in (21) and using (10), we obtain

$$2\cos^2\theta\, Z(\ln f)\,g(X,Y) = g(h(Y,Z),FPX) - g(h(PX,Z),FY). \tag{22}$$

Hence, the result follows from (22) by using (6). □

A warped product manifold $M = M_1 \times_f M_2$ is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$.

The following corollary is an immediate consequence of the above lemma.

Corollary 4.5. *There does not exist any proper warped product mixed totally geodesic submanifold of the form $M = M_\perp \times_f M_\theta$ of a Kaehler manifold $\widetilde{M}$ such that $M_\perp$ is a totally real submanifold and M_θ is a proper pointwise slant submanifold of $\widetilde{M}$.*

Proof. The proof of the corollary follows from (22) by using the mixed totally geodesic condition. □

We note that the above corollary is also given in [18] as a remark.

Lemma 4.6. *Let $M = M_\perp \times_f M_\theta$ be a proper warped product pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

$$g(h(X,Z), FY) - g(h(Y,Z), FX) = 2\tan\theta\, Z(\theta)\, g(PX, Y) \tag{23}$$

for any $Z \in \Gamma(TM_\perp)$ and $X, Y \in \Gamma(TM_\theta)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(\widetilde{\nabla}_Z X, Y) = g\nabla_Z X, Y) = Z(\ln f)\, g(X, Y). \tag{24}$$

On the other hand, for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we also have

$$g(\widetilde{\nabla}_Z X, Y) = g(J\widetilde{\nabla}_Z X, JY) = g(\widetilde{\nabla}_Z JX, JY).$$

Using (2),(4), (7) and (15), we get

$$\begin{aligned} g(\widetilde{\nabla}_Z X, Y) &= g(\widetilde{\nabla}_Z PX, PY) + g(\widetilde{\nabla}_Z PX, FY) + g(\widetilde{\nabla}_Z FX, JY) \\ &= \cos^2\theta\, Z(\ln f)\, g(X, Y) + g(h(Z, PX), FY) - g(\widetilde{\nabla}_Z JFX, Y). \end{aligned}$$

Then from (8), we derive

$$\begin{aligned} g(\widetilde{\nabla}_Z X, Y) &= \cos^2\theta\, Z(\ln f)\, g(X, Y) + g(h(Z, PX), FY) - g(\widetilde{\nabla}_Z tFX, Y) \\ &\quad - g(\widetilde{\nabla}_Z fFX, Y). \end{aligned}$$

Using (12), we obtain

$$\begin{aligned} g(\widetilde{\nabla}_Z X, Y) &= \cos^2\theta\, Z(\ln f)\, g(X, Y) + g(h(Z, PX), FY) - g(\widetilde{\nabla}_Z \sin^2\theta X, Y) \\ &\quad + g(\widetilde{\nabla}_Z FPX, Y) \\ &= \cos^2\theta\, Z(\ln f)\, g(X, Y) + g(h(Z, PX), FY) + \sin^2\theta\, g(\widetilde{\nabla}_Z X, Y) \\ &\quad + \sin 2\theta\, Z(\theta)\, g(X, Y) - g(A_{FPX} Z, Y), \end{aligned}$$

which on using (6), the above equation takes the form

$$\begin{aligned} \cos^2\theta\, g(\widetilde{\nabla}_Z X, Y) &= \cos^2\theta\, Z(\ln f)\, g(X, Y) + g(h(Z, PX), FY) \\ &\quad + \sin 2\theta\, Z(\theta)\, g(X, Y) - g(h(Y, Z), FPX). \end{aligned} \tag{25}$$

From (24) and (25), we derive

$$g(h(Y, Z), FPX) - g(h(Z, PX), FY) = \sin 2\theta\, Z(\theta)\, g(X, Y). \tag{26}$$

Interchanging X by PX in (26) and then using (9), we get the desired result. Hence, the proof is complete. □

Now, we have the following useful theorem.

Theorem 4.7. *Let $M = M_{\perp} \times {}_f M_{\theta}$ be a warped product pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$ such that $M_{\perp}$ is a totally real submanifold and M_{θ} is a pointwise slant submanifold with slant function θ of $\widetilde{M}$. Then*

$$Z(\ln f) = \tan\theta\, Z(\theta)$$

for any $Z \in \Gamma(TM_{\perp})$.

Proof. From (21) and (23), we have

$$(\tan\theta\, Z(\theta) - Z(\ln f))\; g(PX, Y) = 0. \tag{27}$$

Interchanging Y by PY in (27) and using (10), we obtain

$$\cos^2\theta\, (\tan\theta\, Z(\theta) - Z(\ln f))\; g(X, Y) = 0. \tag{28}$$

Since M is proper, therefore $\cos^2\theta \neq 0$, thus the proof follows from (28) . □

As an application, we have the following consequences of the above theorem.

1. If we assume $\theta = 0$ in Theorem 4.7, then the warped product is of the form $M = M_{\perp} \times_f M_T$, where M_T and $M_{\perp}$ are holomorphic and totally real submanifolds of a Kaehler manifold $\widetilde{M}$, respectively. Thus, the Theorem 3.1 of [4] is a special case of Theorem 4.7 as follows.

Corollary 4.8. *(Theorem 3.1 [4]). If $M = M_{\perp} \times_f M_T$ be a warped product CR-submanifold of a Kaehler manifold $\widetilde{M}$ such that $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\widetilde{M}$, then M is a CR-product.*

2. Also, if we assume that the slant function θ is a constant, i.e., M_{θ} is a proper slant submanifold, then the warped product $M = M_{\perp} \times_f M_{\theta}$ is a hemi-slant warped product submanifold of a Kaehler manifold $\widetilde{M}$, where $M_{\perp}$ and M_{θ} are totally real and proper slant submanifolds of $\widetilde{M}$, respectively. Then, Theorem 4.2 of [16] is a special case of Theorem 4.7 as follows.

Corollary 4.9. *(Theorem 4.2 [16]). Let $\widetilde{M}$ be a Kaehler manifold. Then there exist no warped product submanifolds $M = M_{\perp} \times_f M_{\theta}$ of $\widetilde{M}$ such that $M_{\perp}$ is a totally real submanifold and M_{θ} is a proper slant submanifold of $\widetilde{M}$.*

In order to give another characterization we need the following well known result of S. Hiepko [12].

Hiepko's Theorem. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ both are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

The following result gives a characterization of warped product pointwise hemi-slant submanifolds.

Theorem 4.10. *[18] Let M be a pointwise hemi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then M is locally a warped product submanifold of the form $M_{\perp} \times_f M_{\theta}$ if and only if*

$$A_{FPX}V - A_{JV}PX = V(\mu)\left(\cos^2\theta\right)X,\ \ \forall\, X \in \Gamma(\mathcal{D}^{\theta}),\ V \in \Gamma(\mathcal{D}^{\perp}) \tag{29}$$

for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in \Gamma(\mathcal{D}^{\theta})$.

Remark 4.11. *The inequality for second fundamental form of these kind of warped products may not be evaluated. The reason is that: To evaluate the squared norm of the second fundamental form $\|h\|^2$ from Lemma 4.3 and the relations (17)-(19), we have to assume that either M is mixed totally geodesic or to discuss the equality case M must be a mixed totally geodesic warped product and in both cases M is mixed totally geodesic, but in case of mixed totally geodesic, these warped products do not exist (Corollary 4.5).*

Remark 4.12. *Theorem 4.10 is valid only for the pointwise slant fiber. For example, if θ is constant i.e., M_θ is proper slant, then this is the case of non-existence of warped products (see Theorem 4.2 of [16]) and if $\theta = 0$, i.e., the fiber is a holomorphic submanifold, then again from Theorem 3.1 of [4], this is a case of non-existence of warped products.*

References

[1] R.L. Bishop and B. O'Neill, *Manifolds of Negative curvature,* Trans. Amer. Math. Soc. **145** (1969), 1-49.
[2] B.-Y. Chen, *Slant immersions,* Bull. Austral. Math. Soc. **41** (1990), 135-147.
[3] B.-Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, 1990.
[4] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133** (2001), 177–195.
[5] B.-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. **134** (2001), 103–119.
[6] B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, Hackensack, NJ, 2011.
[7] B.-Y. Chen, *Geometry of warped product submanifolds: a survey*. J. Adv. Math. Stud. **6** (2013), no. 2, 1–43.
[8] B.-Y. Chen and O. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds,* Turk. J. Math. **36** (2012), 630–640.
[9] B.-Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, Hackensack, NJ, 2017.
[10] B.-Y. Chen and S. Uddin *Warped Product Pointwise Bi-slant Submanifolds of Kaehler Manifolds*, Publ. Math. Debrecen 92 (1) (2018), 1-16.
[11] F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold,* Publ. Math. Debrecen **53** (1998), 217–223.
[12] S. Hiepko, *Eine inner kennzeichungder verzerrten produkte*, Math. Ann. 241 (1979), 209-215.
[13] N. Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, Ann. St. Univ. Iasi **9** (1994), 55-61.
[14] K.S. Park, *Pointwise almost h-semi-slant submanifolds*, Int. J. Math., **26**, 1550099 (2015), 26 pages.
[15] B. Sahin, *Nonexistence of warped product semi-slant submanifolds of Kaehler manifolds,* Geom. Dedicata **117** (2006), 195–202.
[16] B. Sahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Pol. Math. **95** (2009), 207-226.
[17] B. Sahin, *Warped product pointwise semi-slant submanifolds of Kehler manifolds*, Port. Math. **70** (2013), 252-268.
[18] S. K. Srivastava and A. Sharma, *Geometry of pointwise pseudo-slant warped product submanifolds in a Kaehler manifold*, arXiv:1601.01714v2 [math.DG].
[19] S. Uddin and A.Y.M. Chi, *Warped product hemi-slant submanifolds of nearly Kaehler manifolds,* An. St. Univ. Ovidius Constanta. 19 (3) (2011), 195–204.
[20] S. Uddin, F.R. Al-Solamy and K.A. Khan, *Geometry of warped product pseudo-slant submanifolds in nearly Kaehler manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat (N.S.) Tome LXIII (2016), f_2 vol. 3, 927–938.
[21] S. Uddin, B.-Y. Chen and F.R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math., 14: 95. https://doi.org/10.1007/s00009-017-0896-8.
[22] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore, 1984.