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The Expression for the Group Inverse of the Anti–triangular Block Matrix

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Abstract. In this paper, we present the explicit expression for the group inverse of the sum of two matrices. As an application, the explicit expression for the group inverse of the anti-triangular block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ are obtained without any conditions on sub-blocks.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrix on complex field $\mathbb{C}$ and let I denote the unit of $M_n(\mathbb{C})$. For an element $A \in M_n(\mathbb{C})$, if there is an element $B \in M_n(\mathbb{C})$ which satisfies $ABA = A$, then B is called a $\{1\}$– inverse of A. If $ABA = A$ and $BAB = B$ hold , then B is called a $\{2\}$–inverse of A, denoted by A^+. An element B is called the Drazin inverse of A, if B satisfies

$$A^kBA = A^k; \quad BAB = B; \quad AB = BA \quad \text{for some integer } k.$$

B is denoted by A^D. The least such integer k is called the index of A, denoted by $ind(A)$. We denote by $A^\pi = I - AA^D$ the spectral idempotent of A. In the case $ind(A) = 1$, A^D reduces to the group inverse of A, denoted by $A^\#$.

The Drazin inverse has various applications in singular differential equations and singular difference equations, Markov chains, and iterative methods (see[4–7, 9–11, 15, 16, 20]). In 1979, S. Campbell and C. Meyer proposed an open problem to find an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ in terms of its sub-blocks, where A and D are supposed to be square matrices(see [4]). A simplified problem to find an explicit representation for the Drazin inverse of $\begin{pmatrix} A & C \\ B & 0 \end{pmatrix}$ was proposed by S. Campbell in 1983(see [7]). Until now, both problems have not been solved. However, many authors have considered the two problems under certain conditions on the sub-blocks(see [3, 9, 12, 13, 15, 17, 18, 25]). As

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a special case, the expression for the group inverse of 2×2 block matrix also has been studied under some conditions(see [1, 2, 8, 14, 18, 19, 23]).

In this paper, we give the explicit expression for the group inverse of the sum of two matrices. As an application, the expression for the group inverse of the anti-triangular matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ are presented without any conditions on sub-blocks.

2. Preliminaries

In this section, we present some important lemmas and investigate the expression of the group inverse in term of its generalized inverse. Let us begin with a familiar lemma.

Lemma 2.1. [19] *Let $A, B \in M_n(\mathbb{C})$, then $I+AB$ is invertible iff $I+BA$ is invertible and $(I+AB)^{-1} = I-A(I+BA)^{-1}B$.*

Lemma 2.2. *Let P be an idempotent matrix in $M_n(\mathbb{C})$, then $I - P$ is invertible iff $P = 0$.*

Proof. Since $I - P$ is invertible, there is an $X \in M_n(\mathbb{C})$ such that $X(I - P) = I$, that is, $X - XP = I$. So, $0 = XP - XP = P$. □

Lemma 2.3. [24, Theorem 4.5.9] *Let $A \in M_n(\mathbb{C})\backslash\{0\}$. Then the following conditions are equivalent:*

(1) $A^\#$ exists.

(2) $AA^+ + A^+A - I$ is invertible for some A^+.

(3) $A^2A^+ + I - AA^+$ is invertible for some A^+.

(4) $A^2A^+ + I - AA^+$ is invertible for any A^+.

(5) $A + I - AA^+$ is invertible for some A^+.

(6) $A + I - AA^+$ is invertible for any A^+.

(7) $A^+A^2 + I - A^+A$ is invertible for some A^+.

(8) $A^+A^2 + I - A^+A$ is invertible for any A^+.

(9) $A + I - A^+A$is invertible for some A^+.

(10) $A + I - A^+A$is invertible for any A^+.

Proof. The equivalence of (1) to (4) are presented in [24, Theorem 4.5.9]. Noting that $A^2A^+ + I - AA^+ = I + (A - I)AA^+$ and $A + I - AA^+ = I + A(I - A^+)$, by Lemma 2.1, we have $(3) \Leftrightarrow (5), (4) \Leftrightarrow (6), (7) \Leftrightarrow (9), (8) \Leftrightarrow (10), (5) \Leftrightarrow (9), (6) \Leftrightarrow (10)$. □

Lemma 2.4. *Let $A \in M_n(\mathbb{C})\backslash\{0\}$. If $A^\#$ exists then*

$$\begin{aligned} A^\# &= A(I + A - A^+A)^{-2} \\ &= (I + A - AA^+)^{-2}A \\ &= (I + A - AA^+)^{-1}A(I + A - A^+A)^{-1}, \end{aligned}$$

independent of the choice of A^+.

Proof. Put $W = A^2A^+ + I - AA^+$. By Lemma 2.3, W is invertible. Set $B = W^{-2}A$. Noting that $AA^+W = WAA^+ = A^2A^+, WA = A^2$, we have

$$\begin{aligned} AB &= AW^{-2}A = AW^{-2}AA^+A = A^2A^+W^{-2}A = AA^+WW^{-2}A = W^{-1}A, \\ BA &= W^{-2}A^2 = W^{-2}WA = W^{-1}A, \\ ABA &= W^{-1}A^2 = W^{-1}WA = A, \\ BAB &= W^{-1}AW^{-2}A = W^{-1}AAA^+W^{-2}A = W^{-1}AA^+W^{-1}A = W^{-2}A. \end{aligned}$$

The above indicate that $B = A^\#$ and independent of the choice of A^+.

Noting that

$$(I + A - AA^+)^{-1}A = A(I + A - A^+A)^{-1},$$

by Lemma 2.1, we have

$$\begin{aligned} A^\# &= (A^2A^+ + I - AA^+)^{-2}A \\ &= [I + (A - I)AA^+]^{-2}A \\ &= [I - (A - I)(I + A - AA^+)^{-1}AA^+]^2A \\ &= [I - (A - I)(I + A - AA^+)^{-1}AA^+](2I - AA^+)(I + A - AA^+)^{-1}A \\ &= [I - (A - I)(I + A - AA^+)^{-1}AA^+]A(I + A - A^+A)^{-1} \\ &= A(I + A - A^+A)^{-2} \\ &= (I + A - AA^+)^{-2}A \\ &= (I + A - AA^+)^{-1}A(I + A - A^+A)^{-1}. \end{aligned}$$

□

Proposition 2.5. *Let $A, B \in M_n(\mathbb{C})$. Then $(AB)^\#$ exists iff $I + AB - ABB^+(A^+ABB^+)^+A^+$ is invertible iff $I + AB - B^+(A^+ABB^+)^+A^+AB$ is invertible. In this case,*

$$\begin{aligned} (AB)^\# &= \{I + AB - ABB^+(A^+ABB^+)^+A^+\}^{-2}AB \\ &= AB\{I + AB - B^+(A^+ABB^+)^+A^+AB\}^{-2}, \end{aligned}$$

Proof. Noting that $B^+(A^+ABB^+)^+A^+$ is a {1,2}–inverse of AB. By Lemma 2.4, we get the results. □

Corollary 2.6. *Let $A, B \in M_n(\mathbb{C})$ with $(AB)^\#$ exists.*

(1) If B is invertible, then $(AB)^\#$ exists iff $I + AB - AA^+$ is invertible. In this case,

$$(AB)^\# = (I + AB - AA^+)^{-2}AB.$$

(2) If A is invertible, then $(AB)^\#$ exists iff $I + AB - B^+B$is invertible. In this case,

$$(AB)^\# = AB(I + AB - B^+B)^{-2}.$$

3. Main Results

Let $A, B, C, D \in M_n(\mathbb{C})$. Throughout of this paper, we denote $E_A = I - AA^+, F_A = I - A^+A$.

Lemma 3.1. *Let $A, X, Y \in M_n(\mathbb{C})$ and $Z = I - YA^+X, U = E_AX, V = YF_A, S = E_VZF_U$. Let*

$$G = A^+ - F_AV^+YA^+ + (F_AV^+Z + A^+X)[F_US^+E_VYA^+ - (I - F_US^+E_VZ)U^+E_A].$$

Then G is a {1}–inverse of $A - XY$.

Proof.

$$
\begin{aligned}
(A-XY)G &= AA^+ + AA^+X[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&- XE_VYA^+ - (XVV^+Z + XYA^+X)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&= AA^+ + (X-U)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&- XE_VYA^+ - (XVV^+Z + XYA^+X)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&= AA^+ + X[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A] + UU^+E_A\\
&- XE_VYA^+ - (XVV^+Z + X(I-Z))[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&= AA^+ + X[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A] + UU^+E_A\\
&- XE_VYA^+ - (X - XE_VZ)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]\\
&= AA^+ + UU^+E_A - XE_SE_VYA^+ - XE_VZU^+E_A + XSS^+E_VZU^+E_A\\
&= AA^+ + UU^+E_A - XE_SE_VYA^+ - XE_SE_VZU^+E_A\\
&= AA^+ + UU^+E_A - XE_SE_V(YA^+ + ZU^+E_A)\\
&= I - E_UE_A - XE_SE_V(YA^+ + ZU^+E_A).
\end{aligned}
$$

$$
\begin{aligned}
G(A-XY) &= A^+A - F_AV^+YA^+A + (F_AV^+Z + A^+X)F_US^+E_VYA^+A - A^+XY\\
&+ F_AV^+YA^+XY - (F_AV^+Z + A^+X)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]XY\\
&= A^+A - F_AV^+Y + F_AV^+V + (F_AV^+Z + A^+X)F_US^+E_VY - A^+XY\\
&+ F_AV^+YA^+XY - (F_AV^+Z + A^+X)[F_US^+E_VYA^+ - (I-F_US^+E_VZ)U^+E_A]XY\\
&= A^+A - F_AV^+Y + F_AV^+V + (F_AV^+Z + A^+X)F_US^+E_VY - A^+XY\\
&+ F_AV^+(I-Z)Y - (F_AV^+Z + A^+X)[F_US^+E_V(I-Z)Y - (I-F_US^+E_VZ)U^+UY]\\
&= A^+A + F_AV^+V - (A^+X + F_AV^+Z)Y\\
&+ (F_AV^+Z + A^+X)[F_US^+E_VZY + (I-F_US^+E_VZ)U^+UY]\\
&= A^+A + F_AV^+V + (F_AV^+Z + A^+X)[F_US^+E_VZY + (I-F_US^+E_VZ)U^+UY - Y]\\
&= A^+A + F_AV^+V + (F_AV^+Z + A^+X)[F_US^+E_VZ - F_US^+E_VZU^+U - F_U]Y\\
&= A^+A + F_AV^+V - (F_AV^+Z + A^+X)F_UF_SY\\
&= I - F_AF_V - (F_AV^+Z + A^+X)F_UF_SY.
\end{aligned}
$$

It is easy to verify $(A-XY)G(A-XY) = A-XY$. This shows G is a $\{1\}$–inverse of $A-XY$. □

Corollary 3.2. *Let $A, X \in M_n(\mathbb{C})$ and $Z = I + A^+X, U = E_AX, S = A^+AZF_U$. Let*

$$G = A^+ + (F_A - A^+X)[F_US^+A^+ + (I - F_US^+A^+AZ)U^+E_A],$$

and

$$
\begin{aligned}
(A+X)G &= I - E_UE_A + XE_S(A^+ - A^+AZU^+E_A),\\
G(A+X) &= I - F_UF_S.
\end{aligned}
$$

Proof. Replacing Y by $-I$ in Lemma 3.1, we get easily the first part of the results. Noting that $Z = I + A^+X$ and $S = A^+AZF_U$, we have

$$
\begin{aligned}
G(A+X) &= I - (F_A - A^+X)F_UF_S\\
&= I - (I - A^+A - A^+X)F_UF_S\\
&= I - [I - A^+A(I + A^+X)]F_UF_S\\
&= I - (I - A^+AZ)F_UF_S\\
&= I - F_UF_S.
\end{aligned}
$$

□

Similarly, we have

Corollary 3.3. *Let* $A, X \in M_n(\mathbb{C})$ *and* $Z = I + XA^+, V = XF_A, S = E_V ZAA^+$. *Let*

$$G = A^+ - F_A V^+ XA^+ + (F_A V^+ Z + A^+ X)[AA^+ S^+ E_V XA^+ + (I - AA^+ S^+ E_V Z)E_A].$$

Then G *is a* $\{1\}$*–inverse of* $A + X$ *and*

$$(A + X)G = I - E_S E_V,$$
$$G(A + X) = I - F_A F_V - (F_A V^+ ZAA^+ - A^+)F_S X.$$

Theorem 3.4. *Let* $A, X, Y \in M_n(\mathbb{C})$ *and* $Z = I - YA^+X, U = E_A X, V = YF_A, S = E_V ZF_U$. *Then* $(A - XY)^{\#}$ *exists iff*

$$A - XY + E_U E_A + XE_S E_V (YA^+ + ZU^+ E_A)$$

is invertible iff

$$A - XY + F_A F_V + (F_A V^+ Z + A^+ X)FuF_S Y$$

is invertible and

$$\begin{aligned}(A - XY)^{\#} &= \{A - XY + E_U E_A + XE_S E_V (YA^+ + ZU^+ E_A)\}^{-2}(A - XY)\\ &= (A - XY)\{A - XY + F_A F_V + (F_A V^+ Z + A^+ X)FuF_S Y\}^{-2}\\ &= \{A - XY + E_U E_A + XE_S E_V (YA^+ + ZU^+ E_A)\}^{-1}(A - XY)\\ &\times \{A - XY + F_A F_V + (F_A V^+ Z + A^+ X)F_U F_S Y\}^{-1}.\end{aligned}$$

Proof. By Lemma 3.1, G is a $\{1\}$–inverse of $A - XY$. Thus, $(A - XY)^+ = G(A - XY)G$ is a generalized inverse of $A - XY$. Hence,

$$(A - XY)(A - XY)^+ = (A - XY)G = I - E_U E_A - XE_S E_V (YA^+ + ZU^+ E_A,$$
$$(A - XY)^+(A - XY) = G(A - XY) = I - F_A F_V - (F_A V^+ Z + A^+ X)F_U F_S Y.$$

So, the results follow by Lemma 2.3 and Lemma 2.4. □

Using Lemma 2.3, Lemma 2.4 and Corollary 3.2, Corollary 3.3, we have the following corollaries:

Corollary 3.5. *Let* $A, X \in M_n(\mathbb{C})$ *and* $Z = I + A^+X, U = E_A X, S = A^+AZF_U$. *Then* $(A + X)^{\#}$ *exists iff*

$$A + X + E_U E_A - XE_S (A^+ - A^+ AZU^+ E_A)$$

is invertible iff $A + X + F_U F_S$ *is invertible and*

$$\begin{aligned}(A + X)^{\#} &= (A + X)(A + X + F_U F_S)^{-2}\\ &= \{A + X + E_U E_A - XE_S (A^+ - A^+ AZU^+ E_A)\}^{-2}(A + X)\\ &= \{A + X + E_U E_A - XE_S (A^+ - A^+ AZU^+ E_A)\}^{-1}(A + X)(A + X + F_U F_S)^{-1}.\end{aligned}$$

Corollary 3.6. *Let* $A, X \in M_n(\mathbb{C})$ *and* $Z = I + XA^+, V = XF_A, S = E_V ZAA^+$. *Then* $(A + X)^{\#}$ *exists iff*

$$A + X + F_A F_V + (F_A V^+ ZAA^+ - A^+)F_S X$$

is invertible iff $A + X + E_S E_V$ *is invertible and*

$$\begin{aligned}(A + X)^{\#} &= (A + X + E_S E_V)^{-2}(A + X)\\ &= (A + X)\{A + X + F_A F_V + (F_A V^+ ZAA^+ - A^+)F_S X\}^{-2}\\ &= (A + X + E_S E_V)^{-1}(A + X)\{A + X + F_A F_V + (F_A V^+ ZAA^+ - A^+)FsX\}^{-1}.\end{aligned}$$

In the following, we investigate the expression for the group inverse of anti-triangular matrix. First, we cite a lemma which comes from [21].

Lemma 3.7. [21] *Suppose $M, X \in M_n(\mathbb{C})$. Then $N = M - MXM$ has a {1}–inverse Y iff M has a {1}–inverse $X + (I - XM)Y(I - MX)$.*

Lemma 3.8. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Z = D - CA^+B, P = E_AB, Q = CF_A, R = ZF_P, W = E_RQQ^+$. Then there exists a {1}–inverse G of M such that*

$$I - MG = \begin{pmatrix} E_PE_A & 0 \\ -E_WE_R(ZP^+E_A + CA^+) & E_WE_R \end{pmatrix}.$$

Proof. Taking $X = \begin{pmatrix} A^+ & 0 \\ 0 & 0 \end{pmatrix}$ and $N = M - MXM$. Let Y be a {1}–inverse of N. Then by Lemma 3.7, $G = X + (I - XM)Y(I - MX)$ be a {1}–inverse of M and

$$I - MG = I - MX - M(I - XM)Y(I - MX) = (I - NY)(I - MX). \tag{1}$$

Let $N_1 = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$. Then $N = N_1 + N_2 = \begin{pmatrix} 0 & E_AB \\ CF_A & D - CA^+B \end{pmatrix}$.

Note that

$$N_1^+ = \begin{pmatrix} 0 & Q^+ \\ P^+ & 0 \end{pmatrix}, \qquad T = I + N_2N_1^+ = \begin{pmatrix} I & 0 \\ ZP^+ & I \end{pmatrix},$$

$$N_1N_1^+ = \begin{pmatrix} PP^+ & 0 \\ 0 & QQ^+ \end{pmatrix}, \qquad N_1^+N_1 = \begin{pmatrix} Q^+Q & 0 \\ 0 & P^+P \end{pmatrix},$$

$$V = N_2F_{N_1} = \begin{pmatrix} 0 & 0 \\ 0 & ZF_P \end{pmatrix}, \qquad S = E_VTN_1N_1^+ = \begin{pmatrix} PP^+ & 0 \\ E_RZP^+ & E_RQQ^+ \end{pmatrix}$$

and $S^+ = \begin{pmatrix} PP^+ & 0 \\ -W^+E_RZP^+ & W^+ \end{pmatrix}$. Thus, by Corollary 3.3, we have

$$I - NY = E_SE_V = \begin{pmatrix} E_P & 0 \\ -E_WE_RZP^+ & E_WE_R \end{pmatrix}.$$

Hence, by Eq.(1), we have

$$\begin{aligned} I - MG &= (I - NY)(I - MX) \\ &= \begin{pmatrix} E_P & 0 \\ -E_WE_RZP^+ & E_WE_R \end{pmatrix} \begin{pmatrix} E_A & 0 \\ -CA^+ & I \end{pmatrix} \\ &= \begin{pmatrix} E_PE_A & 0 \\ -E_WE_R(ZP^+E_A + CA^+) & E_WE_R \end{pmatrix}. \end{aligned}$$

□

Theorem 3.9. *Let $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ with $A^\#, D^\#$ exist. Then $M^\#$ exists iff $D^\pi CA^\pi = 0$. In this case,*

$$M^\# = \begin{pmatrix} A^\# & 0 \\ (D^\#)^2CA^\pi + D^\pi C(A^\#)^2 - D^\#CA^\# & D^\# \end{pmatrix}.$$

Proof. Take $A^+ = A^\#, D^+ = D^\#$ and $B = 0$ in Lemma 3.8, we have $Z = D, P = 0, Q = CA^\pi, R = Z = D, W = D^\pi QQ^+$ and

$$M + I - MG = \begin{pmatrix} A + A^\pi & 0 \\ C - E_W D^\pi CA^+ & D + E_W D^\pi \end{pmatrix}.$$

Since $A + A^\pi$ and $D + D^\pi$ are invertible, $M + I - MG$ is invertible iff $I - WW^+D^\pi$ is invertible. Thus, by Lemma 2.2, $I - WW^+D^\pi$ is invertible iff $WW^+D^\pi = 0$ iff $W = 0$ iff $D^\pi Q = D^\pi CA^\pi = 0$. Hence, if $M^\#$ exists, then

$$(M + I - MG)^{-1} = \begin{pmatrix} A^\# + A^\pi & 0 \\ D^\pi CA^+ - C & D^\# + D^\pi \end{pmatrix}.$$

By simple calculation, we get

$$M^\# = \begin{pmatrix} A^\# & 0 \\ (D^\#)^2 CA^\pi + D^\pi C(A^\#)^2 - D^\# CA^\# & D^\# \end{pmatrix}.$$

□

Theorem 3.10. *Let* $M = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$ *and* $R = DF_B, W = E_R CC^+$*. Then* $M^\#$ *exists iff*

$$DF_B - CB + E_W E_R (F_B + DB^+B)$$

is invertible. In this case,

$$M^\# = \begin{pmatrix} I + B\xi\eta & -B\xi \\ B^+ - F_B\xi\eta & F_B\xi \end{pmatrix}^2 \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

Here,

$$\xi = \{DF_B - CB + E_W E_R (F_B + DB^+B)\}^{-1},$$
$$\eta = C + DB^+ + E_W E_R (I - D)B^+.$$

Proof. Take $A = 0$ in Lemma 3.8, we have $Z = D, P = B, Q = C, R = DF_B, W = E_R CC^+$ and

$$\begin{aligned} M + I - MG &= \begin{pmatrix} E_B & B \\ C - E_W E_R DB^+ & D + E_W E_R \end{pmatrix} \\ &= \begin{pmatrix} I & B \\ C + DB^+ + E_W E_R (I - D)B^+ & D + E_W E_R \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^+ & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ C + DB^+ + E_W E_R (I - D)B^+ & DF_B - CB + E_W E_R (F_B + DB^+B) \end{pmatrix} \begin{pmatrix} E_B & B \\ -B^+ & I \end{pmatrix}. \end{aligned}$$

By Lemma 2.3, we have $M^\#$ exists iff $DF_B - CB + E_W E_R (F_B + DB^+B))$ is invertible.
Put

$$\xi = \{DF_B - CB + E_W E_R (F_B + DB^+B)\}^{-1},$$
$$\eta = C + DB^+ + E_W E_R (I - D)B^+.$$

Then

$$(M + I - MG)^{-1} = \begin{pmatrix} I + B\xi\eta & -B\xi \\ B^+ - F_B\xi\eta & F_B\xi \end{pmatrix}.$$

Thus, by Lemma 3.1, we have

$$M^\# = \begin{pmatrix} I + B\xi\eta & -B\xi \\ B^+ - F_B\xi\eta & F_B\xi \end{pmatrix}^2 \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

□

We find it is very difficulty to calculate $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{\#}$ if take $D = 0$ in Lemma 3.8. So, we present another method as following:

Theorem 3.11. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ *and* $P = AF_C, W = E_P BB^+$. *Then* $M^{\#}$ *exists iff*

$$AF_C - BC + E_W E_P (F_C + AC^+C)$$

is invertible. In this case,

$$M^{\#} = \begin{pmatrix} F_C\xi & C^+ - F_C\xi\eta \\ -C\xi & I + C\xi\eta \end{pmatrix}^2 \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Here,

$$\xi = \{AF_C - BC + E_W E_P (F_C + AC^+C)\}^{-1},$$
$$\eta = B + AC^+ + E_W E_P (I - A)C^+.$$

Proof. Let $M_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, M_2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $M = M_1 + M_2$. Let I_2 be the identity of $M_2(\mathbb{C})$ and $P = AF_C, W = E_P BB^+$. Noting that

$$M_1^+ = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad V = M_2 F_{M_1} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

$$E_V = I_2 - VV^+ = \begin{pmatrix} E_P & 0 \\ 0 & I \end{pmatrix}, \qquad Z = I_2 + M_2 M_1^+ = \begin{pmatrix} I & AC^+ \\ 0 & I \end{pmatrix},$$

$$S = E_V Z M_1 M_1^+ = \begin{pmatrix} W & E_P AC^+ \\ 0 & CC^+ \end{pmatrix}, \qquad S^+ = \begin{pmatrix} W^+ & -W^+ E_P AC^+ \\ 0 & CC^+ \end{pmatrix},$$

$$E_S = \begin{pmatrix} E_W & -E_W E_P AC^+ \\ 0 & E_C \end{pmatrix}, \qquad M + E_S E_V = \begin{pmatrix} A + E_W E_P & B - E_W E_P AC^+ \\ C & E_C \end{pmatrix}$$

and

$$\begin{aligned} M + E_S E_V &= \begin{pmatrix} A + E_W E_P & B - E_W E_P AC^+ \\ C & E_C \end{pmatrix} \\ &= \begin{pmatrix} A + E_W E_P & B - E_W E_P AC^+ + AC^+ + E_W E_P C^+ \\ C & I \end{pmatrix} \begin{pmatrix} I & -C^+ \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} AF_C - BC + E_W E_P (F_C + AC^+C) & B + AC^+ + E_W E_P (I - A)C^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & -C^+ \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} AF_C - BC + E_W E_P (F_C + AC^+C) & B + AC^+ + E_W E_P (I - A)C^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -C^+ \\ C & E_C \end{pmatrix}. \end{aligned}$$

Thus, by Corollary 3.6, we have $M^{\#}$ exists iff $AF_C - BC + E_W E_P (F_C + AC^+C)$ is invertible.

Put

$$\xi = \{AF_C - BC + E_W E_P (F_C + AC^+C)\}^{-1},$$
$$\eta = B + AC^+ + E_W E_P (I - A)C^+.$$

Then, by Simple calculation, we have

$$(M + E_S E_V)^{-1} = \begin{pmatrix} F_C & C^+ \\ -C & I \end{pmatrix} \begin{pmatrix} \xi & -\xi\eta \\ 0 & I \end{pmatrix} = \begin{pmatrix} F_C\xi & C^+ - F_C\xi\eta \\ -C\xi & I + C\xi\eta \end{pmatrix}.$$

Thus, by Corollary 3.6,

$$M^{\#} = (M + E_S E_V)^{-2} M = \begin{pmatrix} F_C\xi & C^+ - F_C\xi\eta \\ -C\xi & I + C\xi\eta \end{pmatrix}^2 \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

□

Corollary 3.12. *Let* $M = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. *Then* $M^{\#}$ *exists iff* $E_B F_C - BC$ *is invertible. Put* $\xi = (E_B F_C - BC)^{-1}$ *, then*

$$M^{\#} = \begin{pmatrix} 0 & -(I - F_C\xi)C^+ C\xi B \\ -C\xi BB^+(I - F_C\xi)C^+ C & 0 \end{pmatrix}.$$

Proof. Let $\xi = (E_B F_C - BC)^{-1}$. Then

$$E_B \xi^{-1} = E_B F_C = \xi^{-1} F_C, \xi BCC^+ = -C^+, B^+ BC\xi = -B^+.$$

Thus,

$$\xi E_B = F_C \xi, C\xi E_B = 0, F_C \xi B = 0, BC\xi = -BB^+$$

and

$$\begin{aligned} (F_C\xi - I)C^+ C &= (F_C\xi - I)(I - F_C) \\ &= (F_C\xi - I) - (F_C\xi F_C - F_C) \\ &= (F_C\xi - I) - (\xi E_B F_C - F_C) \\ &= (F_C\xi - I). \end{aligned}$$

Associate with Theorem 3.11, we get the results. □

Corollary 3.13. [9] *Let* $M = \begin{pmatrix} I & I \\ C & 0 \end{pmatrix}$ *and* $C^{\#}$ *exists. Then*

$$\begin{pmatrix} I & I \\ C & 0 \end{pmatrix}^{\#} = \begin{pmatrix} C^\pi & C^\pi + C^{\#} \\ CC^{\#} & -C^{\#} \end{pmatrix}.$$

Now, we consider the group inverse of $M = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}$. From the proof of Theorem 3.11, we have

$$I_2 - \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ = E_S E_V = \begin{pmatrix} E_W E_P & -E_W E_P A C^+ \\ 0 & E_C \end{pmatrix}.$$

Here, $P = AF_C, W = E_P BB^+$.

Since

$$\begin{pmatrix} B & A \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

by Corollary 2.6, we have $\begin{pmatrix} B & A \\ 0 & C \end{pmatrix}^{\#}$ exists iff

$$\begin{pmatrix} B & A \\ 0 & C \end{pmatrix} + I_2 - \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^+ = \begin{pmatrix} B + E_W E_P & A - E_W E_P A C^+ \\ 0 & C + E_C \end{pmatrix}$$

is invertible.

Assume that $B^\#, C^\#$ exist. Then $P = AC^\pi, W = E_P BB^\#$ and $\begin{pmatrix} B + E_W E_P & A - E_W E_P AC^+ \\ 0 & C + E_C \end{pmatrix}$ is invertible iff $B + E_W E_P$ is invertible. Noting that $E_W E_P B^\pi = E_W E_P$ and $B^\# + B^\pi$ is invertible, we have $B + E_W E_P$ is invertible iff

$$(B + E_W E_P B^\pi)(B^\# + B^\pi) = BB^\# + E_W E_P B^\pi = I - (I - E_W E_P)B^\pi$$

is invertible iff $I - B^\pi(I - E_W E_P)$ is invertible by Lemma 2.1. So, by Lemma 2.2, we have $I - B^\pi(I - E_W E_P)$ is invertible iff $B^\pi(I - E_W E_P) = 0$ iff $B^\pi P = 0$. Thus, we get the following theorem by Corollary 2.6:

Theorem 3.14. [4, 8, 22] *Let $M = \begin{pmatrix} B & A \\ 0 & C \end{pmatrix}$ with $B^\#, C^\#$ exist. Then $M^\#$ exists iff $B^\pi AC^\pi = 0$. In this case,*

$$M^\# = \begin{pmatrix} B^\# & (B^\#)^2 AC^\pi + B^\pi A(C^\#)^2 - B^\# AC^\# \\ 0 & C^\# \end{pmatrix}.$$

Example 3.15. *Let $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Put $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $M = \begin{pmatrix} 0 & B \\ I & D \end{pmatrix}$. Taking $B^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $R = DF_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, W = E_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $DF_B - CB + E_W E_R(F_B + DB^+B) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ is invertible. Thus, by Theorem 3.10, we have $M^\#$ exists and $M^\# = M^2$.*

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