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On the High Order Convergence of the Difference Solution of Laplace's Equation in a Rectangular Parallelepiped

Adiguzel A. Dosiyev[a], **Ahlam Abdussalam**[b]

[a]*Near East University, Department of Mathematics, Nicosia, KKTC, Mersin 10, Turkey*
[b]*Azzaytuna University, Department of Mathematics, Tripoli, Libya*

Abstract. The boundary functions φ_j of the Dirichlet problem, on the faces Γ_j, $j = 1, 2, ..., 6$ of the parallelepiped R are supposed to have seventh derivatives satisfying the Hölder condition and on the edges their second, fourth and sixth order derivatives satisfy the compatibility conditions which result from the Laplace equation. For the error $u_h - u$ of the approximate solution u_h at each grid point (x_1, x_2, x_3), a pointwise estmation $O(\rho h^6)$ is obtained, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current grid point to the boundary of R; h is the grid step. The solution of difference problems constructed for the approximate values of the first and pure second derivatives converge with orders $O(h^6 |\ln h|)$ and $O(h^{5+\lambda})$, $0 < \lambda < 1$, respectivly.

1. Introduction

The construction and justification of highly accurate approximate methods for the solution and its derivatives of PDEs in a rectangle or in a rectangular parallelepiped are important not only for the development of theory of these methods, but also to improve some version of domain decomposition methods for more complicated domains [3–5, 8, 14, 16, 17, 19, 22].

It is obvious that the accuracy of the approximate derivatives depends on the accuracy of the approximate solution. As is proved in [9], the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the $2D$ Laplace equation in any strictly interior subdomain with the same order h, (h is the grid step), with which the difference solution converges on the given domain. In [18], $O(h^2)$ order uniform convergence of the solution of the difference equation, and its first and pure second difference derivatives over the whole grid domain to the solution, and corresponding derivatives of solution for the $2D$ Laplace equation was proved. In [6] three difference schemes were constructed to approximate the solution and its first and pure second derivatives of $2D$ Laplace's equation with order of $O(h^4)$.

In [20], for the 3D Laplace equation in a rectangular parallelepiped the constructed difference schemes converge with order of $O(h^2)$ to the first and pure second derivatives of the exact solution of the Dirichlet problem. It is assumed that the fourth derivatives of the boundary functions on the faces of a parallelepiped satisfy the Hölder condition, and on the edges their second derivatives satisfy the compatibility condition that is implied by the Laplace equation. Whereas in [21], the convergence with order $O(h^2)$ of the difference

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Email addresses: adiguzel.dosiyev@emu.edu.tr (Adiguzel A. Dosiyev), o_mohammed27@yahoo.com (Ahlam Abdussalam)

derivatives to the corresponding first order derivatives was proved, when the third derivatives of the boundary functions on the faces satisfy the Hölder condition. Further, in [7] by assuming that the boundary functions on the faces have the sixth order derivatives satisfying the Hölder condition, and the second and fourth derivatives satisfy the compatibility conditions on the edges, for the uniform error of the approxmate solution $O(h^6 |\ln h|)$ order, and for the first and pure second derivatives $O(h^4)$ order was obtained.

In this paper, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. The boundary functions on the faces of a parallelepiped are supposed to have the seventh order derivatives satisfying the Hölder condition, and on the edges the second, fourth and sixth order derivatives satisfy the compatibility conditions. We present and justify difference schemes on a cubic grid for obtaining the solution of the Dirichlet problem, its first and pure second derivatives. For the approximate solution a pointwise estimation for the error of order $O(\rho h^6)$ with the weight function ρ, where $\rho = (x_1, x_2, x_3)$ is the distance from the current grid point (x_1, x_2, x_3) to the boundary of the parallelepiped, is obtained. This estimation gives an additional accuracy of the finite difference solution near the boundary of the parallelepiped, which is used to approximate the first derivatives with uniform error of order $O(h^6 |\ln h|)$. The approximate pure second derivatives are obtained with uniform accuracy $O(h^{5+\lambda})$ $0 < \lambda < 1$.

2. The Dirichlet Problem in a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : = 0 < x_i < a_i,\ i = 1, 2, 3\}$ be an open rectangular parallelepiped ; Γ_j $(j = 1, 2, ..., 6)$ be its faces including the edges; Γ_j for $j = 1, 2, 3$ (for $j = 4, 5, 6$) belongs to the plane $x_j = 0$ (to the plane $x_{i-3} = a_{i-3}$), let $\Gamma = \cup_{j=1}^{6} \Gamma_j$ be the boundary and $\gamma_{\mu\nu} = \Gamma\mu \cap \Gamma\nu$ be the edges of parallelepiped R. We say that $f \in C^{k,\lambda}(D)$, if f has kth derivatives on D satisfying a Hölder condition with exponent $\lambda \in (0, 1)$.

We consider the boundary value problem

$$\triangle u = 0 \text{ on } R,\ u = \varphi_j \text{ on } \Gamma_j,\ j = 1, 2, ..., 6, \tag{1}$$

where $\triangle \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, φ_j are given functions. Assume that

$$\varphi_j \in C^{7,\lambda}(\Gamma_j),\ 0 < \lambda < 1,\ j = 1, 2, ..., 6,\ \varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu}, \tag{2}$$

$$\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{3}$$

$$\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}, \tag{4}$$

$$\frac{\partial^6 \varphi_\mu}{\partial t_\mu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_{\mu\nu}^2} + \frac{\partial^6 \varphi_\mu}{\partial t_\mu^4 \partial t_\nu^2} = \frac{\partial^6 \varphi_\nu}{\partial t_\mu^2 \partial t_\nu^4} + \frac{\partial^6 \varphi_\nu}{\partial t_\nu^6} + \frac{\partial^6 \varphi_\mu}{\partial t_\nu^4 \partial t_{\mu\nu}^2} \text{ on } \gamma_{\mu\nu}. \tag{5}$$

where $1 \le \mu < \nu \le 6$, $\nu - \mu \ne 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

Lemma 2.1. *Let $\rho(x_1, x_2, x_3)$ be the distance from the current point of the open parallelepiped R to its boundary and let $\partial/\partial l \equiv \alpha_1 \partial/\partial x_1 + \alpha_2 \partial/\partial x_2 + \alpha_3 \partial/\partial x_3$, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Then the next inequality holds*

$$\left| \frac{\partial^8 u(x_1, x_2, x_3)}{\partial l^8} \right| \le c\rho^{\lambda-1}(x_1, x_2, x_3),\ (x_1, x_2, x_3) \in R, \tag{6}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$, u is a solution of problem (1).

Proof. We choose an arbitrary point $(x_{10}, x_{20}, x_{30}) \in R$. Let $\rho_0 = \rho(x_{10}, x_{20}, x_{30})$, and $\bar{\sigma}_0 \subset \bar{R}$ be the closed ball of radius ρ_0 centred at (x_{10}, x_{20}, x_{30}). Consider the harmonic function on R

$$v(x_1, x_2, x_3) = \partial^7 u(x_1, x_2, x_3)/\partial l^7 - \partial^7 u(x_{10}, x_{20}, x_{30})/\partial l^7. \tag{7}$$

As it follows from Theorem 2.1 in [15] the solution u of problem (1), which satisfies the conditions (2)-(5) belongs to the class $C^{7,\lambda}(\overline{R})$, $0 < \lambda < 1$. Then for the function (7) we have

$$\max_{(x_1,x_2,x_3)\in\bar{\sigma}_0} |v(x_1, x_2, x_3)| \leq c_0 \rho_0^{\lambda}, \tag{8}$$

where c_0 is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$. Using estimate (8) and applying Lemma 3 in [11] (see Chap. 4, Sec. 3), we obtain

$$\left| \frac{\partial^8 u(x_1, x_2, x_3)}{\partial l^8} \right| \leq c \rho_0^{\lambda-1}(x_1, x_2, x_3), \tag{9}$$

where c is a constant independent of the point $(x_{10}, x_{20}, x_{30}) \in R$ or the direction of $\partial/\partial l$. Since the point $(x_{10}, x_{20}, x_{30}) \in R$ is arbitrary, inequality (6) holds true. □

Let v be a solution of the problem

$$\triangle v = 0 \text{ on } R, \ v = \Psi_j \ \text{ on } \Gamma_j, \ j = 1, 2, ..., 6, \tag{10}$$

where Ψ_j, $j = 1, 2, ..., 6$, are given functions and

$$\Psi_j \in C^{5,\lambda}, \ 0 < \lambda < 1, \ j = 1, 2, ..., 6, \ \Psi_\mu = \Psi_\nu \text{ on } \gamma_{\mu\nu}, \tag{11}$$

$$\frac{\partial^2 \Psi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \Psi \nu}{\partial t_\nu^2} + \frac{\partial^2 \Psi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{12}$$

$$\frac{\partial^4 \Psi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \Psi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 \Psi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \Psi_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}. \tag{13}$$

Lemma 2.2. *The next inequality is true*

$$\left| \frac{\partial^8 v(x_1, x_2, x_3)}{\partial l^8} \right| \leq c_1 \rho^{\lambda-3}(x_1, x_2, x_3), \ \ (x_1, x_2, x_3) \in R, \tag{14}$$

where c is a constant independent of the direction of differentiation $\partial/\partial l$.

Proof. The proof of Lemma 2.2 is provided by analogy with the proof of Lemma 2.1, by taking into account that the fifth order derivatives of v are harmonic and $v \in C^{5,\lambda}(\overline{R})$. □

3. Finite Difference Method for the Dirichlet Problem

Let $h > 0$, and $a_i/h \geq 6$, $i = 1, 2, 3, ..., 6$ integers. We assign R^h a cubic grid on R, with step h, obtained by the planes $x_i = 0, h, 2h, ..., i = 1, 2, 3$. Let D^h be a set of nodes of this grid, $R^h = R \cap D^h$, $\Gamma_j^h = \Gamma_j \cap D^h$, and $\Gamma^h = \Gamma_1^h \cup \Gamma_2^h \cup ... \cup \Gamma_6^h$.

Let the operator $\mathfrak{R}$ be defined as follows (see [10])

$$\Re u(x_1,x_2,x_3)=\frac{1}{128}\left(14\sum_{p=1^{(1)}}^{6}u_p+3\sum_{q=7^{(2)}}^{18}u_q+\sum_{r=19^{(3)}}^{26}u_r\right),(x_1,x_2,x_3)\in R, \tag{15}$$

where the sum $\sum_{(k)}$ is taken over the grid nodes that are at a distance of $\sqrt{k}h$ from the point (x_1,x_2,x_3), and u_p,u_q and u_r are the values of u at the corresponding grid points.

We consider the finite difference approximations of problem (1):

$$u_h=\Re u_h \text{ on } R^h,\ u_h=\varphi_j \text{ on } \Gamma_j^h,\ j=1,2,...,6. \tag{16}$$

By the maximum principle system (16) has a unique solution.

Hereafter $c,c_1,c_2,...$ are constants independent of the multipliers standing by them, and the same notation may be used for different constants.

Let R^{kh} be the set of the grid nodes R^h whose distance from Γ is kh. It is obvious that $1\le k\le N(h)$,

$$N(h)=\left[\min\{a_1,a_2,a_3\}/2h\right], \tag{17}$$

where, $[d]$ is the integer part of d.

We define for $1\le k\le N(h)$

$$f_h^k=\begin{cases}1, & \rho(x_1,x_2,x_3)=kh,\\ 0, & \rho(x_1,x_2,x_3)\ne kh.\end{cases} \tag{18}$$

Lemma 3.1. *The solution of the system*

$$v_h^k=\Re v_h^k+f_h^k \text{ on } R^h,\ v_h^k=0 \text{ on } \Gamma^h \tag{19}$$

satisfies the inequality

$$v_h^k(x_1,x_2,x_3)\le Q_h^k,\ 1\le k\le N(h), \tag{20}$$

where Q_h^k is defined as follows

$$Q_h^k=Q_h^k(x_1,x_2,x_3)=\begin{cases}\frac{6\rho}{h}, & 0\le\rho(x_1,x_2,x_3)\le kh,\\ 6k, & (x_1,x_2,x_3)>kh.\end{cases} \tag{21}$$

Proof. By virtue of (15) *and* (21), *we have*

$$Q_h^k=\Re Q_h^k+q_h^k \text{ on } R^h,\ Q_h^k=0 \text{ on } \Gamma^h,\ k=1,...,N(h), \tag{22}$$

where $\left|q_h^k\right|\ge 1$. *On the basis of (18), (19), (22) and the Comparison Theorem (see [13], Chap. 4), we obtain*

$$v_h^k\le Q_h^k, \text{ on } \overline{R}^h.$$

□

Remark 3.2. From (20) and (21) it follows that

$$\max_{(x_1,x_2,x_3)\in R^h}\left|v_h^k(x_1,x_2,x_3)\right|\le 6k,\ 1\le k\le N(h).$$

Theorem 3.3. *Assume that the boundary function φ_j satisfy the conditions (2)-(5). Then at each point $(x_1, x_2, x_3) \in R^h$*

$$|u_h - u| \leq c\rho h^6, \tag{23}$$

where u_h is the solution of the finite difference problem (16), u is the exact solution of problem (1) and $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R^h$ to the boundary of rectangular parallelepiped R.

Proof. Let

$$\varepsilon_h = u_h - u \text{ on } \bar{R}^h. \tag{24}$$

By (16) and (24) the error function ε_h satisfies the system of equations

$$\varepsilon_h = \Re\varepsilon_h + (\Re u - u) \text{ on } R^h,\ \varepsilon = 0 \text{ on } \Gamma^h. \tag{25}$$

On the basis of Lemma 2.1, by analogy with the proof of Lemma 3 and 4 in [24] for difference $\Re u - u$ in (25) we have

$$\max_{(x_1,x_2,x_3)\in R^{kh}} \left|\Re u - u\right| \leq c\frac{h^{7+\lambda}}{k^{1-\lambda}},\ k = 1, 2, ..., N(h) \tag{26}$$

Let ε_h^k, $1 \leq k \leq N(h)$ be a solution of the system

$$\varepsilon_h^k = \Re\varepsilon_h^k + r_h^k \text{ on } R^h,\ \varepsilon_h^k = 0 \text{ on } \Gamma^h \tag{27}$$

where

$$r_h^k = \begin{cases} \Re u - u \text{ on } (x_1, x_2, x_3) \in R^{kh}, \\ 0 \text{ on } (x_1, x_2, x_3) \in R^h \backslash R^{kh}. \end{cases} \tag{28}$$

It is obvious that

$$\varepsilon_h = \sum_{k=1}^{N(h)} \varepsilon_h^k, \tag{29}$$

where ε_h is a solution of system (25).

By virtue of (27), (28) and Lemma 3.1, for each k, $1 \leq k \leq N(h)$, follows the inequality

$$\left|\varepsilon_h^k(x_1, x_2, x_3)\right| \leq Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in R^{kh}} \left|\Re u - u\right|, \text{ on } \overline{R}^h. \tag{30}$$

On the basis of (24), (26), (29) and (30), we obtain

$$\begin{aligned} |\varepsilon_h| &\leq \sum_{k=1}^{N(h)} \left|\varepsilon_h^k\right| \leq \sum_{k=1}^{N(h)} Q_h^k(x_1, x_2, x_3) \max_{(x_1,x_2,x_3)\in R^{kh}} \left|\Re u - u\right| \\ &\leq 6ch^8 \sum_{k=1}^{\rho/h-1} \frac{k}{(kh)^{1-\lambda}} + 6ch^8 \sum_{k=\rho/h}^{N(h)} \frac{\rho/h}{(kh)^{1-\lambda}} \\ &\leq ch^6\rho. \end{aligned} \tag{31}$$

□

4. Approximation of the First Derivatives

Let u be a solution of the boundary value problem (1). We put $v = \frac{\partial u}{\partial x_1}$. It is obvious that the function v is a solution of boundary value problem

$$\triangle v = 0 \text{ on } R,\ v = F_j \text{ on } \Gamma_j,\quad j = 1,2,...,6, \tag{32}$$

where $F_j = \frac{\partial u}{\partial x_1}$ on $\Gamma_j,\ j = 1,2,...,6.$

Let u_h be the solution of finite difference problem (16). We define the following operators $F_{\nu h}$, $\nu = 1,2,3,...,6$, as

$$\begin{aligned}
F_{1h}(u_h) &= \frac{1}{60h}[-147\varphi_1(x_2,x_3) + 360u_h(h,x_2,x_3) - 450u_h(2h,x_2,x_3) \\
&\quad +400u_h(3h,x_2,x_3) - 225u_h(4h,x_2,x_3) + 72u_h(5h,x_2,x_3) \\
&\quad -10u_h(6h,x_2,x_3)] \text{ on } \Gamma_1^h, && (33)\\
F_{4h}(u_h) &= \frac{1}{60h}[147\varphi_1(x_2,x_3) - 360u_h(a_1-h,x_2,x_3) + 450u_h(a_1-2h,x_2,x_3) \\
&\quad -400u_h(a_1-3h,x_2,x_3) + 225u_h(a_1-4h,x_2,x_3) \\
&\quad -72u_h(a_1-5h,x_2,x_3) + 10u_h(a_1-6h,x_2,x_3)] \text{ on } \Gamma_4^h, && (34)\\
F_{ph}(u_h) &= \frac{\partial \varphi_p}{\partial x_1} \text{ on } \Gamma_p^h,\ p = 2,3,5,6. && (35)
\end{aligned}$$

Consider the finite difference boundary value problem

$$v_h = \mathfrak{R}v_h \text{ on } R^h,\ v_h = F_{jh} \text{ on } \Gamma_j^h,\quad j = 1,2,...,6, \tag{36}$$

where $F_{kh},\ j = 1,2,...,6$ are defined by (33)-(35).

Theorem 4.1. *The estimation is true*

$$\max_{(x_1,x_2,x_3)\in\bar{R}^h} \left| v_h - \frac{\partial u}{\partial x_1} \right| \le ch^6(1 + |\ln h|), \tag{37}$$

where v_h is the solution of finite difference problem (36), u is the solution of problem (1).

Proof. Let

$$\varepsilon_h = v_h - v \text{ on } \bar{R}^h, \tag{38}$$

where $v = \frac{\partial u}{\partial x_1}$. From (36) and (38), we have

$$\varepsilon_h = \mathfrak{R}\varepsilon_h + (\mathfrak{R}v - v) \text{ on } R^h,$$

$$\varepsilon_h = F_{kh}(u_h) - v \text{ on } \Gamma_k^h,\ ,k = 1,4,\ \ \varepsilon_h = 0 \text{ on } \Gamma_p^h,\ p = 2,3,5,6.$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2 \tag{39}$$

where

$$\varepsilon_h^1 = \mathfrak{R}\varepsilon_h^1 \text{ on } R^h, \tag{40}$$

$$\varepsilon_h^1 = F_{kh}(u_h) - v \ \text{ on } \ \Gamma_k^h, \ \ k = 1,4, \ \ \varepsilon_h^1 = 0 \ \text{ on } \ \Gamma_p^h, \ \ p = 2,3,5,6, \tag{41}$$

$$\varepsilon_h^2 = \mathfrak{R}\varepsilon_h^2 + (\mathfrak{R}v - v) \text{ on } R^h, \ \ \varepsilon_h^2 = 0 \text{ on } \Gamma_j^h, \ \ j = 1,2...,6. \tag{42}$$

By the maximum principle, for the solution of system (40), (41), we have

$$\max_{(x_1,x_2,x_3)\in \bar{R}^h} \left|\varepsilon_h^1\right| \leq \max_{q=1,4} \max_{(x_1,x_2,x_3)\in\Gamma_q^h} \left|F_{qh}(u_h) - v\right|. \tag{43}$$

We estimate $F_{qh}(u_h) - v$ by estimating the differences $F_{qh}(u_h) - F_{qh}(u)$ and $F_{qh}(u) - v$. By using the linearity of oprators $F_{qh}, q = 1,4$, and Theorem 3.3 when the distance function $\rho(x_1, x_2, x_3)$ in the estimation (23) takes the values $\rho(x_1, x_2, x_3) = \nu h$, $0 \leq \nu \leq 6$, we obtain

$$\max_{q=1,4} \max_{(x_1,x_2,x_3)\in\Gamma_q^h} \left|F_{qh}(u_h) - F_{qh}(u)\right| \leq c_1 h^6. \tag{44}$$

Since, by Lemma 2.2, $u \in C^{7,\lambda}(\bar{R})$ the formulae (33) and (34) give the sixth order approximation of $\frac{\partial u}{\partial x_1}$ on Γ_1^h and, Γ_4^h respectively (see [2, 12]). Then, for the difference $F_{qh}(u) - v$ we have

$$\max_{q=1,4} \max_{(x_1,x_2,x_3)\in\Gamma_k^h} \left|F_{qh}(u) - v\right| \leq c_2 h^6, k = 1,4. \tag{45}$$

On the basis of (43) – (45) we obtain

$$\max_{(x_1,x_2,x_3)\in \bar{R}^h} \left|\varepsilon_h^1\right| \leq c_3 h^6. \tag{46}$$

The solution ε_h^2 of system (42) is the error of the approximate solution obtained by the finite difference method for problem (32), when on the boundary nodes Γ_{jh} approximate values are defined as the exact values of the functions F_j. It is obvious that F_j, $j = 1,2,...,6$, satisfy the conditions

$$F_j \in C^{6,\lambda}, \ 0 < \lambda < 1, \ \ j = 1,2,...,6, \tag{47}$$

$$F_\mu = F_\nu \text{ on } \gamma_{\mu\nu}, \tag{48}$$

$$\frac{\partial^2 F_\mu}{\partial t_\mu^2} + \frac{\partial^2 F v}{\partial t_\nu^2} + \frac{\partial^2 F_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu},$$

$$\frac{\partial^4 F_\mu}{\partial t_\mu^4} + \frac{\partial^4 F_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 F_\nu}{\partial t_\nu^4} + \frac{\partial^4 F_\mu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}. \tag{49}$$

Since the function $v = \frac{\partial u}{\partial x_1}$ is harmonic on R with the boundary values of F_j, $j = 1,2,...,6$, on the basis of (47)-(49) and by Lemma 2.6 in [7], we have

$$\max_{(x_1,x_2,x_3)\in \bar{R}^h} \left|\varepsilon_h^2\right| \leq c_4 h^6(1 + |\ln h|). \tag{50}$$

By virtue of (38), (39), (46) and (50), we obtain

$$\max_{(x_1,x_2,x_3)\in \bar{R}^h} |\varepsilon_h| \leq c h^6(1 + |\ln h|).$$

□

5. Approximation of the Pure Second Derivatives

We consider the finite difference problem

$$\omega_h = \mathfrak{R}\omega_h \text{ on } R^h,\ \omega_h = \Psi_j \text{ on } \Gamma_j^h,\ j = 1, 2, ..., 6, \tag{51}$$

where the boundary functions Ψ_j are defined through the second derivatives of the bondary functions φ_j of problem (1) as follows

$$\Psi_p = \frac{\partial^2 \varphi_p}{\partial x_1^2},\ p = 2, 3, 5, 6;\ \ \Psi_q = -\frac{\partial^2 \varphi_q}{\partial x_2^2} - \frac{\partial^2 \varphi_q}{\partial x_3^2},\ q = 1, 4. \tag{52}$$

Theorem 5.1. *The estimation holds*

$$\max_{(x_1,x_2,x_3)\in \overline{R}^h} \left| \omega_h - \frac{\partial^2 u}{\partial x_1} \right| \leq c_2 h^{5+\lambda},\ 0 < \lambda < 1, \tag{53}$$

where v_h is a solution of finite difference problem (51), *u is a solution of problem (1).*

Proof. By virtue of (2)-(5) and (52) the functions Ψ_j, $j = 1, ..., 6$ in (51) satisfy conditions (11)-(13).Then, the harmonic function $\omega = \partial^2 u/\partial x_1^2 \in C^{5,\lambda}(\overline{R})$ and is the unique solution of the problem

$$\triangle\omega = 0 \text{ on } R,\ \omega = \Psi_j \ \text{ on } \Gamma_j^h,\ j = 1, 2, ..., 6. \tag{54}$$

Let

$$\epsilon_h = \omega_h - \omega, \tag{55}$$

where ω_h, and ω is a solution of problems (51) and (54) respectively. Then for ϵ_h, we have

$$\epsilon_h = \mathfrak{R}\epsilon_h + (\mathfrak{R}\omega - \omega) \ \text{ on } R^h,\ \epsilon_h = 0 \text{ on } \ \Gamma^h. \tag{56}$$

Using Lemma 2.2 instead of Lemma 2.1, we obtain

$$\max_{(x_1,x_2,x_3)\in R^{kh}} \left| \mathfrak{R}\omega - \omega \right| \leq c_3 \frac{h^{5+\lambda}}{k^{3-\lambda}},\ k = 1, 2, ..., N(h). \tag{57}$$

On the basis of (56), (57), Remark 3.2, by analogy with the proof of Theorem 3.3, we have

$$\begin{aligned}
\max_{(x_1,x_2,x_3)\in R^h} |\epsilon_h| &\leq \sum_{k=1}^{N(h)} 6k \max_{(x_1,x_2,x_3)\in R^h} \left| \mathfrak{R}\omega - \omega \right| \leq \sum_{k=1}^{N(h)} 6kc_3 \frac{h^{5+\lambda}}{k^{3-\lambda}} \\
&\leq 6c_3 h^{5+\lambda} \left\{ 1 + \int_1^{N(h)} x^{\lambda-2} dx \right\} \leq c_4 h^{5+\lambda}.
\end{aligned} \tag{58}$$

From (55) and (58), follows (53). □

6. Conclusion

For the approximate solution of the 3D Laplace equation in a rectangular parallelepiped R, a new pointwise error of order $O(\rho h^6)$, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point to the boundary of R, is obtained. This estimation shows the additional downturn of the error near the boundary as ρ, which is used to get $O(h^6(1 + |\ln h|))$ order of accuracy for the approximate value of the first derivatives. For the approxmation of the pure second derivatives, $O(h^{5+\lambda})$ order of estimate is obtained by the requarements on the functions given on the faces of R which cannot be essentially lowered in $C^{k,\lambda}$ (see [1]).

The obtained results can be applied for the approximation of a solution and its derivatives of problems in more complicated domains when different version of domain decomposition methods are used (see [14, 17, 19]).

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