Filomat 32:4 (2018), 1209–1220
https://doi.org/10.2298/FIL1804209S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Multivalued φ Contractions and Fixed Point Theorems

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Abstract. In this paper we establish some fixed point theorems for multivalued mappings satisfying contractive condition involving gauge function when the underlying primary structure is b-metric space. Our proposed iterative scheme converges to the the fixed point with higher order. Moreover, we also calculate priori and posteriori estimates for the fixed point. Our main results generalize/extend many per-existing results in literature. Consequently, to substantiate the validity of our result we obtain an existence result for the solution of integral inclusion.

1. Introduction

The main source of the existence of metric fixed point theory is due to the famous mathematician Banach. Many mathematical problems can be formulated equivalently as fixed point problems, i,e., the existence of solutions of many problems of differential, integral and integro-differential equations become equivalent to fixed point problems of suitable mappings.
Banach contraction principle proposes an iterative method which converges to the fixed point linearly. In order to obtain a high order of convergence, Proinov [23] extended/generalized Banach contraction theorem by generalizing the contractive condition which involves a gauge function of order $r \geq 1$. Later on his work was extended and generalized to b-metric space by the authors in [25]. In this context, φ contractions were studied without the assumptions of gauge function in the setting of b-metric space by authors in [26]. Interesting developments on the subject of b-metric can be found in [2–4, 7, 9–12, 16, 20]. Some details for multivalued mappings and their fixed points are included in [1, 7, 8, 20].
The research in fixed point theory moved forward when Nadler generalized Banach contraction principle for multivalued mappings [21]. In [17] authors undertook further investigations in this direction and generalized Nadler's result by introducing a contractive condition which involves gauge function. The obtained results in [17] also extend and generalize main results of Proinov [23] to the multivalued mappings.
In this paper we establish some fixed point theorems for multivalued mappings satisfying contractive condition involving a gauge function when the underlying space is endowed with b-metric. Our results

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25, 05C40
Keywords. Fixed point, b-metric, gauge function, Hausdorff metric.
Received: 27 December 2016; Revised: 24 May 2017; Accepted: 29 May 2017
Communicated by Erdal Karapinar
The first author is thankful to Higher Education Commision of Pakistan for providing financial support under the project no. 21-928/SRGP/R&D/HEC/2016.
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extend/generalize main results of [17, 23, 25] and in turn many well known results in literature become special cases of our results e.g., Rheinboldt [24], Kornstaedt [19, Satz 4.1], Hicks and Rhoades [14, Theorem 3], Park [22, Theorem 2], Gel'man [13, Theorem 3]. We also calculate priori and posteriori estimates to approach the fixed point and the proposed iterative scheme converges to the fixed point with higher order. Consequently, we also furnish with an application for integral inclusion where the kernel of inclusion may not satisfy usual Lipschitz condition.

We start by recollecting some notions and preliminaries in b-metric space that are found in [5, 11]. Let $\mathbb{R}$ denote the real line whereas $\mathbb{R}^+$ denote the set of all non-negative real numbers.

Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied: $d(x, y) = 0 \iff x = y$; $d(x, y) = d(y, x)$; $d(x, z) \leq s[d(x, y) + d(y, z)]$. The pair (X, d) is called a b-metric space with the coefficient $s \geq 1$. For convenience we represent with the triplet (X, d, s) a b-metric space with coefficient s.

Definition 1.1. *[5, 11] A sequence $\{x_n\}$ in a b-metric space X is:*

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

A b-metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Lemma 1.2. *[12] Let (X, d, s) be a b-metric space, then a convergent sequence has a unique limit; every convergent sequence is Cauchy; and in general the b-metric d is not a continuous functional.*

Subsequently, throughout this paper let X be a nonempty set endowed with a b-metric d unless specified otherwise. We denote by $N(X)$ the class of all nonempty subsets of b-metric space X, $CB(X)$ the class of all nonempty closed and bounded subsets of X. Let J denote an interval on $\mathbb{R}^+$ containing 0, i.e., an interval of the form $[0, R], [0, R)$ or $[0, \infty)$ and $([0, 0] = \{0\})$. Let $P_n(t)$ denote the polynomial of the form $P_n(t) = 1 + t + ... + t^{n-1}$ and $P_0(t) = 0$. We use φ^n to denote the nth iterate of a function $\varphi : J \to J$. The closed ball centered at $x \in X$ and radius r is denoted by $\overline{B}(x; r)$.

Definition 1.3. *[11] Let (X, d, s) be a b-metric space. The generalized Pompeiu-Hausdorff metric $H : CB(X) \times CB(X) \to \mathbb{R}^+$ is defined as*

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\} \text{ for every } A, B \in CB(X). \tag{1}$$

If the b-metric space (X, d, s) is complete then the induced Hausdorff b-metric space $(CB(X), H)$ is also complete [11].

Let $T : D \subset X \to X$ and there exists $x \in D$ such that the set $O(x) = \{x, Tx, T^2x, ...\} \subset D$. The set $O(x)$ is known as the orbit of $x \in D$ under T. We recall that a function G from D into the set of real numbers is said to be T-orbitally lower semi-continuous at $t \in D$ if $\{x_n\} \subset O(x)$ and $x_n \to t$ implies $G(t) \leq \liminf G(x_n)$ [14].

Definition 1.4. *[23] Let $r \geq 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:*

1. $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ *for all* $\lambda \in (0, 1)$ *and* $t \in J$;
2. $\varphi(t) < t$ *for all* $t \in J - \{0\}$.

The first condition of the Definition 1.4 infers that $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$ [**?**].

Definition 1.5. *[6] A gauge function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function if $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \in J$.*

Proinov[23] proved his main results by assuming Bianchini-Grandolfi gauge function φ and the mapping $T : D \subset X \to X$ satisfying the contractive condition $d(Tx, T^2x) \leq \varphi(d(x, Tx))$ when the underlying space is endowed with a metric. But in the setting of b-metric space for some technical reasons authors in [25] introduced the following class of gauge functions.

Definition 1.6. *[25] A gauge function $\varphi : J \to J$ is said to be a b-Bianchini-Grandolfi gauge function if*

$$\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty \text{ for all } t \in J, \tag{2}$$

where the fixed constant $s \geq 1$ is the coefficient of b-metric space. Note that a b-Bianchini-Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(t) = s\sigma(\varphi(t)) + t. \tag{3}$$

Furthermore, in order to calculate the prior and posterior estimates in the setting of b-metric space the authors in [25] considered gauge functions of the form

$$\varphi(t) = t\frac{\phi(t)}{s} \quad \text{for all } t \in J, \tag{4}$$

where $s \geq 1$ is the coefficient of b-metric d and ϕ is nonnegative nondecreasing function on J satisfying

$$0 \leq \phi(t) < 1 \quad \text{for all } t \in J. \tag{5}$$

Remark 1.7. *[25] For a given gauge function φ the nonnegative nondecreasing function ϕ on J satisfying (4) and (5) can be obtained as follows:*

$$\phi(x) = \begin{cases} \frac{s\varphi(t)}{t}, & \text{if } t \in J \setminus \{0\} \\ 0, & \text{if } t=0, \end{cases}$$

where $s \geq 1$ is the coefficient of b-metric d.

Let $s \geq 1$, be a fixed real number then:

1. $\varphi(t) = \frac{ct}{s}, 0 < c < 1$ is a gauge function of order 1 on $J = [0, \infty)$;
2. $\varphi(t) = \frac{ct^r}{s}(c > 0, r > 1)$ is a gauge function of order r on $J = [0, h)$ where $h = (\frac{1}{c})^{\frac{1}{(r-1)}}$;

are b-Bianchini-Grandolfi gauge functions [25].

Lemma 1.8. *[25] Let φ be a Gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (4) and (5) then:*

1. $0 \leq \dfrac{\phi(t)}{s} < 1$ *for all $t \in J$;*
2. $\phi(\mu t) \leq \mu^{r-1}\phi(t)$ *for all $\mu \in (0, 1)$ and $t \in J$.*

Lemma 1.9. *[25] Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (4) and (5) then for every $n \geq 0$ we have:*

1. $\varphi^n(t) \leq t[\dfrac{\phi(t)}{s}]^{P_n(r)}$ *for all $t \in J$;*
2. $\phi(\varphi^n(t)) \leq s[\dfrac{\phi(t)}{s}]^{r^n}$ *for all $t \in J$.*

2. Main results

We start with the following intuitive lemmas.

Lemma 2.1. *Let (X, d, s) be a b-metric space. Let $B \in CB(X)$ and $a \in X$ be a fixed element. Then for every $\epsilon > 0$ there exists $b \in B$ such that*

$$d(a, b) \leq d(a, B) + \epsilon.$$

Lemma 2.2. *Let (X, d, s) be a b-metric space. Let $A, B \in CB(X)$ and $a \in A$ be a fixed element. Then for every $\epsilon > 0$ there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \epsilon.$$

It is essential to mention here that to establish the fixed point theorem we do not necessarily require the gauge functions φ satisfying (4) and (5). But we consider the gauge function φ such that $\Sigma_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$, where s is the coefficient of b-metric space.

Theorem 2.3. *Let (X, d, s) be a complete b-metric space such that b-metric d is a continuous functional. Let D be a closed subset of X, φ a b-Bianchini Grandolfi gauge function on an interval J. Assume that $T : D \to CB(X)$ satisfies $Tx \cap D \neq \phi$ and*

$$H(Tx \cap D, Ty \cap D) \quad \leq \quad \varphi(d(x, y)), \tag{6}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $x \neq y$. Suppose that $x_0 \in D$ be such that $d(x_0, w) \in J$ for some $w \in Tx_0 \cap D$. Then the following assertions hold:

(i) *there exists a sequence $\{x_n\}$ in D with $x_{n+1} \in Tx_n; n = 0, 1, 2, \cdots$ and $\xi \in D$ so that $\lim_{n\to\infty} x_n = \xi$;*

(ii) *ξ is a fixed point of T in D if and only if the function $f(x) = d(x, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.*

Proof. Setting $x_1 = w \in Tx_0 \cap D$ we have $d(x_0, x_1) \neq 0$, otherwise x_0 is a fixed point of T. Let $\rho_0 = \sigma(d(x_0, x_1)$ where σ is defined by (2). From (3) we have $\sigma(t) \geq t$ so that

$$d(x_0, x_1) \leq \rho_0. \tag{7}$$

Thus x_1 belongs to the closed ball $\overline{B}(x_0; \rho_0)$. Since $d(x_0, x_1) \in J$ so that from (6) it follows that

$$H(Tx_0 \cap D, Tx_1 \cap D) \quad < \quad \varphi(d(x_0, x_1)).$$

Choose an $\epsilon_1 > 0$ such that

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \leq \varphi(d(x_0, x_1)). \tag{8}$$

Since D is closed and Tx_1 is closed and bounded, by Lemma 2.2 there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \leq H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1. \tag{9}$$

We assume that $d(x_1, x_2) \neq 0$, otherwise x_1 is a fixed point of T. From inequalities (9) and (8), we obtain

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{10}$$

Further, $d(x_1, x_2) \leq \varphi(d(x_0, x_1)) < d(x_0, x_1)$ implies

$$d(x_1, x_2) \in J. \tag{11}$$

By using triangular inequality for b-metric, we obtain

$$\begin{aligned}
d(x_0, x_2) &\leq s(d(x_0, x_1) + sd(x_1, x_2)) \\
&\leq sd(x_0, x_1) + s^2 d(x_1, x_2) \\
&\leq sd(x_0, x_1) + s^2 \varphi(d(x_0, x_1)) \quad \text{(using (10))} \\
&= s[d(x_0, x_1) + s\varphi(d(x_0, x_1))] \\
&\leq s\sigma(d(x_0, x_1)) \\
&\leq s\sigma(d(x_0, x_1)) + d(x_0, x_1) \\
&= \sigma(d(x_0, x_1)) = \rho_0. \qquad \text{(using (3))}
\end{aligned}$$

Thus, $x_2 \in \overline{B}(x_0, \rho_0)$. Since, $d(x_1, x_2) \in J$ so that from (22) it follows that

$$H(Tx_1 \cap D, Tx_2 \cap D) \quad < \quad \varphi(d(x_1, x_2)).$$

Choose $\epsilon_2 > 0$ such that

$$H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2 \leq \varphi(d(x_1, x_2)). \tag{12}$$

Since D is closed and Tx_1 is closed and bounded, by Lemma 2.2 there exists $x_3 \in Tx_2 \cap D$ such that

$$d(x_2, x_3) \leq H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2. \tag{13}$$

We assume that $d(x_2, x_3) \neq 0$, otherwise x_2 is a fixed point of T. From inequalities (10), (12) and (13) we obtain

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)). \tag{14}$$

Further, $d(x_2, x_3) \leq \varphi(d(x_1, x_2)) < d(x_1, x_2)$ implies

$$d(x_2, x_3) \in J. \tag{15}$$

By using triangular inequality for b-metric, we obtain

$$\begin{aligned}
d(x_0, x_3) &\leq sd(x_0, x_1) + s^2 d(x_1, x_2) + s^3 d(x_2, x_3) \\
&\leq s[d(x_0, x_1) + sd(x_1, x_2) + s^2 d(x_2, x_3)] \\
&\leq s[d(x_0, x_1) + s\varphi(d(x_1, x_2)) + s^2\varphi^2(d(x_2, x_3))] \quad \text{(using (14))} \\
&\leq s\sigma(d(x_0, x_1)) \\
&\leq s\sigma(d(x_0, x_1)) + d(x_0, x_1) \\
&= \sigma(d(x_0, x_1)) = \rho_0. \qquad \text{(using (3))}
\end{aligned}$$

Thus $x_3 \in \overline{B}(x_0, \rho_0)$. Continuing in the same way we get a sequence $\{x_n\}$ in $\overline{B}(x_0, \rho_0)$ such that $x_n \in Tx_{n-1} \cap D, x_{n-1} \neq x_n; n = 1, 2, 3, \cdots$ with $d(x_{n-1}, x_n) \in J$ and

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \quad \leq \quad \varphi^n(d(x_0, x_1)). \tag{16}$$

For any $p \geq 1$, by using triangular inequality for b-metric we have

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq s^n d(x_n, x_{n+1}) + s^{n+1} d(x_{n+1}, x_{n+2}) + ... + s^{n+p-1} d(x_{n+p-1}, x_{n+p}) \\
&\leq s^n \varphi^n(d(x_0, x_1)) + s^{n+1}\varphi^{n+1}(d(x_0, x_1)) + ... + s^{n+p-1}\varphi^{n+p-1}(d(x_0, x_1)).
\end{aligned} \tag{17}$$

Since, φ is a b-Bianchini Grandolfi gauge function then $\sum_{i=1}^{\infty} s^i \varphi^i(d(x_0, x_1)) < \infty$. Assume that

$$S_n = \sum_{i=1}^{n} s^i \varphi^i(d(x_0, x_1)) \quad \text{and} \quad \lim_{n\to\infty} S_n = S. \tag{18}$$

From (17) and (18), we obtain

$$d(x_n, x_{n+p}) \leq \left[S_{n+p-1} - S_n\right]. \tag{19}$$

In view of (18), relation (19) implies $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. Which shows that $\{x_n\}$ is a Cauchy sequence in the closed ball $\overline{B}(x_0, \rho_0)$. Since $\overline{B}(x_0, \rho_0)$ is closed in X, there exists an $\xi \in \overline{B}(x_0, \rho_0)$ such that $x_n \to \xi$. Further, observe that $\xi \in D$. Since $x_n \in Tx_{n-1} \cap D$ and $d(x_{n-1}, x_n) \in J$ for $n = 1, 2, \cdots$. It follows from (22) that

$$\begin{aligned}
d(x_n, Tx_n \cap D) &\leq H(Tx_{n-1} \cap D, Tx_n \cap D) \\
&\leq \varphi(d(x_{n-1}, x_n) \\
&< d(x_{n-1}, x_n).
\end{aligned} \tag{20}$$

Letting $n \to \infty$ from (20) we have

$$\lim_{n\to\infty} d(x_n, Tx_n \cap D) = 0. \tag{21}$$

Assume that $f(x) = d(x, Tx \cap D)$ is T-orbitally lower continuous at ξ, then

$$d(\xi, T\xi \cap D) = f(\xi) \leq \liminf_n f(x_n) = \liminf_n d(x_n, Tx_n \cap D) = 0.$$

Hence, $\xi \in T\xi$ as $T\xi$ is closed. Conversely, if ξ is a fixed point of T then $f(\xi) = 0 \leq \lim_n \inf f(x_n)$, since $\xi \in D$. □

Now we proceed to establish another variant.

Theorem 2.4. *Let (X, d, s) be a complete b-metric space such that b-metric d is a continuous functional. Let D be a closed subset of X, φ a b-Bianchini Grandolfi gauge function on an interval J satisfying (4) and (5). Assume that $T : D \to CB(X)$ satisfies $Tx \cap D \neq \phi$ and*

$$H(Tx \cap D, Ty \cap D) \quad \leq \quad \varphi(d(x, y)) \tag{22}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $x \neq y$. Suppose that $x_0 \in D$ be such that $d(x_0, w) \in J$ for some $w \in Tx_0 \cap D$. Then the following assertions hold:

(i) *there exists a sequence $\{x_n\}$ with $x_{n+1} \in Tx_n; n = 0, 1, \cdots$ in $\overline{B}(x_0, \rho_0)$ that converges to a point $\xi \in \overline{B}(x_0, \rho_n)$;*

(ii) *for all $n \geq 0$ the following priori estimate holds,*

$$\begin{aligned} d(x_n, \xi) &\leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(d(x_0, x_1))^{P_j(r)} \\ &= d(x_0, Tx_0) \frac{\phi(d(x_0, x_1))^{P_n(r)}}{s^{n-1}[1 - \phi(d(x_0, x_1))^{r^n}]}; \end{aligned} \tag{23}$$

(iii) *for all $n \geq 1$ the following posteriori estimate holds,*

$$\begin{aligned} d(x_n, \xi) &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{P_j(r)} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(d(x_n, x_{n-1}))[\frac{\phi(d(x_n, x_{n-1}))}{s}]^{r-1}}; \end{aligned} \tag{24}$$

(iv) *for all $n \geq 1$ we have*

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})) \leq \mu^{P_n(r)} d(x_0, Tx_0), \tag{25}$$

where $\mu = \frac{\phi(d(x_0, x_1))}{s}$;

(v) *ξ is a fixed point of T if and only if the function $f(x) = d(x, Tx \cap D)$ is T-orbitally lower semi continuous at ξ.*

Proof. **(i)** Its proof follows from Theorem 2.3.

(ii) For $m \geq n$, using triangle inequality for b-metric we obtain

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{m-n} d(x_{m-1}, x_m)) \\ &\leq s\varphi(d(x_n, x_{n+1})) + ... + s^{m-n}\varphi^{m-n}(d(x_{m-1}, x_m)) \\ &\leq \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j \varphi^j (d(x_0, x_1)) \quad \text{(using (16))} \\ &\leq \sum_{j=n}^{m-1} s^j d(x_0, x_1) [\frac{\phi(d(x_0, x_1))}{s}]^{P_j(r)} \quad \text{(using Lemma 1.9)} \\ &\leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{m-1} \lambda^{P_j(r)}, \end{aligned}$$

where $\lambda = \phi(d(x_0, x_1))$. Keeping n fixed and letting $m \to \infty$ we get

$$d(x_n, \xi) \leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)} = \frac{d(x_0, Tx_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)}. \tag{26}$$

We note that

$$r^n + r^{n+1} \geq 2r^n, \qquad r^n + r^{n+1} + r^{n+2} \geq 3r^n, \quad ...,$$

and we deduce that

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n}, \quad \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n}, \quad ....$$

Thus, we obtain

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{P_j(r)} &= \lambda^{P_j(r)} + \lambda^{P_{j+1}(r)} + ... \\ &= \lambda^{P_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + ...] \\ &\leq \lambda^{P_j(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + ...] \\ &= \frac{\lambda^{P_n(r)}}{1 - \lambda^{r^n}}. \end{aligned}$$

Hence from (26) we have

$$d(x_n, \xi) \leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(d(x_0, x_1))^{P_j(r)} = d(x_0, Tx_0) \frac{\phi(d(x_0, x_1))^{P_n(r)}}{s^{n-1}[1 - \phi(d(x_0, x_1))^{r^n}]}.$$

(iii) From (26), for $n \geq 0$ we have

$$d(x_n, \xi) \quad \leq \quad \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{P_j(r)}.$$

Setting $n = 0$, $y_0 = x_0$ and $y_1 = x_1$, we have

$$d(y_0, \xi) \quad \leq \quad sd(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{P_j(r)}.$$

Setting again $y_0 = x_n$ and $y_1 = x_{n+1}$, we obtain

$$\begin{aligned} d(x_n, \xi) &\leq sd(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{P_j(r)} \\ &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{P_j(r)} \\ &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{j} \\ &= \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}. \end{aligned} \tag{27}$$

From Lemma 1.9(2) we obtain

$$\begin{aligned} \phi(\varphi(d(x_n, x_{n-1}))) &\leq s[\frac{\phi(d(x_n, x_{n-1}))}{s}]^r \\ &= \phi(d(x_n, x_{n-1}))[\frac{\phi(d(x_n, x_{n-1}))}{s}]^{r-1}. \end{aligned}$$

Which implies

$$\frac{1}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \leq \frac{1}{1 - \phi(\varphi(d(x_n, x_{n-1})))[\frac{\phi(d(x_n, x_{n-1}))}{s}]^{r-1}}. \tag{28}$$

Thus from (27) and (28) for $n \geq 1$ we deduce

$$\begin{aligned} d(x_n, \xi) &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}[\frac{\phi(d(x_n, x_{n-1}))}{s}]^{r-1}. \end{aligned}$$

(iv) Since,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)) \\ &= d(x_{n-1}, x_n)\frac{\phi(d(x_{n-1}, x_n))}{s} \\ &\leq d(x_0, x_1)\mu^{P_{n-1}(r)}\mu^{r^{n-1}} \quad \text{(using Lemma 1.8)} \\ &= d(x_0, x_1)\mu^{P_{n-1}(r)+r^{n-1}} \\ &= d(x_0, x_1)\mu^{P_n(r)} = \mu^{P_n(r)}d(x_0, Tx_0). \end{aligned}$$

(v) Its proof is analogue to the proof of Theorem 2.3.

□

Remark 2.5.

1. *Theorem 2.3 and 2.4 generalize [25, Theorem 3.7 & 3.10] to the case of multivalued mappings.*
2. *Theorem 2.3 and 2.4 generalize [18, Theorem 2.1 & 2.8] to the case of b-metric space.*

3. *Theorem 2.3 and 2.4 extend/generalize [17, Theorem 2.11 & 2.15] when $s = 1$ and range of T is taken to be $CB(X)$ instead of the space of all nonempty proximinal closed subsets of X.*
4. *Theorem 2.3 and 2.4 extend/generalize [23, Theorem 4.1 & 4.2] when $s = 1$ and T is a single-valued mapping.*

Corollary 2.6. *Let (X, d, s) be a complete b-metric space such that b-metric d is a continuous functional. Let φ be a b-Bianchini Grandolfi gauge function of order $r \geq 1$ on an interval J satisfying (4) and (5). Assume that $T : X \to CB(X)$ satisfies*

$$H(Tx, Ty) \quad \leq \quad \varphi(d(x, y)) \tag{29}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$. Suppose that $x_0 \in X$ is such that $d(x_0, w) \in J$ for some $w \in Tx_0$. Then the following assertions hold:

(i) *there exists a sequence $\{x_n\}$ in X with $x_n \in Tx_{n-1}; n = 1, 2, \cdots$ that converges to the fixed point $\xi \in S = \{x \in X : d(x, \xi) \in J\}$ of T;*

(ii) *the estimates (23)-(25) are valid.*

Proof. From (29) we have

$$H(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y) \text{ for all } x, y \in X, x \neq y.$$

Thus T is continuous. Hence the conclusions (i) and (ii) follow form Theorem 2.4. □

Remark 2.7. *Note that Corollary 2.6 extends [18, Corollary 2.11] to the case of b-metric. It also includes [17, Corollary 2.18] with the exception that the range of T is $CB(X)$ instead of nonempty proximinal subsets of X.*

3. Application

In this section we shall establish the existence of solution for the integral inclusion as an important consequence of Corollary 2.6.

Theorem 3.1. *Consider the following integral inclusion*

$$\begin{aligned} x(t) &\in P \int_{t_0}^{t} k(\tau, x(\tau)) d\tau + \gamma \\ &\in PK_x(t) + \gamma, \end{aligned} \tag{30}$$

where $K_x(t) = \int_{t_0}^{t} k(\tau, x(\tau)) d\tau$, P is a compact subset of real line $\mathbb{R}$ and $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

1. *k is continuous on the rectangle $R = \left\{(t, x) : |t - t_0| \leq \frac{a^{r-2}}{b^{r-1}}, |x - \gamma| \leq \frac{b}{2a}\right\}$, where $a = \max_{p \in P} |p|, 0 < b < a$ and $r \geq 2$;*

2. *k is bounded as $|k(t, x)| < \frac{1}{2}\left(\frac{b}{a}\right)^r$ for all $(t, x) \in R$;*

3.

$$|k(t, x(t)) - k(t, y(t))| \leq \frac{b}{a}|x(t) - y(t)|^r. \tag{31}$$

Then the integral inclusion (30) has a solution on the interval $I = [t_0 - \frac{a^{r-2}}{b^{r-1}}, t_0 + \frac{a^{r-2}}{b^{r-1}}]$.

Proof. Let $C(I)$ denote the space of all continuous functions with the metric $d(x,y) = \sup_{t\in I}|x(t) - y(t)|$. Consider the set $\tilde{C} := \left\{x \in C(I) : d(x,\gamma) \le \frac{b}{2a}\right\}$. Since, $\tilde{C}$ is closed subset of $C(I)$ and hence is complete. Define $T : \tilde{C} \to K(\tilde{C})$ as

$$Tx(t) = P\int_{t_0}^{t} k(\tau, x(\tau))d\tau + \gamma = PK_x(t) + \gamma. \tag{32}$$

The problem of finding the solution of (30) becomes equivalent to the fixed point problem of the operator T defined in (32). We show that T is well defined that is (i) T is defined for each $x \in \tilde{C}$, (ii) Tx is a compact subset of $\tilde{C}$ for each $x \in \tilde{C}$.
For $s \in I$, $|s - t_0| \le \frac{a^{r-2}}{b^{r-1}}$ and by definition of $\tilde{C}$ we obtain $|x(s) - \gamma| \le \frac{b}{2a}$. Thus $(s, x(s)) \in R$. Since k is continuous on R therefore integral in (32) exists, so that T is defined for each $x \in \tilde{C}$. Now we show that for each $x \in \tilde{C}$, Tx is a compact subset of $\tilde{C}$. Let $y(t) \in Tx(t)$. Then $y(t) = pK_x(t) + \gamma$ for some $p \in P$ and

$$\begin{aligned}
|y(t) - \gamma| &= |pK_x(t)| \\
&= |p||K_x(t)| \\
&\le a\int_{t_0}^{t} |K(\tau, x(\tau))d\tau| \\
&\le a\int_{t_0}^{t} |K(\tau, x(\tau))|d\tau \\
&< a\frac{1}{2}\Big(\frac{b}{a}\Big)^r \le \frac{b}{2a}.
\end{aligned}$$

Which infers that $d(y,\gamma) \le \frac{b}{2a}$ then $y \in \tilde{C}$. Since $y \in Tx$ was arbitrary hence $Tx \subset \tilde{C}$ for each $x \in \tilde{C}$. Next we show that Tx is compact. Consider a sequence $\{w_n\} \subset Tx$ then $w_n = p_nK_x(t) + \gamma$ for $p_n \in P; n = 1, 2, \cdots$. Since P is compact, there exists a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ such that $p_{n_j} \to \tilde{p} \in P$. Let $w = \tilde{p}K_x(t) + \gamma$, then we obtain

$$d(w_{n_j}, w) = \sup_{t\in I}(|p_{n_j} - \tilde{p}||K_x(t)|) \le |p_{n_j} - \tilde{p}|\sup_{t\in I}|K_x(t)| \to 0 \quad \text{as } j \to \infty.$$

Further we note that

$$H(Tx, Ty) = H(PK_x(t) + \gamma, PK_y(t) + \gamma) \le H(PK_x(t), PK_y(t)). \tag{33}$$

We have

$$H(PK_x(t), PK_y(t)) = \max\Big\{\max_{a'\in PK_x(t)} d(a', PK_y(t)), \max_{b'\in PK_y(t)} d(b', PK_x(t))\Big\}.$$

Now,

$$\begin{aligned}
\max_{a'\in PK_x(t)} d(a', PK_y(t)) &= \max_{a'\in PK_x}\min_{b'\in PK_y} d(a', b') \\
&= \max_{p\in P}\min_{p^*\in P} d(pk(t,x), p^*k(t,y)) \\
&= \max_{p\in P}\min_{p^*\in P}\sup_{t\in I}|pk(t,x) - p^*k(t,y)| \\
&\le \max_{p\in P}\min_{p^*\in P}\sup_{t\in I}\Big[|pk(t,y) - p^*k(t,y)| + |pk(t,y) - pk(t,x)|\Big] \\
&\le \max_{p\in P}\min_{p^*\in P}\Big[|p|\sup_{t\in I}|k(t,y) - k(t,x)| + |p - p^*|\sup_{t\in I}|k(t,y)|\Big] \\
&= \max_{p\in P}|p|\sup_{t\in I}|k(t,y) - k(t,x)| \\
&= a\sup_{t\in I}|k(t,y) - k(t,x)|.
\end{aligned}$$

And we have,

$$\begin{aligned}
|k(t,y)-k(t,x)| &\leq \int_{t_0}^{t} |k(\tau,y(\tau))-k(\tau,x(\tau))|d\tau \\
&\leq \frac{b}{a}\int_{t_0}^{t} |y(\tau)-x(\tau)|^r d\tau \\
&\leq \frac{b}{a}\sup_{s\in I}|y(s)-x(s)|\int_{t_0}^{t} d\tau \\
&= \frac{b}{a}|t-t_0|[d(x,y)]^r \\
&\leq \frac{1}{a}\Big(\frac{a}{b}\Big)^{r-2}[d(x,y)]^r.
\end{aligned}$$

Thus we obtain

$$\max_{a'\in PK_x(t)} d(a',PK_y(t)) \leq a\frac{1}{a}\Big(\frac{a}{b}\Big)^{r-2}[d(x,y)]^r = \Big(\frac{a}{b}\Big)^{r-2}[d(x,y)]^r. \tag{34}$$

Interchanging x and y in inequality (34) we have

$$\max_{b'\in PK_y(t)} d(b',PK_x(t)) \leq \Big(\frac{a}{b}\Big)^{r-2}[d(x,y)]^r. \tag{35}$$

Thus (33) implies

$$H(Tx,Ty) \leq \Big(\frac{a}{b}\Big)^{r-2}[d(x,y)]^r. \tag{36}$$

We note that $d(x,y) \leq \frac{b}{a}$ for ever $x,y \in \tilde{C}$. Setting $\varphi(t) = \left(\frac{a}{b}\right)^{r-2} t^r$ for $t \in J = [0,\frac{b}{a})$ where $0 < b < a$. Now for $0 < \lambda < 1$ and $t \in J$ we have

$$\varphi(\lambda t) = \lambda^r\Big(\frac{a}{b}\Big)^{r-2}t^r \leq \lambda^r\varphi(t). \tag{37}$$

Moreover, for $t \in (0,\frac{b}{a})$ we have $t < \frac{b}{a} < 1$. Thus

$$\varphi t = \Big(\frac{a}{b}\Big)^{r-2}t^r = \Big(\frac{a}{b}\Big)^{r-2}t^{r-2}t^2 < \Big(\frac{a}{b}\Big)^{r-2}\Big(\frac{b}{a}\Big)^{r-2}t^2 = t^2 < t. \tag{38}$$

From (37) and (38) it follows that φ is a gauge function of order $r \geq 2$. Hence we conclude that

$$H(Tx,Ty) \leq \varphi(d(x,y)) \quad \text{for all } x,y \in \tilde{C} \text{ with } d(x,y) \in J. \tag{39}$$

Which from Corollary 2.6 yields that starting from initial approximate $x_0 = \gamma$ the iterative sequence $x_n \in Tx_{n-1}; n = 1,2,\cdots$ converges to the fixed point ξ of T with the rate of convergence $r \geq 2$. □

Remark 3.2. *Observe that in most existence theorems for the solutions of integral equations or inclusions the kernel of equation $k(t,x(t))$ usually satisfies Lipschitz condition in some sense. Unlike to this, in our result the kernel satisfies condition (31) which is not Lipschitz since $r \geq 2$. Theorem 3.1 not only guarantees the existence of solution but it also proposes an iterative scheme with higher rate of convergence.*

We also include the following variant. Its proof can easily be established.

Theorem 3.3. *Consider the following integral inclusion*

$$x(t) \in P \int_{t_0}^{t} k(\tau, x(\tau))d\tau + Q \tag{40}$$

$$\in PK_x(t) + Q,$$

where $K_x(t) = \int_{t_0}^{t} k(\tau, x(\tau))d\tau$*,* P*,* Q *are compact subsets of real line* $\mathbb{R}$ *and* $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies conditions (1-3) of Theorem 3.1.*
Then for every $\gamma \in Q$ *the integral inclusion (40) has a solution on the interval* $I = [t_0 - \frac{a^{r-2}}{b^{r-1}}, t_0 + \frac{a^{r-2}}{b^{r-1}}]$.

Remark 3.4. *Setting* $P = \{1\}$ *Theorem 3.1 reduces to [25, Theorem 4.2].*

References

[1] H. Afshari, H. Aydi, E. Karapinar, Existence of fixed points of set-valued mappings in *b*-metric spaces, *East Asian Mathematical Journal* 32 (2016) 3 319–332 DOI:10.7858/eamj.2016.024.
[2] U. Aksoy, E. Karapinar, I. M. Erhan, Fixed points of generalized α-admissible contractions on *b*-metric spaces with an application to boundary value problems, *J. Nonlinear Convex Anal.* 17 (2016) 6 1095–1108.
[3] H. Alsulami, S. Glyaz, E. Karapinar, I. M. Erhan, An Ulam stability result on quasi-*b*-metric-like spaces, *Open Mathematics* 14 (2016) 1 DOI 10.1515/math-2016-0097.
[4] H. Aydi, M. F. Bota, E. Karapinar, S. Moradi, A common fixed point for weak-ϕ-Contractions on *b*-metric spaces, *Fixed Point Theory* 13 (2012) 2 337-346.
[5] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Functional Analysis, *U1'yanovsk Gos. Ped. Inst., U1'yanovsk* 30 (1989) 26–37.
[6] R. M. Bianchini, M. Grandolfi, Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* 45 (1968) 212–216.
[7] M. Boriceanu, M. Bota, A. Petruşel, Multivalued fractals in *b*-metirc spaces, *Cent. Eur. J. Math.* 8 (2010) 367.
[8] M. F. Bota, C. Chifu, E. Karapinar, Fixed point theorems for generalized $\alpha - \psi$-Ciric-type contractive multivalued operators in *b*-metric spaces, *J. Nonlinear Sci. Appl.* 9 (2016) 3 1165–1177.
[9] M. F. Bota, E. Karapinar, A note on "Some results on multi-valued weakly Jungck mappings in *b*-metric space", *Cent. Eur. J. Math.* 11 (2013) 9 1711–1712.
[10] M. F. Bota, E. Karapinar, O. Mlesnite, Ulam-Hyers stability results for fixed point problems via $\alpha - \psi$-contractive mapping in *b*-metric space, *Abstr. Appl. Anal.* (2013) Article Id: 825293.
[11] S. Czerwik, Nonlinear set-valued contraction mappings in *b*-metric spaces, *Atti. Semin. Mat. Fis. Univ. Modena.* 46 (1998) 263–276.
[12] S. Czerwik, Functional equations and inequalities in several variables, *World Scientific, Singapore* (2002).
[13] A. Gel'man, A certain fixed point principle, *Kokl. Akad. Nauk. SSSR* 198 (1971) 506–508 (in Russian); *Soviet Math. Kokl.* 12 (1971) 813–816 (Engl. translation).
[14] T. L. Hicks, B. E. Rhoades, A Banach type fixed point theorem, *Math. Jponica.* 24 (1979) 327–330.
[15] Selma GULYAZ OZYURT, On some alpha-admissible contraction mappings on Branciari b-metric spaces, 1(1),1-13 (2017), Article Id: 2017:1:1
[16] E. Karapinar, H. Piri, H. AlSulami, Fixed points of generalized F-Suzuki type contraction in complete *b*-metric spaces, *Discrete. Dyn. Nat. Soc.* (2015) Article ID 969726 8 pages.
[17] Q. Kiran, T. Kamran, Nadler's type principle with higher order of convergence, *Nonlinear Anal. TMA* 69 (2008) 4106–4120.
[18] Q. Kiran, T. Kamran, Fixed point theorems for generalized contractive multi-valued maps, *Comput. Math. Appl.* 59 (2010) 3812–3823.
[19] H. J. Kornstaedt, Funktionalungleichungen and iterationsverfahren, *Aequationes Math.* 13 (1975) 95–105.
[20] M. A. Kutbi , E. Karapinar, J. Ahmed, A. Azam, Some fixed point results for multi-valued mappings in *b*-metric spaces, *J. Inequal. Appl.* (2014) 126.
[21] S. B. Nadler Jr., Multivalued contraction mappings, *Pacific. J. Math.* 25 (1976) 279–285.
[22] S. Park, A unified approach to fixed points of contractive maps, *J. Korean. Math. Soc.* 16 (1980) 95–105.
[23] P. D. Proinov, A generalization of the Banach principle with high order of convergence of successive approximations, *Nonlinear Anal.TMA* 67 (2007) 2361–2369.
[24] W. C. Reinboldt, A unified convergence theory for a class of iterative processes, *SIAM, J. Numer. Anal.* 5 (1968) 42–63.
[25] M. Samreen, Q. Kiran, T. Kamran, Fixed point theorems for φ-contractions, *J. Inequal. Appl.* (2014) 1–16.
[26] M. Samreen, T. Kamran, N. Shahzad, Some fixed point theorems in *b*-metric space endowed with graph, *Abstract Applied Anal.* (2013) Artical ID: 967132.