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Topological Properties of a Pair of Relation-Based Approximation Operators

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Abstract. Rough set theory is an important tool for data mining. Lower and upper approximation operators are two important basic concepts in the rough set theory. The classical Pawlak rough approximation operators are based on equivalence relations and have been extended to relation-based generalized rough approximation operators. This paper presents topological properties of a pair of relation-based generalized rough approximation operators. A topology is induced by the pair of generalized rough approximation operators from an inverse serial relation. Then, connectedness, countability, separation property and Lindelöf property of the topological space are discussed. The results are not only beneficial to obtain more properties of the pair of approximation operators, but also have theoretical and actual significance to general topology.

1. Introduction

Rough set theory was proposed by Pawlak to conceptualize, organize and analyze various types of data in data mining. The rough set method is especially useful for dealing with vagueness and granularity in information systems. It deals with the approximation of an arbitrary subset of a universe by two definable subsets which are referred to as the lower and upper approximations. By using the lower and upper approximations of decision classes, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The lower and upper approximation operators in the Pawlak's rough set model [15] are induced by equivalence relations or partitions. However, the requirement of an equivalence relation or partition in the Pawlak's rough set model may limit the applications of the rough set model. Then, many authors have generalized the notion of approximation operators by using more general binary relations [3, 23, 25, 30, 31], by employing coverings [1, 2, 32, 35], by utilizing adjoint operators [14], or by considering the fuzzy environment [5, 12, 29].

Topology is a branch of mathematics. There exist near connections between topology and rough set theory. Many authors investigated topological structures of rough sets [8–10, 16–22, 27, 28, 31, 33, 35].

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Skowron explored the topic of topology in information systems [22]. Wiweger extended the Pawlak rough sets to topological rough sets [27]. Yao discussed the Pawlak's rough sets through topological properties of lower and upper approximation operations [31]. Lin and Liu investigated axioms for approximation operators within the framework of topological spaces [10]. Wu and Mi examined topological structure of generalized rough sets in infinite universes of discourse [28]. Polkowski defined the hit-or-miss topology on rough sets and proposed a scheme to approximate mathematical morphology within the general paradigm of soft computing [17, 18]. Kondo presented topological properties of a type of relation-based rough sets [8]. Qin et al. [20], Zhang et al. [33] and Li et al. [9] presented a further investigation of the pair of relation-based approximation operators studied in [8]. Pomykala studied topological properties of two pairs of covering-based rough set approximation oprators [19]. Zhu explored a type of covering-based rough sets by topological approach [35]. Zhang et al. presented topological properties of four pairs of relation-based generalized approximation operators [34].

The purpose of this paper is to discuss topological properties of a pair of relation-based generalized approximation operators. In Section 2, we present definitions and properties of the operators. In Section 3, we investigate connectedness, countability, separation property and Lindelöf property of the topological space induced by the operators, and present relationships between the connectedness of topological space and the existence of definable sets in rough sets to show an application of the results on topological structure of the relation-based generalized rough approximation operators.

2. Definitions and Properties of Generalized Approximation Operators

Suppose U is a non-empty set called the universe, and $\mathcal{P}(U)$ is the power set of U. For $X \subseteq U$, $-X$ is the complement of X in U. We do not restrict the universe to be finite.

Let U be a nonempty set and R a binary relation on U. For any $(x, y) \in U \times U$, if $(x, y) \in R$, then we say x has relation R with y, and denote this relationship as xRy. For a binary relation R, $\{(y, x)|(x, y) \in R\}$ is denoted by R^{-1}. For any $x \in U$, we call the set $\{y \in U|xRy\}$ the successor neighborhood of x in R and denote it as $R_s(x)$, and the set $\{y \in U|yRx\}$ the predecessor neighborhood of x in R and denote it as $R_p(x)$. Let R be a binary relation on U.

If for any $x \in U$, there exists a $y \in U$ such that yRx, then R is referred to as an inverse serial relation. In other words, if $\cup\{R_s(x)|x \in U\} = U$, then R is inverse serial.

If for any $x \in U$, xRx, then R is referred to as a reflexive relation. In other words, if for any $x \in U$, $x \in R_s(x)$, then R is reflexive.

If for any $x, y \in U$, $xRy \Rightarrow yRx$, then R is referred to as symmetric. In other words, if for any $x, y \in U$, $y \in R_s(x) \Rightarrow x \in R_s(y)$, then R is symmetric.

If for any $x, y, z \in U$, xRy and $yRz \Rightarrow xRz$, then R is referred to as transitive. In other words, if for any $x, y \in U$, $y \in R_s(x) \Rightarrow R_s(y) \subseteq R_s(x)$, then R is transitive.

If R is reflexive, symmetric and transitive, then R is referred to as an equivalence relation on U.

Clearly, a binary relation R is inverse serial if and only if $\{R_s(x)|x \in U\}$ is a cover of U. It is easy to see that a reflexive relation is inverse serial, but the converse does not hold. Besides reflexive, symmetric and transitive relations, inverse serial relation is ubiquitous in real life. We give two examples.

Example 2.1. An incomplete information system $S = (U, AT)$ is presented in Table 1, where $U = \{x_1, x_2, \cdots, x_6\}$, $AT = \{a, b\}$ is the conditional attribute set, a, b stand for systolic pressure, diastolic pressure, respectively. $V_a = \{\text{H, N, L}\}$, $V_b = \{\text{H, N, L}\}$, where H, N, and L stand for high, normal and low, respectively. For any $c \in AT$, $c : U \to V_c$, i.e., $c(x) \in V_c$ for all $x \in U$.

Table 1: An information system.

U	a	b
x_1	{H, N}	{N}
x_2	{H, N}	{N}
x_3	{L, N}	{H, N}
x_4	{L}	{L}
x_5	{H}	{H,N}
x_6	{H, N}	{H}

Define a binary relation R^1_A on U by $x\ R^1_A\ y$ if and only if $(-c(x)) \subseteq c(y)$ for all $c \in A$ ($A \subseteq AT$), which is called strong left orthogonality. Define a binary relation R^2_A on U by $x\ R^2_A\ y$ if and only if $c(x) \cap (-c(y)) \neq \emptyset$ for all $c \in A$ ($A \subseteq AT$), which is called weak right negative similarity [14].

In Table 1, we have $R^1_{\{a\}} = \{(x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_1), (x_3, x_2), (x_3, x_5), (x_3, x_6), (x_4, x_1), (x_4, x_2), (x_4, x_6), (x_5, x_3), (x_6, x_3), (x_6, x_4)\}$. $R^2_{\{b\}} = \{(x_1, x_4), (x_1, x_6), (x_2, x_4), (x_2, x_6), (x_3, x_1), (x_3, x_2), (x_3, x_4), (x_3, x_6), (x_4, x_1), (x_4, x_2), (x_4, x_3), (x_4, x_5), (x_5, x_1), (x_5, x_2), (x_5, x_4), (x_5, x_6), (x_6, x_1), (x_6, x_2), (x_6, x_4)\}$. Hence $R^1_{\{a\}}$ is inverse serial and symmetric, and $R^1_{\{a\}}$ is neither reflexive nor transitive. $R^2_{\{b\}}$ is inverse serial, and $R^2_{\{b\}}$ is not reflexive, symmetric or transitive.

Example 2.2. Let $U = \{a, b, c, d, e, f, g\}$ be a small class of students. The class has elected its leader. Define a binary relation on U by:

xRy if and only if x chooses y as the leader.

Suppose that the election result is $R = \{(a, a), (a, c), (b, a), (b, c), (c, b), (c, d), (d, e), (d, f), (e, f), (e, e), (f, a), (f, c), (g, b), (g, g)\}$. Then R is inverse serial, and R is not reflexive, symmetric or transitive.

Yao has generalized the Pawlak rough set model by using general binary relations [31], and presented the pair of relation-based generalized approximation operators.

Definition 2.3. ([31]) Let R be a binary relation on U. Define a pair of approximation operators $(\underline{apr}'', \overline{apr}'')$ by: for any $X \subseteq U$,

$$\underline{apr}''(X) = -\overline{apr}''(-X) = \{x | x \in R_s(y) \Rightarrow R_s(y) \subseteq X\} \cup (- \cup \{R_s(x) | x \in U\}),$$
$$\overline{apr}''(X) = \cup\{R_s(x) | R_s(x) \cap X \neq \varnothing\}.$$

If R is an inverse serial relation, then $\underline{apr}''(X) = \{x | x \in R_s(y) \Rightarrow R_s(y) \subseteq X\}$. It is easy to obtain that the approximation operators $(\underline{apr}'', \overline{apr}'')$ are the classical Pawlak approximation operators if R is an equivalence relation. We employ the next example to show that the approximation operators $(\underline{apr}'', \overline{apr}'')$ have practical applications.

Example 2.4. Continued from Example 2.2. For any $z \in U$, $\overline{apr}''(\{z\}) = \cup\{R_s(x) | R_s(x) \cap \{z\} \neq \varnothing\}$. Then, for any $y \in \overline{apr}'(\{z\}) \backslash \{z\}$, there exists an $x \in U$ such that $y \in R_s(x)$ and $R_s(x) \cap \{z\} \neq \emptyset$. Hence $\{y, z\} \subseteq R_s(x)$, which implies that x chooses z and y in the meantime. Therefore, y is a competitor of z.

It is easy to obtain some properties of the pairs of approximation operators $(\underline{apr}'', \overline{apr}'')$.

Proposition 2.5. *[31] Let R be a binary relation on the universe U. Then, for any $X, Y \subseteq U$,*

(1) $\underline{apr}''(U) = U$, $\overline{apr}''(\varnothing) = \varnothing$,

(2) if R is inverse serial, then $\underline{apr}''(X) \subseteq X \subseteq \overline{apr}''(X)$,

(3) $\underline{apr}''(X \cap Y) = \underline{apr}''(X) \cap \underline{apr}''(Y)$, $\overline{apr}''(X \cup Y) = \overline{apr}''(X) \cup \overline{apr}''(Y)$,

(4) $\overline{apr}''(\underline{apr}''(X)) \subseteq \underline{apr}''(X)$, $\overline{apr}''(X) \subseteq \overline{apr}''(\overline{apr}''(X))$,

(5) $\overline{apr}''(X) = \cup_{x \in X} \overline{apr}''(\{x\})$ for all $X \neq \emptyset$.

(6) $\underline{apr}''(X) = X \Leftrightarrow \overline{apr}''(X) = X$.

The properties (∗∗) of a binary relation R was introduced in [34] as a necessary and sufficient condition for $\overline{apr}''$ to be idempotent.

Definition 2.6. ([34]) R is said to have property (∗∗) if for any $x, y \in U$, whenever $\{u, v\} \subseteq R_s(x)$ and $\{v, w\} \subseteq R_s(y)$, there exists a $z \in U$ such that $\{u, w\} \subseteq R_s(z)$.

Theorem 2.7. ([34]) *Let R be a binary relation on the universe U. Then the following are equivalent:*
(1) R satisfies (∗∗),
(2) $\overline{apr}''(\overline{apr}''(X)) = \overline{apr}''(X)$ for all $X \subseteq U$,
(3) $\underline{apr}''(\underline{apr}''(X)) = \underline{apr}''(X)$ for all $X \subseteq U$.

3. Topological Properties of the Generalized Upper (Lower) Approximation Operators

Topology is a theory with many applications not only in almost all branches of mathematics, but also in many real life applications. Binary relation on a set is a simple mathematical model to which many real-life data can be connected. There exist many results on the relationships between topological spaces and binary relations. McCord and Stong presented that there is an isomorphism between partially ordered sets and Alexandroff T_0 topologies [11, 26]. Naturman extended the result to a duality between Alexandroff spaces and preorders [13]. Skowron discussed topologies induced by binary relations in information systems [22]. Šlapal represented ternary relations on a given set by topologies [24]. Girish and John discussed the multiset topologies induced by multiset relations [7]. There are also some results on the topological properties of the generalized approximation operators induced by binary relations. For example, Yao [30], Kondo [8], Qin et al. [20], Zhang et al. [33] and Li et al. [9] discussed topological properties of the type of approximation operators $(\underline{apr}, \overline{apr})$. Zhang et al. presented topological properties of four pairs of relation-based generalized approximation operators [34], and gave necessary and sufficient conditions for the relation-based generalized upper (lower) approximation operator to be a topological closure (interior) operator. In this section, we will discuss topological properties of the pair of approximation operators $(\underline{apr}'', \overline{apr}'')$ induced by an inverse serial relation. For the basic topological concepts, we refer to [6].

Definition 3.1. ([4, 6]) Let U be a non-empty set and $cl : \mathcal{P}(U) \to \mathcal{P}(U)$. For any $X, Y \subseteq U$, consider the following axioms:
(1) $cl(\emptyset) = \emptyset$,
(2) $X \subseteq cl(X)$,
(3) $cl(X \cup Y) = cl(X) \cup cl(Y)$,
(4) $cl(cl(X)) = cl(X)$,
(5) $cl(X) = \cup_{x \in X} cl(\{x\})$.
If cl satisfies (1)–(3), then cl is called a closure operator, and (U, cl) is called a closure space [4]. If cl satisfies (1)–(4), that is, cl satisfies Kuratowski closure axiom, then cl is called a topological closure operator [6]. If a closure operator cl satisfies (5), then cl is called a quasi-discrete closure operator [4].

In fact, in a closure space (U, cl), it is easy to prove that $\tau(cl) = \{-X | cl(X) = X\}$ is a topology. Similarly, the topological interior operator can be defined by corresponding axioms.

Proposition 3.2. ([34]) *Let R be a binary relation on U. Then the following are equivalent:*
(1) $\tau(\overline{apr}'') = \{X \subseteq U | \underline{apr}''(X) = X\} = \{-X | \overline{apr}''(X) = X\}$ is a topology,
(2) R is inverse serial,
(3) $\overline{apr}''$ is a closure operator,
(4) $\overline{apr}''$ is a quasi-discrete closure operator.

To present a necessary and sufficiency condition for $\overline{apr}''$ ($\underline{apr}''$) being a topological closure (interior) operator, we define a binary relation R' from the binary relation R by:

$xR'y$ if and only if there exists a $z \in U$ such that $\{x, y\} \subseteq R_s(z)$.

Proposition 3.3. ([34]) *Let R be a binary relation on U. Then the following are equivalent:*
(1) R' is an equivalence relation,
(2) R is inverse serial and satisfies (**),
(3) $\underline{apr}''$ is a topological interior operator,
(4) $\overline{apr}''$ is a topological closure operator.

By Proposition 3.3, the inverse seriality and the property (**) are necessary and sufficiency conditions for $\overline{apr}''$ ($\underline{apr}''$) to be a topological closure (interior) operator. In order to present more properties of the topological space $(U, \tau(\overline{apr}''))$, we define another binary relation.

Definition 3.4. Let R be a binary relation on U. For any $x, y \in U$, x, y are said to be linked if there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of U such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. In this case, we write $x\widetilde{R}y$.

Proposition 3.5. *Let R be a binary relation on U.*
(1) If R is inverse serial, then $\widetilde{R}$ is reflexive.
(2) $\widetilde{R}$ is symmetric.
(3) $\widetilde{R}$ is transitive.

Proof. (1) For any $x \in U$, since R is inverse serial, there exists a $y \in U$ such that $x \in R_s(y)$. Then $xR^{-1}y$ and yRx. Hence $x\widetilde{R}x$. It means that $\widetilde{R}$ is reflexive.

(2) For any $x, y \in U$, if $x\widetilde{R}y$, then there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of U such that $x = b_0R^{-1}a_1$, a_1Rb_1, $b_1R^{-1}a_2$, a_2Rb_2, $\cdots$, $b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Then we have $y = b_nR^{-1}a_n$, a_nRb_{n-1}, $b_{n-1}R^{-1}a_{n-1}$, $\cdots$, $b_1R^{-1}a_1$, a_1Rx. It follows that $y\widetilde{R}x$. Thus, $\widetilde{R}$ is symmetric.

(3) For any $x, y, z \in U$, if $x\widetilde{R}y$ and $y\widetilde{R}z$, then there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$, and we can find $c_1, c_2, \cdots, c_m$ and $d_1, d_2, \cdots, d_{m-1}$ of U such that $yR^{-1}c_1, c_1Rd_1, d_1R^{-1}c_2, c_2Rd_2, \cdots, d_{n-1}R^{-1}c_n$ and c_nRz. Then we have $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2$, a_2Rb_2, $\cdots$, $b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$, $yR^{-1}c_1$, c_1Rd_1, $d_1R^{-1}c_2$, c_2Rd_2, $\cdots$, $d_{n-1}R^{-1}c_n$ and c_nRz. Consequently, $x\widetilde{R}z$. □

From Proposition 3.5, we can obtain that if R is inverse serial, then $\widetilde{R}$ is an equivalence relationship, i.e., $U/\widetilde{R} \triangleq \{\widetilde{R}_s(x) | x \in U\}$ is a partition of U and $[x]_{\widetilde{R}} = \widetilde{R}_s(x)$ is an equivalence class.

Proposition 3.6. *If R is inverse serial, then for any $x \in U$*
(1) $\underline{apr}''([x]_{\widetilde{R}}) = [x]_{\widetilde{R}}$,
(2) $[x]_{\widetilde{R}}$ is an open and closed subset of $(U, \tau(\overline{apr}''))$.

Proof. (1) According to Proposition 2.5(2), we obtain $\underline{apr}''([x]_{\widetilde{R}}) \subseteq [x]_{\widetilde{R}}$. For any $y \in [x]_{\widetilde{R}}$, there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of U such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. For any $z \in U$ with $y \in R_s(z)$, we have $R_s(z) \subseteq [x]_{\widetilde{R}}$. Indeed, for any $u \in R_s(z)$, we get $yR^{-1}z$ and zRu. Then, there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}, y$ of U such that $x = b_0R^{-1}a_1, a_1Rb_1, b_1R^{-1}a_2, a_2Rb_2, \cdots, b_{n-1}R^{-1}a_n$, a_nRy, $yR^{-1}z$ and zRu. It implies that $u \in [x]_{\widetilde{R}}$. Hence we have $y \in \underline{apr}''([x]_{\widetilde{R}})$. Therefore, $[x]_{\widetilde{R}} \subseteq \underline{apr}''([x]_{\widetilde{R}})$.

(2) By (1), $[x]_{\widetilde{R}}$ is an open set. Then, according to Proposition 2.5(6), we deduce that $[x]_{\widetilde{R}}$ is a closed set. □

Proposition 3.7. *If R is inverse serial, then*
(1) for any $X \in \tau(\overline{apr}'')$ and $x \in X$, $[x]_{\widetilde{R}} \subseteq X$,
(2) $\{[x]_{\widetilde{R}}\}$ is an open neighborhood base of $x \in U$.

Proof. (1) For any $y \in [x]_{\widetilde{R}}$, there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of U such that $x = b_0R^{-1}a_1$, a_1Rb_1, $b_1R^{-1}a_2$, a_2Rb_2, $\cdots$, $b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Then $\{x, b_1\} \subseteq R_s(a_1)$, $\{b_1, b_2\} \subseteq R_s(a_2)$, $\cdots$, $\{b_{n-1}, y\} \subseteq R_s(a_n)$. Since $X \in \tau(\overline{apr}'')$, we have $\overline{apr}''(X) = X$. Hence $x \in \overline{apr}''(X)$. By Definition 2.3, we get $R_s(a_1) \subseteq X$. Then $b_1 \in X = \overline{apr}''(X)$, which implies $R_s(a_2) \subseteq X$. Thus $b_2 \in X = \overline{apr}''(X)$. In the same way, we have $y \in X$. Hence, we can conclude that $[x]_{\widetilde{R}} \subseteq X$.

(2) By Proposition 3.6(2) and (1), it is easy to see that $\{[x]_{\widetilde{R}}\}$ is an open neighborhood base of x. □

Proposition 3.8. *If R is inverse serial, then for any $x \in U$,*

(1) $\overline{\{x\}} = [x]_{\widetilde{R}}$.

(2) $[x]_{\widetilde{R}}$ is a connected component that contains x.

Proof. (1) According to Proposition 2.5(6), $X \subseteq U$ is an open set in $(U, \tau(\overline{apr}''))$, if and only if X is a closed set. Then, by Proposition 3.7, we have

$$\begin{aligned}\overline{\{x\}} &= \cap\{B | x \in B, \text{ and } B \text{ is closed}\}\\ &= \cap\{B | x \in B, \text{ and } B \text{ is open}\} = [x]_{\widetilde{R}}.\end{aligned}$$

(2) Assume that C_x is a connected component containing x. Then C_x is closed. It follows that C_x is open. By Proposition 3.7, we get that $[x]_{\widetilde{R}} \subseteq C_x$. Then $[x]_{\widetilde{R}} = C_x$. Otherwise, $[x]_{\widetilde{R}} \neq C_x$. Hence $[x]_{\widetilde{R}}$ is an open and closed proper subset of C_x, which contradicts that C_x is a connected component. Therefore, $[x]_{\widetilde{R}}$ is a connected component that contains x. □

Proposition 3.9. *If R is inverse serial, then $(U, \tau(\overline{apr}''))$ is a locally connected space.*

Proof. For any $x \in U$ and $A \in \tau(\overline{apr}'')$ with $x \in A$, we have $[x]_{\widetilde{R}} \subseteq A$. By Proposition 3.8, $[x]_{\widetilde{R}}$ is a connected set. Then $(U, \tau(\overline{apr}'')$ is a locally connected space. □

Theorem 3.10. *If R is an inverse serial relation on U, then $(U, \tau(\overline{apr}''))$ is connected if and only if $x\widetilde{R}y$ for all $x, y \in U$.*

Proof. "$\Leftarrow$". Suppose $U = X \cup Y$ and $X \cap Y = \emptyset$. Let $x \in X$ and $y \in Y$. By the assumption, we have $x\widetilde{R}y$. Then, there exist $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_{n-1}$ of U such that $x = b_0R^{-1}a_1$, a_1Rb_1, $b_1R^{-1}a_2$, a_2Rb_2, $\cdots$, $b_{n-1}R^{-1}a_n$ and $a_nRy = b_n$. Since $x = b_0 \in X$ and $y = b_n \notin X$, there exists a greatest i for which $b_i \in X$. Then $b_{i+1} \in Y$. By Definition 7, $a_{i+1}Rb_{i+1}$ and $b_iR^{-1}a_{i+1}$. It follows that $\{b_{i+1}\} \subseteq R_s(a_{i+1}) \cap Y \neq \emptyset$ and $\{b_i\} \subseteq R_s(a_{i+1}) \cap X \neq \emptyset$. Then $b_i \in \overline{apr}''(Y)$ and $b_i \notin Y$, $b_{i+1} \in \overline{apr}''(X)$ and $b_{i+1} \notin X$. Hence $\overline{apr}''(X) \neq X$ and $\overline{apr}''(Y) \neq Y$, which implies that X and Y are not closed. Thus, U is not the union of two disjoint closed sets, that is, U is connected.

"$\Rightarrow$". Suppose that there exist $x, y \in U$ such that $x\widetilde{R}y$ does not hold. Let $X = \{z \in U | x\widetilde{R}z\}$.

Then X is closed. In fact, for any $u \in \overline{apr}''(X)$, there exists a $v \in U$ such that $u \in R_s(v)$ and $R_s(v) \cap X \neq \emptyset$. Let $w \in R_s(v) \cap X$. Then $wR^{-1}v$ and vRu. Since $w \in X$, we get $u \in X$. Hence $\overline{apr}''(X) \subseteq X$. It follows that $\overline{apr}''(X) = X$, which implies that X is closed.

We are going to prove that $-X$ is closed. If not, $\overline{apr}''(-X) \nsubseteq -X$. Then there exists a $u \in U$ such that $u \in \overline{apr}''(-X)$ and $u \notin -X$. Hence there exists a $v \in U$ such that $u \in R_s(v)$ and $R_s(v) \cap (-X) \neq \emptyset$. Let $w \in R_s(v) \cap (-X)$. It follows that $uR^{-1}v$ and vRw. Since $u \in X$, we have $w \in X$, which contradicts the fact $w \in -X$.

Since $x \in X$ and $y \in -X$, U is the union of two disjoint non-empty closed sets, that is, U is not connected. □

Definition 3.11. ([6]) Let (U, τ) be a topological space. If $A \subseteq U$ is open in U if and only if A is closed in U, then (U, τ) is called a pseudo-discrete space. If the intersection of arbitrarily many open sets in U is still open, then τ is called an Alexandrov topology, and (U, τ) is said to be an Alexandrov space.

Proposition 3.12. *If R is an inverse serial relation on U, then*

(1) $(U, \tau(\overline{apr}''))$ is pseudo-discrete,

(2) $(U, \tau(\overline{apr}''))$ is an Alexandrov space.

Proof. (1) It is easy to prove that $\underline{apr}''(X) = X \Leftrightarrow \overline{apr}''(X) = X$ for all $X \subseteq U$. Then $(U, \tau(\overline{apr}'')$ is quasi-discrete.

(2) Since each open set in U is closed, the intersection of arbitrarily many open sets in U is still open. Hence $(U, \tau(\overline{apr}'')$ is an Alexandrov space. □

If R is inverse serial, by Proposition 3.12, for any $X \subseteq U$,

$X \subseteq U$ is definable by $\underline{apr}''$ and $\overline{apr}''$

$\Leftrightarrow \underline{apr}''(X) = X = \overline{apr}''(X)$

$\Leftrightarrow X$ is an open and closed set in $(U, \tau(\overline{apr}''))$

$\Leftrightarrow X \in \tau(\overline{apr}'')$.

Then the family of all definable subsets of U is $\tau(\overline{apr}'')$. On the other hand,

$(U, \tau(\overline{apr}''))$ is not connected

$\Leftrightarrow (U, \tau(\overline{apr}''))$ has non-empty open and closed proper subsets

$\Leftrightarrow (U, \tau(\overline{apr}''))$ has other definable sets besides $\emptyset$ and U.

$(U, \tau(\overline{apr}''))$ is connected

$\Leftrightarrow (U, \tau(\overline{apr}''))$ do not have non-empty open and closed proper subsets

$\Leftrightarrow$ the definable sets by $\underline{apr}''$ and $\overline{apr}''$ are no other than $\emptyset$ and U.

Hence we can note that $(U, \tau(\overline{apr}''))$ is connected, if and only if the definable sets by $\underline{apr}''$ and $\overline{apr}''$ are no other than $\emptyset$ and U. $(U, \tau(\overline{apr}''))$ is not connected, if and only if $(U, \tau(\overline{apr}''))$ has other definable sets besides $\emptyset$ and U. Thus, there exist relationships between the connectedness of topological spaces and the existence of definable sets in approximation spaces.

Proposition 3.13. *Let R be an inverse serial relation on U. Then*

(1) $(U, \tau(\overline{apr}''))$ is a first countable space.

(2) $(U, \tau(\overline{apr}''))$ is a locally separable space.

Proof. (1) By Proposition 3.7(2), we have that $\{[x]_{\widetilde{R}}\}$ is an open neighborhood base of x. Then $(U, \tau(\overline{apr}''))$ is first countable.

(2) According to Proposition 3.8(1), $\{x\}$ is a dense subset of $[x]_{\widetilde{R}}$, then $[x]_{\widetilde{R}}$ is separable. Hence, by Proposition 3.7(1), each neighborhood of x has separable subset $[x]_{\widetilde{R}}$. It implies that $(U, \tau(\overline{apr}''))$ is locally separable. □

Proposition 3.14. *Let R be an inverse serial relation on U. Then*

(1) $(U, \tau(\overline{apr}''))$ is a regular space,

(2) $(U, \tau(\overline{apr}''))$ is a normal space.

Proof. (1) For any $x \in U$ and closed set B with $x \notin B$, by Proposition 3.12, we have that B is open. Then there exist two disjoint open sets $U \backslash B$ and B such that $x \in U \backslash B$ and $B \subseteq B$. Hence $(U, \tau(\overline{apr}''))$ is regular.

(2) For each pair A, B of disjoint closed subsets of U, by Proposition 3.12, we have that A and B are open sets. Then there exist disjoint open sets A and B such that $A \subseteq A$ and $B \subseteq B$. It follows that $(U, \tau(\overline{apr}''))$ is normal. □

Proposition 3.15. *Let R be an inverse serial relation on U. Then the following are equivalent:*

(1) $U/\widetilde{R}$ is countable,

(2) $(U, \tau(\overline{apr}''))$ is second-countable,

(3) $(U, \tau(\overline{apr}''))$ is separable,

(4) $(U, \tau(\overline{apr}''))$ is a Lindelöf space.

Proof. (1)⇒(2). By Proposition 3.7, $\{[x]_{\widetilde{R}} | x \in U\}$ is a base of $(U, \tau(\overline{apr}''))$. Since $U/\widetilde{R} = \{[x]_{\widetilde{R}} | x \in U\}$ is countable, we obtain that $(U, \tau(\overline{apr}''))$ is second-countable.

(2)⇒(3). It is clear.

(3)⇒(4). Let C be an open cover of U, and D be a countable dense subset of U. For any $x \in D$, there exists $K_x \in C$ such that $x \in K_x$. Let $C_0 = \{K_x | x \in D\}$. Then C_0 is countable. Now we prove that C_0 is a cover of U. For any $y \in U$, there exists $K \in \mathcal{C}$ such that $y \in K$. Hence $[y]_{\widetilde{R}} \subseteq K$. Since D is a dense subset of U, we get $[y]_{\widetilde{R}} \cap D \neq \emptyset$. Let $x \in [y]_{\widetilde{R}} \cap D$. Then there exists a $K_x \in \mathcal{C}_0$ such that $x \in K_x$. It follows that $[x]_{\widetilde{R}} \subseteq K_x$. Since $x \in [y]_{\widetilde{R}}$ and $\widetilde{R}$ is an equivalence relation, we obtain $[x]_{\widetilde{R}} = [y]_{\widetilde{R}}$. Then $y \in [y]_{\widetilde{R}} = [x]_{\widetilde{R}} \subseteq K_x$. Therefore, we can conclude that $(U, \tau(\overline{apr}''))$ is a Lindelöf space.

(4)⇒(1). $U/\widetilde{R} = \{[x]_{\widetilde{R}} | x \in U\}$ is an open cover of U. Since $\widetilde{R}$ is an equivalence relation, $U/\widetilde{R}$ is the only subcover of $U/\widetilde{R}$. Since $(U, \tau(\overline{apr}''))$ is a Lindelöf space, we obtain that $U/\widetilde{R}$ is countable. □

Example 3.16. Continued from Example 2.2. We have
$\tau(\overline{apr}'') = \{\emptyset, \{a, c\}, \{e, f\}, \{b, d, g\}, \{a, c, e, f\}, \{a, c, b, d, g\}, \{e, f, b, d, g\}, U\}$,
$\widetilde{R} = \{(a, a), (a, c), (c, c), (c, a), (b, b), (b, d), (d, b), (d, d), (b, g), (g, b), (g, g), (g, d), (d, g), (e, e), (f, f), (e, f), (f, e)\}$.

Then $\widetilde{R}$ is an equivalence relation, and
$U/\widetilde{R} = \{\{a, c\}, \{e, f\}, \{b, d, g\}\}$.

We obtain that $(U, \tau(\overline{apr}''))$ is quasi-discrete, regular, normal and non-connected. $\tau(\overline{apr}'')$ is the family of definable sets by $\underline{apr}''$ and $\overline{apr}''$.

4. Conclusion

In this paper, we have investigated topological properties of a pairs of relation-based generalized approximation operators. We have discussed connectedness, countability, separation property and Lindelöf property of the topological space induced by the approximation operators. We have described relationships between the connectedness of topological space and the existence of definable sets in rough sets to show an application of the discussion of topological structure of the relation-based generalized rough approximation operators.

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