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# Differential Polynomials of Meromorphic Functions Sharing a Set of Values

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**Abstract.** In this paper, we investigate the uniqueness problem of meromorphic functions when nonlinear differential polynomials generated by them share a set of values with finite weight and obtain some results which generalize the results due to H.Y. Xu [ J. Computational Analysis and Applications 16 (2014) 942–954].

## 1. Introduction, Definitionnitions and Results

In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We shall use the standard notations of the Nevanlinna's theory of meromorphic functions as explained in [4], [5] and [21]. For a nonconstant meromorphic function $f$, we denote by $T(r, f)$ the Nevanlinna Characteristic function of $f$ and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of finite logarithmic measure. For $a \in \mathbb{C} \cup \{\infty\}$ and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \text{ counting multiplicity}\},$$
$$\overline{E}(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \text{ ignoring multiplicity}\}.$$

Let $f$ and $g$ be two nonconstant meromorphic functions. If $E(S, f) = E(S, g)$, we say that $f$ and $g$ share the set $S$ CM and if $\overline{E}(S, f) = \overline{E}(S, g)$, we say that $f$ and $g$ share the set $S$ IM. Especially if $S = \{a\}$, we say that $f$ and $g$ share the value $a$ CM when $E(S, f) = E(S, g)$ and $a$ IM when $\overline{E}(S, f) = \overline{E}(S, g)$.

Let $k$ be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_{k)}(a, f)$ the set of all $a$-points of $f$ whose multiplicities are not greater than $k$. Also by $\overline{E}_{k)}(a, f)$, we denote the set of all distinct $a$-points of $f$ whose multiplicities are not greater than $k$. If $E_{\infty)}(a, f) = E_{\infty)}(a, g)$, we say that $f$ and $g$ share the value $a$ CM and if $\overline{E}_{\infty)}(a, f) = \overline{E}_{\infty)}(a, g)$, we say that $f$ and $g$ share the value $a$ IM. For $a \in \mathbb{C} \cup \{\infty\}$ and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_{k)}(S, f) = \bigcup_{a\in S} E_{k)}(a, f) \text{ and } \overline{E}_{k)}(S, f) = \bigcup_{a\in S} \overline{E}_{k)}(a, f).$$

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Many research works on differential polynomials of meromorphic functions sharing certain value have been done by many mathematicians worldwide (see [1], [3], [10], [12], [14], [15], [16], [17], [19], [20]). Recently, there have been an increasing interest in studying differential polynomials of meromorphic functions sharing a set of values. In this direction we need the following definitions.

**Definition 1.1.** *[6, 7] Let $f$ be a nonconstant meromorphic function and $k$ be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{k)}(r, a; f)$ the counting function of those a-points of $f$ whose multiplicities are not greater than $k$ and by $\overline{N}_{k)}(r, a; f)$ the corresponding reduced counting function of $f$. We denote $N_{(k+1}(r, a; f)$ by the counting function of those a-points of $f$ whose multiplicities are greater than $k$ and $\overline{N}_{(k+1}(r, a; f)$ by the corresponding reduced counting function of $f$.*

**Definition 1.2.** *[7] Let $f$ be a nonconstant meromorphic function and $k$ be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_k(r, a; f)$ the counting function of a-points of $f$, where an a-point of multiplicity $m$ is counted $m$ times if $m \le k$ and $k$ times if $m > k$. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_{(2}(r, a; f) + \ \ldots \ + \overline{N}_{(k}(r, a; f).$$

*Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$.

**Definition 1.3.** *[21] Let $f$ and $g$ be two nonconstant meromorphic functions such that $f$ and $g$ share the value 1 IM. We denote by $N_E^{1)}(r, 1; f)$ the counting function of common simple 1-points of $f$ and $g$.*

In 1997, C.C. Yang, X.H. Hua [18] proved the following theorem.

**Theorem 1.1.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $n$ $(\ge 11)$ be an integer. If $f^n f'$ and $g^n g'$ share the value $a$ CM where $a \in \mathbb{C}\backslash\{0\}$, then either $f = tg$ for $t^{n+1} = 1$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where $c$, $c_1$ and $c_2$ are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Regarding Theorem 1.1, one may ask the following question.

**Question 1.1.** *Whether there exists a differential polynomial $d$ such that for any pair of nonconstant meromorphic functions $f$ and $g$ we can get $f \equiv g$ when $d(f)$ and $d(g)$ share one value CM?*

In 2002, C.Y. Fang, M.L. Fang [2] and in 2004, W.C. Lin, H.X. Yi [9] gave a positive answer to the above question and proved the following results respectively.

**Theorem 1.2.** *[2] Let $f$ and $g$ be two nonconstant meromorphic functions, and $n$ be a positive integer. If $E_{k)}(1, f^n(f-1)^2 f') = E_{k)}(1, g^n(g-1)^2 g')$ and one of the following conditions is satisfied: (a) $k \ge 3$ and $n \ge 13$, (b) $k = 2$ and $n \ge 15$, (c) $k = 1$ and $n \ge 23$, then $f \equiv g$.*

**Theorem 1.3.** *[9] Let $f$ and $g$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n+1}$ and $n$ $(\ge 12)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In 2006, I. Lahiri, R. Pal [8] also proved the following results corresponding to the above question.

**Theorem 1.4.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$ $(\ge 13)$ be an integer. If $E_{3)}(1, f^n(f-1)^2 f') = E_{3)}(1, g^n(g-1)^2 g')$, then $f \equiv g$.*

**Theorem 1.5.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$ $(\ge 14)$ be an integer. If $E_{3)}(1, f^n(f^3-1)f') = E_{3)}(1, g^n(g^3-1)g')$, then $f \equiv g$.*

In 2014, H.Y. Xu [13] investigated the uniqueness of meromorphic functions when differential polynomials generated by them share a set $S_m = \{1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{m-1}\}$, where $\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$, $m$ is an integer and obtained the following results.

**Theorem 1.6.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ $(\geq 2)$ be two positive integers. For $a,\ b \in \mathbb{C}\backslash\{0\}$, let $E_{k)}(S_m, f^n(f-a)(f-b)f') = E_{k)}(S_m, g^n(g-a)(g-b)g')$ and let $f$ or $g$ be meromorphic function having only multiple poles. If the expressions $\frac{a+b}{n+2} g \sum_{s=0}^{n+1} \left(\frac{f}{g}\right)^s - \frac{ab}{n+1} \sum_{s=0}^{n} \left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+2} \left(\frac{f}{g}\right)^s$ have no common simple zeros and one of the following conditions holds:*

*(i) $k \geq 3$: $n > 4 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;*

*(ii) $k = 2$: $n > 4 + \frac{11}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;*

*(iii) $k = 1$: $n > 4 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$,*

*then $f \equiv g$.*

**Theorem 1.7.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ $(\geq 2)$ be two positive integers. If $E_{k)}(S_m, f^n(f-a)^2 f') = E_{k)}(S_m, g^n(g-a)^2 g')$ and one of the following conditions holds:*

*(i) $k \geq 3$ and $n > 4 + \frac{8}{m}$;*

*(ii) $k = 2$ and $n > \max\{4 + \frac{4}{m}, 2 + \frac{10}{m}\}$;*

*(iii) $k = 1$: $n > 4 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$,*

*then $f \equiv g$.*

**Note 1.1.** *There are some lacuna in the lower bound of $n$ in Theorem 1.7. In the proof (not given in details) of the theorem, Case (i) of Lemma 2.4 [13] is required where the lower bound of $n$ is taken as $n \geq 8$.*

In this paper we consider the more general differential polynomial namely, $f^n(f-a)(f-b)(f-c)f'$ where $a,\ b,\ c \in \mathbb{C}\backslash\{0\}$ and obtain the following results.

**Theorem 1.8.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ $(\geq 2)$ be two positive integers. Let $E_{k)}(S_m, f^n(f-a)(f-b)(f-c)f') = E_{k)}(S_m, g^n(g-a)(g-b)(g-c)g')$, where $a,\ b,\ c \in \mathbb{C}\backslash\{0\}$ and $a \neq b \neq c$ and let $f$ or $g$ be meromorphic function having only multiple poles. If the expressions $\frac{a+b+c}{n+3} g^2 \sum_{s=0}^{n+2} \left(\frac{f}{g}\right)^s - \frac{ab+bc+ca}{n+2} g \sum_{s=0}^{n+1} \left(\frac{f}{g}\right)^s + \frac{abc}{n+1} \sum_{s=0}^{n} \left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3} \left(\frac{f}{g}\right)^s$ have no common simple zeros and one of the following conditions holds:*

*(i) $k \geq 3$: $n > 5 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$;*

*(ii) $k = 2$: $n > 5 + \frac{23}{2m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$;*

*(iii) $k = 1$: $n > 5 + \frac{22}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$,*

*then $f \equiv tg$, where $t^m = 1$.*

**Theorem 1.9.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ $(\geq 2)$ be two positive integers. Let $E_{k)}(S_m, f^n(f-a)^2(f-b)f') = E_{k)}(S_m, g^n(g-a)^2(g-b)g')$, where $a,\ b \in \mathbb{C}\backslash\{0\}$, $a \neq b$ and let $f$ or $g$ be meromorphic function having only multiple poles. If the expressions $\frac{2a+b}{n+3} g^2 \sum_{s=0}^{n+2} \left(\frac{f}{g}\right)^s - \frac{a^2+2ab}{n+2} g \sum_{s=0}^{n+1} \left(\frac{f}{g}\right)^s + \frac{a^2 b}{n+1} \sum_{s=0}^{n} \left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3} \left(\frac{f}{g}\right)^s$ have no common simple zeros and one of the following conditions holds:*

*(i) $k \geq 3$: $n > 3 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{2}{m}$ when $m \geq 4$;*

*(ii) $k = 2$: $n > 3 + \frac{11}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{2}{m}$ when $m \geq 4$;*

*(iii) $k = 1$: $n > 3 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{2}{m}$ when $m \geq 4$,*

*then $f \equiv tg$, where $t^m = 1$.*

**Theorem 1.10.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ $(\geq 2)$ be two positive integers. For $a \in \mathbb{C}\backslash\{0\}$, let $E_{k)}(S_m, f^n(f-a)^3 f') = E_{k)}(S_m, g^n(g-a)^3 g')$ and let $f$ or $g$ be meromorphic function having only multiple poles. If the expressions $\frac{3a}{n+3} g^2 \sum_{s=0}^{n+2} \left(\frac{f}{g}\right)^s - \frac{3a^2}{n+2} g \sum_{s=0}^{n+1} \left(\frac{f}{g}\right)^s + \frac{a^3}{n+1} \sum_{s=0}^{n} \left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3} \left(\frac{f}{g}\right)^s$ have no common simple zeros and one of the following conditions holds:*

*(i) $k \geq 3$: $n > \max\{10,\ 2 + \frac{8}{m}\}$ when $2 \leq m \leq 3$ and $n > \max\{10,\ \frac{8}{m}\}$ when $m \geq 4$;*

*(ii) $k = 2$: $n > \max\{10,\ 2 + \frac{10}{m}\}$ when $2 \leq m \leq 3$ and $n > \max\{10,\ \frac{10}{m}\}$ when $m \geq 4$;*

*(iii) $k = 1$: $n > \max\{10,\ 2 + \frac{16}{m}\}$ when $2 \leq m \leq 3$ and $n > \max\{10,\ \frac{16}{m}\}$ when $m \geq 4$,*

*then $f \equiv tg$, where $t^m = 1$.*

## 2. Lemmas

We now state some lemmas which will be needed in the sequel.

**Lemma 2.1.** *[11] Let $f$ be a nonconstant meromorphic function and let*

$$R(f)=\frac{a_pf^p+a_{p-1}f^{p-1}+\ \dots\ +a_1f+a_0}{b_qf^q+b_{q-1}f^{q-1}+\ \dots\ +b_1f+b_0}$$

*be an irreducible rational function in $f$ where $a_p$ $(\neq 0)$, $a_{p-1}, \ \dots\ , a_1, a_0$ and $b_q$ $(\neq 0)$, $b_{q-1}, \ \dots\ , b_1, b_0$ are constants. Then*

$$T(r,R(f))=dT(r,f)+S(r,f),$$

*where $d=\max\{p,\ q\}$.*

**Lemma 2.2.** *[21] Let $f$ be a nonconstant meromorphic function and $k$ be a positive integer. Then*

$$T(r,f^{(k)})\leq T(r,f)+k\overline{N}(r,\infty;f)+S(r,f)$$

*and*

$$N(r,0;f^{(k)})\leq N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

**Lemma 2.3.** *[2] Let $f$ and $g$ be two nonconstant meromorphic functions and $k$ be a positive integer. If $E_{k)}(1,f)=E_{k)}(1,g)$, then one of the following cases holds:*

$$\begin{aligned}(i)\ \ T(r,f)+T(r,g)\ \ &\leq\ \ N_2(r,\infty;f)+N_2(r,0;f)+N_2(r,\infty;g)+N_2(r,0;g)+\overline{N}(r,1;f)+\overline{N}(r,1;g)\\ &\quad -N_E^{1)}(r,1;f)+\overline{N}_{(k+1}(r,1;f)+\overline{N}_{(k+1}(r,1;g)+S(r,f)+S(r,g);\end{aligned}$$

*(ii) $f=\frac{(B+1)g+(A-B-1)}{Bg+(A-B)}$, where $A$ $(\neq 0)$, $B$ are two constants.*

**Lemma 2.4.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n>\frac{2}{m}+\frac{4}{m^2}-1$. If one of $f$ and $g$ is meromorphic function having only multiple poles and the expressions $\frac{a+b+c}{n+3}g^2\sum_{s=0}^{n+2}\left(\frac{f}{g}\right)^s-\frac{ab+bc+ca}{n+2}g\sum_{s=0}^{n+1}\left(\frac{f}{g}\right)^s+\frac{abc}{n+1}\sum_{s=0}^{n}\left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3}\left(\frac{f}{g}\right)^s$ have no common simple zeros, and*

$$\begin{aligned}&\left(\frac{1}{n+4}f^{n+4}-\frac{a+b+c}{n+3}f^{n+3}+\frac{ab+bc+ca}{n+2}f^{n+2}-\frac{abc}{n+1}f^{n+1}\right)^m\\ \equiv\ &\left(\frac{1}{n+4}g^{n+4}-\frac{a+b+c}{n+3}g^{n+3}+\frac{ab+bc+ca}{n+2}g^{n+2}-\frac{abc}{n+1}g^{n+1}\right)^m,\end{aligned}$$

*where $a,\ b,\ c\in\mathbb{C}\backslash\{0\}$, then $f\equiv tg$ where $t^m=1$.*

*Proof.* From the assumption of Lemma 2.4, we have

$$\begin{aligned}&\left(\frac{1}{n+4}f^{n+4}-\frac{a+b+c}{n+3}f^{n+3}+\frac{ab+bc+ca}{n+2}f^{n+2}-\frac{abc}{n+1}f^{n+1}\right)\\ \equiv\ \ &t\left(\frac{1}{n+4}g^{n+4}-\frac{a+b+c}{n+3}g^{n+3}+\frac{ab+bc+ca}{n+2}g^{n+2}-\frac{abc}{n+1}g^{n+1}\right),\end{aligned}\tag{1}$$

where $t^m=1$. From (1), we see that $f$ and $g$ share $\infty$ CM. Without loss of generality, we assume that $g$ has some multiple poles. Put $h=\frac{f}{g}$. Suppose that $h$ is not constant. Then from (1), we have

$$Ag^3(h^{n+4}-t)+Bg^2(h^{n+3}-t)+Cg(h^{n+2}-t)+D(h^{n+1}-t)\equiv 0,$$

$$\text{i.e.,}\qquad Ag^3 = -Bg^2\frac{h^{n+3}-t}{h^{n+4}-t} - Cg\frac{h^{n+2}-t}{h^{n+4}-t} - D\frac{h^{n+1}-t}{h^{n+4}-t}, \tag{2}$$

where $A = \frac{1}{n+4}$, $B = -\frac{a+b+c}{n+3}$, $C = \frac{ab+bc+ca}{n+2}$ and $D = -\frac{abc}{n+1}$.

Let $z_0$ be a pole of $g$ with multiplicity $p_0$ ($\geq 2$), which is not a zero of $h - u_{k,r}$ where $u_{k,r}^{n+4} = t = \omega^r$ ($k = 0, 1, \ldots, n+3$; $r = 0, 1, \ldots, m-1$) such that $\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$. From (2), we have $3p_0 = 2p_0$ i.e., $p_0 = 0$. Thus, we get a contradiction. Hence the poles of $g$ are precisely the zeros of $h - u_{k,r}$.

Let $z_1$ be a zero of $h - u_{k,r}$ with multiplicity $q_1$ which is a pole of $g$ with multiplicity $p_1$. From (2), we have $3p_1 = 2p_1 + q_1$ i.e., $p_1 = q_1$. Since $g$ has no simple pole, it follows that such points are multiple zeros of $h - u_{k,r}$. For $r = 0, 1, \ldots, m-1$, we obtain from (2),

$$Ag^3 = -\frac{Bg^2(h^{n+3}-\omega^r) + Cg(h^{n+2}-\omega^r) + D(h^{n+1}-\omega^r)}{h^{n+4}-\omega^r}, \tag{3}$$

where $A = \frac{1}{n+4}$, $B = -\frac{a+b+c}{n+3}$, $C = \frac{ab+bc+ca}{n+2}$ and $D = -\frac{abc}{n+1}$.

Let $z_2$ be a simple zero of $h - u_{k,r}$ ($k = 0, 1, \ldots, n+3$; $r = 0, 1, 2, \ldots, m-1$) which is a zero of multiplicity $q_2$ ($\geq 2$) of numerator of (3). Then from (3), we see that $z_2$ would be a zero of $g^3$ of order $q_2 - 1$. Therefore $z_2$ would be a zero of $h^{n+1} - \omega^r$. We note that the number of common factors of $h^{n+1} - w^r$ and $h^{n+4} - w^r$ are less or equal to the number of common factors of $h^{m(n+1)} - 1$ and $h^{m(n+4)} - 1$ for $r = 0, 1, 2, \ldots, m-1$. Since $\gcd(m(n+1), m(n+4))$ is either $m$ or $3m$, it follows that $h^{n+1} - \omega^r$ and $h^{n+4} - \omega^r$ may have at most $3m$ common factors for $r = 0, 1, 2, \ldots, m-1$, where $\gcd(p, q)$ means greatest common divisor of $p$ and $q$. Again a meromorphic function can not have more than two Picard exceptional values. Therefore, we see that $h - u_{k,r}$ has multiple zeros for at least $(n+4)m - 3m - 2$ values of $k \in \{0, 1, \ldots, m(n+4)-1\}$ when $r = 0, 1, 2, \ldots, m-1$. Thus, $\Theta(u_{k,r}; h) \geq \frac{1}{2}$ for at least $(n+4)m - 3m - 2$ values of $k \in \{0, 1, 2, \ldots, m(n+4)-1\}$ when $r = 0, 1, 2, \ldots, m-1$, which is a contradiction as $n > \frac{2}{m} + \frac{4}{m^2} - 1$. Hence $h$ is a constant. If $h \neq t$, from (2) we see that $g$ will be a constant function which is impossible. Thus $f \equiv tg$ where $t^m = 1$. This proves the Lemma. □

**Lemma 2.5.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n > 2$. Then*

$$(f^n(f-a)(f-b)(f-c)f')^m(g^n(g-a)(g-b)(g-c)g')^m \not\equiv 1,$$

*where $a, b, c \in \mathbb{C}\backslash\{0\}$ and $a \neq b \neq c$.*

*Proof.* In the contrary, we may assume that

$$(f^n(f-a)(f-b)(f-c)f')^m(g^n(g-a)(g-b)(g-c)g')^m \equiv 1.$$

Then

$$f^n(f-a)(f-b)(f-c)f'g^n(g-a)(g-b)(g-c)g' \equiv t, \tag{4}$$

where $t^m = 1$. Let $z_0$ be a zero of $f$ with multiplicity $p_0$. Then from (4), we see that $z_0$ is a pole of $g$ (say with multiplicity $q_0$). Thus, we have $np_0 + p_0 - 1 = nq_0 + 3q_0 + q_0 + 1$, i.e., $3q_0 + 2 = (n+1)(p_0 - q_0) \geq n+1$, i.e., $q_0 \geq \frac{n-1}{3}$. Hence, we obtain

$$(n+1)p_0 \geq \frac{(n+4)(n-1)+6}{3} \quad \text{i.e.,} \quad p_0 \geq \frac{n+2}{3}. \tag{5}$$

Let $z_1$ be a zero of $f - a$ with multiplicity $p_1$. Then from (4), we see that $z_1$ is a pole of $g$ (say with multiplicity $q_1$). Thus, we have $p_1 + p_1 - 1 = (n+4)q_1 + 1$, i.e., $2p_1 = (n+4)q_1 + 2$. Hence, we obtain

$$p_1 \geq \frac{(n+4)q_1 + 2}{2} \geq \frac{n+6}{2}. \tag{6}$$

We can get the similar results for the zeros of $f-b$ and $f-c$. Similarly, we get the same results for the zeros of $g(g-a)(g-b)(g-c)$.

Since a pole of $f$ is either a zero of $g(g-a)(g-b)(g-c)$ or a zero of $g'$, we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g)+\overline{N}(r,a;g)+\overline{N}(r,b;g)+\overline{N}(r,c;g)+\overline{N_0}(r,0;g')+S(r,f)+S(r,g)\\ &\leq \frac{3}{n+2}N(r,0;g)+\frac{2}{n+6}N(r,a;g)+\frac{2}{n+6}N(r,b;g)+\frac{2}{n+6}N(r,c;g)\\ &\quad+\overline{N_0}(r,0;g')+S(r,f)+S(r,g)\\ &\leq \left(\frac{3}{n+2}+\frac{6}{n+6}\right)T(r,g)+\overline{N_0}(r,0;g')+S(r,f)+S(r,g), \qquad (7)\end{aligned}$$

where $\overline{N_0}(r,0;g')$ denotes the reduced counting function of those zeros of $g'$ which are not the zeros of $g(g-a)(g-b)(g-c)$.

By the second fundamental theorem of Nevanlinna and from (5)–(7), we obtain

$$\begin{aligned}3T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,a;f)+\overline{N}(r,b;f)+\overline{N}(r,c;f)+\overline{N}(r,\infty;f)-\overline{N_0}(r,0;f')+S(r,f)\\ &\leq \frac{3}{n+2}N(r,0;f)+\frac{2}{n+6}N(r,a;f)+\frac{2}{n+6}N(r,b;f)+\frac{2}{n+6}N(r,c;f)\\ &\quad+\left(\frac{3}{n+2}+\frac{6}{n+6}\right)T(r,g)+\overline{N_0}(r,0;g')-\overline{N_0}(r,0;f')+S(r,f)+S(r,g)\\ &\leq \left(\frac{3}{n+2}+\frac{6}{n+6}\right)\{T(r,f)+T(r,g)\}+\overline{N_0}(r,0;g')-\overline{N_0}(r,0;f')+S(r,f)+S(r,g). \qquad (8)\end{aligned}$$

Similarly, we have

$$3T(r,g) \leq \left(\frac{3}{n+2}+\frac{6}{n+6}\right)\{T(r,f)+T(r,g)\}+\overline{N_0}(r,0;f')-\overline{N_0}(r,0;g')+S(r,f)+S(r,g). \qquad (9)$$

Adding (8) and (9) we get

$$\left(3-\frac{6}{n+2}-\frac{12}{n+6}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction as $n>2$ and the proof of Lemma 2.5 is complete. □

**Lemma 2.6.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n>4$. Then*

$$(f^n(f-a)^2(f-b)f')^m(g^n(g-a)^2(g-b)g')^m\not\equiv 1,$$

*where $a,\ b\in\mathbb{C}\backslash\{0\}$ and $a\neq b$.*

*Proof.* In the contrary, we may assume that

$$(f^n(f-a)^2(f-b)f')^m(g^n(g-a)^2(g-b)g')^m\equiv 1.$$

Then

$$f^n(f-a)^2(f-b)f'g^n(g-a)^2(g-b)g'\equiv t, \qquad (10)$$

where $t^m=1$. Let $z_0$ be a zero of $f$ with multiplicity $p_0$. Then from (10), we see that $z_0$ is a pole of $g$ (say with multiplicity $q_0$). Thus we have $np_0+p_0-1=nq_0+3q_0+q_0+1$, i.e., $3q_0+2=(n+1)(p_0-q_0)\geq n+1$, i.e., $q_0\geq\frac{n-1}{3}$. Hence we obtain

$$(n+1)p_0\geq\frac{(n+4)(n-1)+6}{3} \quad \textit{i.e.,} \quad p_0\geq\frac{n+2}{3}. \qquad (11)$$

Let $z_1$ be a zero of $f-a$ with multiplicity $p_1$. Then from (10), we see that $z_1$ is a pole of $g$ (say with multiplicity $q_1$). Thus we have $2p_1 + p_1 - 1 = (n+4)q_1 + 1$, i.e., $3p_1 = (n+4)q_1 + 2$. Hence

$$p_1 \geq \frac{(n+4)q_1 + 2}{3} \geq \frac{n+6}{3}. \tag{12}$$

Let $z_2$ be a zero of $f-b$ with multiplicity $p_2$. Then from (10), we see that $z_2$ is a pole of $g$ (say with multiplicity $q_2$). Thus we have $p_2 + p_2 - 1 = (n+4)q_2 + 1$, i.e., $2p_2 = (n+4)q_2 + 2$. Hence we obtain

$$p_2 \geq \frac{(n+4)q_2 + 2}{2} \geq \frac{n+6}{2}. \tag{13}$$

Similarly, we have the same results for the zeros of $g(g-a)(g-b)$.

Since a pole of $f$ is either a zero of $g(g-a)(g-b)$ or a zero of $g'$, we have

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g) \\
&\leq \frac{3}{n+2}N(r,0;g) + \frac{3}{n+6}N(r,a;g) + \frac{2}{n+6}N(r,b;g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g) \\
&\leq \left(\frac{3}{n+2} + \frac{5}{n+6}\right)T(r,g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g),
\end{aligned} \tag{14}$$

where $\overline{N_0}(r,0;g')$ denotes the reduced counting function of those zeros of $g'$ which are not the zeros of $g(g-a)(g-b)$. By the second fundamental theorem of Nevanlinna and from (11)–(14), we obtain

$$\begin{aligned}
2T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) - \overline{N_0}(r,0;f') + S(r,f) \\
&\leq \frac{3}{n+2}N(r,0;f) + \frac{3}{n+6}N(r,a;f) + \frac{2}{n+6}N(r,b;f) + \left(\frac{3}{n+2} + \frac{5}{n+6}\right)T(r,g) \\
&\quad + \overline{N_0}(r,0;g') - \overline{N_0}(r,0;f') + S(r,f) + S(r,g) \\
&\leq \left(\frac{3}{n+2} + \frac{5}{n+6}\right)\{T(r,f) + T(r,g)\} + \overline{N_0}(r,0;g') - \overline{N_0}(r,0;f') + S(r,f) + S(r,g).
\end{aligned} \tag{15}$$

Similarly,

$$2T(r,g) \leq \left(\frac{3}{n+2} + \frac{5}{n+6}\right)\{T(r,f) + T(r,g)\} + \overline{N_0}(r,0;f') - \overline{N_0}(r,0;g') + +S(r,f) + S(r,g). \tag{16}$$

Adding (15) and (16) we get

$$\left(2 - \frac{6}{n+2} - \frac{10}{n+6}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which contradicts to the assumption that $n > 4$. This proves the lemma. □

**Lemma 2.7.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers satisfying $n > 10$. Then*

$$(f^n(f-a)^3 f')^m (g^n(g-a)^3 g')^m \not\equiv 1,$$

*where $a \in \mathbb{C}\backslash\{0\}$.*

*Proof.* If possible, we may assume that

$$(f^n(f-a)^3 f')^m (g^n(g-a)^3 g')^m \equiv 1.$$

Then

$$f^n(f-a)^3 f' g^n (g-a)^3 g' \equiv t, \tag{17}$$

where $t^m = 1$.

Let $z_0$ be a zero of $f$ with multiplicity $p_0$. Then from (17), we see that $z_0$ is a pole of $g$ (say with multiplicity $q_0$). Thus, we have $np_0 + p_0 - 1 = nq_0 + 3q_0 + q_0 + 1$, i.e., $3q_0 + 2 = (n+1)(p_0 - q_0) \geq n+1$, i.e., $q_0 \geq \frac{n-1}{3}$. Hence we obtain

$$(n+1)p_0 \geq \frac{(n+4)(n-1)+6}{3} \quad i.e., \quad p_0 \geq \frac{n+2}{3}. \tag{18}$$

Let $z_1$ be a zero of $f-a$ with multiplicity $p_1$. Then from (17), we see that $z_1$ is a pole of $g$ (say with multiplicity $q_1$). Thus we have $3p_1 + p_1 - 1 = (n+4)q_1 + 1$, i.e., $4p_1 = (n+4)q_1 + 2$. Hence

$$p_1 \geq \frac{(n+4)q_1 + 2}{4} \geq \frac{n+6}{4}. \tag{19}$$

Similarly, we have the same results for the zeros of $g(g-a)$.

Since a pole of $f$ is either a zero of $g(g-a)$ or a zero of $g'$, we have

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,a;g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g)\\
&\leq \frac{3}{n+2}N(r,0;g) + \frac{4}{n+6}N(r,a;g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g)\\
&\leq \left(\frac{3}{n+2} + \frac{4}{n+6}\right)T(r,g) + \overline{N_0}(r,0;g') + S(r,f) + S(r,g),
\end{aligned} \tag{20}$$

where $\overline{N_0}(r,0;g')$ is the reduced counting function of those zeros of $g'$ which are not the zeros of $g(g-a)$.

By the second fundamental theorem of Nevanlinna and from (18)–(20), we obtain

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f) - \overline{N_0}(r,0;f') + S(r,f)\\
&\leq \frac{3}{n+2}N(r,0;f) + \frac{4}{n+6}N(r,a;f) + \left(\frac{3}{n+2} + \frac{4}{n+6}\right)T(r,g) + \overline{N_0}(r,0;g')\\
&\quad -\overline{N_0}(r,0;f') + S(r,f) + S(r,g)\\
&\leq \left(\frac{3}{n+2} + \frac{4}{n+6}\right)\{T(r,f) + T(r,g)\} + \overline{N_0}(r,0;g') - \overline{N_0}(r,0;f') + S(r,f) + S(r,g).
\end{aligned} \tag{21}$$

Similarly,

$$T(r,g) \leq \left(\frac{3}{n+2} + \frac{4}{n+6}\right)\{T(r,f) + T(r,g)\} + \overline{N_0}(r,0;f') - \overline{N_0}(r,0;g') + S(r,f) + S(r,g). \tag{22}$$

Adding (21) and (22) we get

$$\left(1 - \frac{6}{n+2} - \frac{8}{n+6}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction to the assumption that $n > 10$. This proves Lemma 2.7. □

**Lemma 2.8.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n > 5 + \frac{3}{m}$. Let $F = f^n(f-a)(f-b)(f-c)f'$ and $G = g^n(g-a)(g-b)(g-c)g'$ where $a,\ b,\ c \in \mathbb{C}\backslash\{0\}$ and $a \neq b \neq c$. If one of $f$ and $g$ is meromorphic function having only multiple poles and the expressions $\frac{a+b+c}{n+3}g^2\sum_{s=0}^{n+2}\left(\frac{f}{g}\right)^s - \frac{ab+bc+ca}{n+2}g\sum_{s=0}^{n+1}\left(\frac{f}{g}\right)^s + \frac{abc}{n+1}\sum_{s=0}^{n}\left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3}\left(\frac{f}{g}\right)^s$ have no common simple zeros, and*

$$F^m = \frac{(B+1)G^m + (A-B-1)}{BG^m + (A-B)}, \tag{23}$$

*where $A\ (\neq 0)$, $B$ are two constants, then $f \equiv tg$, where $t^m = 1$.*

*Proof.* Let

$$P(z) = \frac{1}{n+4}z^{n+4} - \frac{a+b+c}{n+3}z^{n+3} + \frac{ab+bc+ca}{n+2}z^{n+2} - \frac{abc}{n+1}z^{n+1}. \tag{24}$$

Then we have

$$F = (P(f))' = f^n(f-a)(f-b)(f-c)f',\ G = (P(g))' = g^n(g-a)(g-b)(g-c)g'. \tag{25}$$

By Lemma 2.1, we have

$$\begin{aligned}(n+4)T(r,f) &= T(r,P(f))\\ &\le T(r,(P(f))') + N(r,0;P(f)) - N(r,0;(P(f))') + S(r,f)\\ &\le T(r,F) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + N(r,\gamma_3;f)\\ &\quad -N(r,a;f) - N(r,b;f) - N(r,c;f) - N(r,0;f') + S(r,f),\end{aligned} \tag{26}$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are three roots of the equation $\frac{1}{n+4}z^3 - \frac{a+b+c}{n+3}z^2 + \frac{ab+bc+ca}{n+2}z - \frac{abc}{n+1} = 0$. Similarly, we can get

$$\begin{aligned}(n+4)T(r,g) &= T(r,P(g))\\ &\le T(r,G) + N(r,0;g) + N(r,\gamma_1;g) + N(r,\gamma_2;g) + N(r,\gamma_3;g)\\ &\quad -N(r,a;g) - N(r,b;g) - N(r,c;g) - N(r,0;g') + S(r,g).\end{aligned} \tag{27}$$

We now consider the following three cases.

**Case 2.8.1.** Suppose $B \neq 0,\ -1$. From (23), we have $\overline{N}(r,\frac{B+1}{B};F^m) = \overline{N}(r,\infty;G^m)$. By the second main theorem and Lemma 2.1, we get

$$\begin{aligned}mT(r,F) &= T(r,F^m)\\ &\le \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}(r,\frac{B+1}{B};F^m) + S(r,f)\\ &= \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}(r,\infty;G^m) + S(r,f)\\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) + \overline{N}(r,c;f)\\ &\quad +\overline{N}(r,0;f') + \overline{N}(r,\infty;g) + S(r,f).\end{aligned} \tag{28}$$

From (26), (28) and Lemma 2.1, we can get

$$\begin{aligned}(n+4)T(r,f) &\le \frac{1}{m}\overline{N}(r,\infty;f) + \left(1+\frac{1}{m}\right)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f)\\ &\quad +N(r,\gamma_3;f) + \frac{1}{m}\overline{N}(r,\infty;g) + S(r,f)\\ &\le \left(4+\frac{2}{m}\right)T(r,f) + \frac{1}{m}T(r,g) + S(r,f).\end{aligned} \tag{29}$$

Similarly, we have

$$(n+4)T(r,g) \le \left(4+\frac{2}{m}\right)T(r,g) + \frac{1}{m}T(r,f) + S(r,g). \tag{30}$$

Adding (29) and (30) we have

$$\left(n-\frac{3}{m}\right)\{T(r,f)+T(r,g)\} \le S(r,f) + S(r,g),$$

which is a contradiction as $n > 5 + \frac{3}{m} > \frac{3}{m}$.

**Case 2.8.2.** Suppose $B = 0$. From (23), we have $\overline{N}(r,\frac{A-1}{A};F^m) = \overline{N}(r,0;G^m)$. We consider two subcases as follows.

**Subcase (i).** Let $A \neq 1$. By the second main theorem and Lemma 2.1, we have

$$\begin{aligned} mT(r,F) &= T(r,F^m) \\ &\leq \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}(r,\frac{A-1}{A};F^m) + S(r,f) \\ &= \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}(r,0;G^m) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) \\ &\quad + \overline{N}(r,c;f) + \overline{N}(r,0;f') + \overline{N}(r,0;g) + \overline{N}(r,a;g) \\ &\quad + \overline{N}(r,b;g) + \overline{N}(r,c;g) + \overline{N}(r,0;g') + S(r,f). \end{aligned} \tag{31}$$

From (26) and (31), we get

$$(n+4)T(r,f) \leq \left(4+\frac{2}{m}\right)T(r,f) + \frac{6}{m}T(r,g) + S(r,f). \tag{32}$$

Similarly, we have

$$(n+4)T(r,g) \leq \left(4+\frac{2}{m}\right)T(r,g) + \frac{6}{m}T(r,f) + S(r,g). \tag{33}$$

From (32) and (33), we have

$$\left(n-\frac{8}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction as $n > 5 + \frac{3}{m} > \frac{8}{m}$.

**Subcase (ii).** Let $A = 1$. Then from (23) we obtain $F^m = G^m$, i.e., $F = tG$ where $t^m = 1$. On integration we obtain $P(f) = tP(g) + d$, where $d$ is a constant. If $d \neq 0$, by the second main theorem and Lemma 2.1, we obtain

$$\begin{aligned} (n+4)T(r,f) &= T(r,P(f)) \\ &\leq \overline{N}(r,\infty;P(f)) + \overline{N}(r,0;P(f)) + \overline{N}(r,d;P(f)) + S(r,f) \\ &= \overline{N}(r,\infty;P(f)) + \overline{N}(r,0;P(f)) + \overline{N}(r,0;P(g)) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + N(r,\gamma_3;f) \\ &\quad + N(r,0;g) + N(r,\gamma_1;g) + N(r,\gamma_2;g) + N(r,\gamma_3;g) + S(r,f) \\ &\leq 5T(r,f) + 4T(r,g) + S(r,f). \end{aligned} \tag{34}$$

Similarly,

$$(n+4)T(r,g) \leq 5T(r,g) + 4T(r,f) + S(r,g). \tag{35}$$

Combining (34) and (35), we get

$$(n-5)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

a contradiction as $n > 5 + \frac{3}{m} > 5$. Hence $d = 0$ and so $P(f) \equiv tP(g)$. Therefore by Lemma 2.4, we get $f \equiv tg$, where $t^m = 1$.

**Case 2.8.3.** Let $B = -1$. Arguing similarly as in the proof of Case 2.8.2, we get $F^mG^m = 1$, a contradiction by Lemma 2.5. □

**Lemma 2.9.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n > 5 + \frac{2}{m}$. Let $F = f^n(f-a)^2(f-b)f'$ and $G = g^n(g-a)^2(g-b)g'$ where $a, b \in \mathbb{C}\backslash\{0\}$ and $a \neq b$. If one of $f$ and $g$ is meromorphic function having only multiple poles and the expressions $\frac{2a+b}{n+3}g^2\sum_{s=0}^{n+2}\left(\frac{f}{g}\right)^s - \frac{a^2+2ab}{n+2}g\sum_{s=0}^{n+1}\left(\frac{f}{g}\right)^s + \frac{a^2b}{n+1}\sum_{s=0}^{n}\left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3}\left(\frac{f}{g}\right)^s$ have no common simple zeros, and (23) holds, then $f \equiv tg$, where $t^m = 1$.*

*Proof.* Let
$$P(z) = \frac{1}{n+4}z^{n+4} - \frac{2a+b}{n+3}z^{n+3} + \frac{a^2+2ab}{n+2}z^{n+2} - \frac{a^2b}{n+1}z^{n+1}.$$
Then we have
$$F = (P(f))' = f^n(f-a)^2(f-b)f',\ G = (P(g))' = g^n(g-a)^2(g-b)g'.$$
Proceeding similarly as in Lemma 2.8 and using Lemmas 2.4 and 2.6 we can get the required result. □

**Lemma 2.10.** *Let $f$ and $g$ be two nonconstant meromorphic functions and $n$, $m$ be two positive integers such that $n > 10$. Let $F = f^n(f-a)^3f'$ and $G = g^n(g-a)^3g'$ where $a \in \mathbb{C}\backslash\{0\}$. If one of $f$ and $g$ is meromorphic function having only multiple poles and the expressions $\frac{3a}{n+3}g^2\sum_{s=0}^{n+2}\left(\frac{f}{g}\right)^s - \frac{3a^2}{n+2}g\sum_{s=0}^{n+1}\left(\frac{f}{g}\right)^s + \frac{a^3}{n+1}\sum_{s=0}^{n}\left(\frac{f}{g}\right)^s$ and $\sum_{s=0}^{n+3}\left(\frac{f}{g}\right)^s$ have no common simple zeros, and (23) holds, then $f \equiv tg$, where $t^m = 1$.*

*Proof.* Let
$$P(z) = \frac{1}{n+4}z^{n+4} - \frac{3a}{n+3}z^{n+3} + \frac{3a^2}{n+2}z^{n+2} - \frac{a^3}{n+1}z^{n+1}.$$
Then we have
$$F = (P(f))' = f^n(f-a)^3f',\ G = (P(g))' = g^n(g-a)^3g'.$$
Proceeding similarly as in Lemma 2.8 and applying Lemmas 2.4 and 2.7 we can deduce the required result. □

## 3. Proof of Theorems

*Proof.* [Proof of Theorem 1.8] Let $F$ and $G$ be given by (25) and $P(z)$ by (24). From the hypothesis of the Theorem we have $E_{k)}(S_m, F) = E_{k)}(S_m, G)$ i.e., $E_{k)}(1, F^m) = E_{k)}(1, G^m)$. It is obvious that
$$\begin{aligned} N_2(r,0;F^m) + N_2(r,\infty;F^m) \ \leq\ & 2N(r,0,f) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f) + 2\overline{N}(r,c;f) \\ & +2\overline{N}(r,0;f') + 2\overline{N}(r,\infty;f) + S(r,f), \end{aligned} \tag{36}$$
and
$$\begin{aligned} N_2(r,0;G^m) + N_2(r,\infty;G^m) \ \leq\ & 2N(r,0,g) + 2\overline{N}(r,a;g) + 2\overline{N}(r,b;g) + 2\overline{N}(r,c;g) \\ & +2\overline{N}(r,0;g') + 2\overline{N}(r,\infty;g) + S(r,g). \end{aligned} \tag{37}$$
We now consider the following three cases.

**Case 3.1.** *Let $k \geq 3$. We can easily see that*
$$\begin{aligned} & \overline{N}(r,1;F^m) + \overline{N}(r,1;G^m) + \overline{N}_{(k+1}(r,1;F^m) + \overline{N}_{(k+1}(r,1;G^m) - N_E^{1)}(r,1;F^m) \\ \leq\ & \frac{1}{2}N(r,1;F^m) + \frac{1}{2}N(r,1;G^m) + S(r,F^m) + S(r,G^m) \\ \leq\ & \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + S(r,F) + S(r,G). \end{aligned} \tag{38}$$
*Suppose that $F^m$ and $G^m$ satisfy (i) of Lemma 2.3. Then using Lemma 2.1 and (38), we get*
$$\begin{aligned} mT(r,F) + mT(r,G) \ =\ & T(r,F^m) + T(r,G^m) \\ \leq\ & \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + N_2(r,0;F^m) + N_2(r,\infty;F^m) \\ & +N_2(r,0;G^m) + N_2(r,\infty;G^m) + S(r,F) + S(r,G), \end{aligned}$$

$$\begin{aligned} \text{i.e.,}\quad T(r,F) + T(r,G) \ \leq\ & \frac{2}{m}N_2(r,0;F^m) + \frac{2}{m}N_2(r,\infty;F^m) + \frac{2}{m}N_2(r,0;G^m) \\ & +\frac{2}{m}N_2(r,\infty;G^m) + S(r,F) + S(r,G). \end{aligned} \tag{39}$$

*Now we consider following two subcases.*

**Subcase 3.1.1.** *We assume that* $2 \le m \le 3$. *Then from (26), (27), (36), (37) and (39) we have*

$$\begin{aligned}
& (n+4)T(r,f)+(n+4)T(r,g) \\
\le\ & \left(1+\frac{4}{m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,\gamma_3;f)+N(r,a;f)+N(r,b;f)+N(r,c;f) \\
& +N(r,0;f')+\frac{4}{m}\overline{N}(r,\infty;f)+\left(1+\frac{4}{m}\right)N(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g)+N(r,\gamma_3;g) \\
& +N(r,a;g)+N(r,b;g)+N(r,c;g)+N(r,0;g')+\frac{4}{m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\
\le\ & \left(9+\frac{8}{m}\right)T(r,f)+\left(9+\frac{8}{m}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

$$\text{i.e.,}\quad \left(n-5-\frac{8}{m}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

*which is a contradiction as* $n > 5+\frac{8}{m}$. *Thus, by Lemma 2.3, we have*

$$F^m=\frac{(B+1)G^m+(A-B-1)}{BG^m+(A-B)},$$

*where* $A\ (\neq 0)$, $B$ *are two constants. Therefore* $f \equiv tg$, *where* $t^m = 1$, *by Lemma 2.8.*

**Subcase 3.1.2.** *Next we assume that* $m \ge 4$. *Then from (26), (27), (36), (37) and (39), we have*

$$\begin{aligned}
& (n+4)T(r,f)+(n+4)T(r,g) \\
\le\ & \left(1+\frac{4}{m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,\gamma_3;f)+\frac{4}{m}\overline{N}(r,\infty;f)+\left(1+\frac{4}{m}\right)N(r,0;g) \\
& +N(r,\gamma_1;g)+N(r,\gamma_2;g)+N(r,\gamma_3;g)+\frac{4}{m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\
\le\ & \left(4+\frac{8}{m}\right)T(r,f)+\left(4+\frac{8}{m}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

$$\text{i.e.,}\quad \left(n-\frac{8}{m}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

*a contradiction as* $n > 5+\frac{3}{m} > \frac{8}{m}$. *Thus by Lemma 2.3, we have*

$$F^m=\frac{(B+1)G^m+(A-B-1)}{BG^m+(A-B)},$$

*where* $A\ (\neq 0)$, $B$ *are two constants. Then by Lemma 2.8 and* $n > 5+\frac{3}{m}$, *we have* $f \equiv tg$, *where* $t^m = 1$.

**Case 3.2.** *Let* $k = 2$. *We can easily see that*

$$\begin{aligned}
& \overline{N}(r,1;F^m)+\overline{N}(r,1;G^m)+\frac{1}{2}\overline{N}_{(3}(r,1;F^m)+\frac{1}{2}\overline{N}_{(3}(r,1;G^m)-N_E^{1)}(r,1;F^m) \\
\le\ & \frac{1}{2}N(r,1;F^m)+\frac{1}{2}N(r,1;G^m)+S(r,F^m)+S(r,G^m) \\
\le\ & \frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+S(r,F)+S(r,G). \qquad (40)
\end{aligned}$$

*Suppose that $F^m$ and $G^m$ satisfy (i) of Lemma 2.3. Then from Lemma 2.1 and (40), we get*

$$\begin{aligned} mT(r,F)+mT(r,G) &= T(r,F^m)+T(r,G^m)\\ &\le \frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+N_2(r,0;F^m)+N_2(r,\infty;F^m)+N_2(r,0;G^m)\\ &\quad +N_2(r,\infty;G^m)+\frac{1}{2}\overline{N}_{(3}(r,1;F^m)+\frac{1}{2}\overline{N}_{(3}(r,1;G^m)+S(r,F)+S(r,G), \end{aligned}$$

$$\begin{aligned} \text{i.e.,}\quad T(r,F)+T(r,G) &\le \frac{2}{m}N_2(r,0;F^m)+\frac{2}{m}N_2(r,\infty;F^m)+\frac{2}{m}N_2(r,0;G^m)+\frac{2}{m}N_2(r,\infty;G^m)\\ &\quad +\frac{1}{m}\overline{N}_{(3}(r,1;F^m)+\frac{1}{m}\overline{N}_{(3}(r,1;G^m)+S(r,F)+S(r,G). \end{aligned} \tag{41}$$

*Also we see that*

$$\begin{aligned} \overline{N}_{(3}(r,1;F^m) &\le \frac{1}{2}N\left(r,\infty;\frac{F^m}{(F^m)'}\right)\\ &= \frac{1}{2}N\left(r,\infty;\frac{(F^m)'}{F^m}\right)+S(r,F)\\ &\le \frac{1}{2}\overline{N}(r,\infty;F^m)+\frac{1}{2}\overline{N}(r,0;F^m)+S(r,F)\\ &\le \frac{1}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,a;f)+\frac{1}{2}\overline{N}(r,b;f)\\ &\quad +\frac{1}{2}\overline{N}(r,c;f)+\frac{1}{2}\overline{N}(r,0;f')+S(r,f), \end{aligned} \tag{42}$$

$$\begin{aligned} \text{and}\quad \overline{N}_{(3}(r,1;G^m) &\le \frac{1}{2}\overline{N}(r,\infty;g)+\frac{1}{2}\overline{N}(r,0;g)+\frac{1}{2}\overline{N}(r,a;g)+\frac{1}{2}\overline{N}(r,b;g)\\ &\quad +\frac{1}{2}\overline{N}(r,c;g)+\frac{1}{2}\overline{N}(r,0;g')+S(r,g). \end{aligned} \tag{43}$$

*Now we discuss following two subcases:*

**Subcase 3.2.1.** *We assume that $2 \le m \le 3$. From (26), (27), (36), (37), (41)–(43), we have*

$$\begin{aligned} &(n+4)T(r,f)+(n+4)T(r,g)\\ \le\ & \left(1+\frac{9}{2m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,\gamma_3;f)+\left(1+\frac{1}{2m}\right)N(r,a;f)\\ &+\left(1+\frac{1}{2m}\right)N(r,b;f)+\left(1+\frac{1}{2m}\right)N(r,c;f)+\left(1+\frac{1}{2m}\right)N(r,0;f')+\frac{9}{2m}\overline{N}(r,\infty;f)\\ &+\left(1+\frac{9}{2m}\right)N(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g)+N(r,\gamma_3;g)+\left(1+\frac{1}{2m}\right)N(r,a;g)\\ &+\left(1+\frac{1}{2m}\right)N(r,b;g)+\left(1+\frac{1}{2m}\right)N(r,c;g)+\left(1+\frac{1}{2m}\right)N(r,0;g')+\frac{9}{2m}\overline{N}(r,\infty;g)\\ &+S(r,f)+S(r,g)\\ \le\ & \left(9+\frac{23}{2m}\right)T(r,f)+\left(9+\frac{23}{2m}\right)T(r,g)+S(r,f)+S(r,g), \end{aligned}$$

$$\text{i.e.,}\quad \left(n-5-\frac{23}{2m}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

*which is a contradiction as $n > 5+\frac{23}{2m}$. Thus by Lemma 2.3, we have*

$$F^m=\frac{(B+1)G^m+(A-B-1)}{BG^m+(A-B)},$$

*where $A$ $(\neq 0)$, $B$ are two constants. Then by Lemma 2.8 and $n > 5 + \frac{23}{2m}$, we have $f \equiv tg$, where $t^m = 1$.*

**Subcase 3.2.2.** *Next we assume that $m \geq 4$. Proceeding similarly as in Subcase 3.1.2, we can get*

$$\begin{aligned}
&(n+4)T(r,f) + (n+4)T(r,g) \\
\leq\ & \left(1 + \frac{9}{2m}\right)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) + N(r,\gamma_3;f) + \frac{1}{2m}N(r,a;f) + \frac{1}{2m}N(r,b;f) \\
&+ \frac{1}{2m}N(r,c;f) + \frac{1}{2m}N(r,0;f') + \frac{9}{2m}\overline{N}(r,\infty;f) + \left(1 + \frac{9}{2m}\right)N(r,0;g) + N(r,\gamma_1;g) \\
&+ N(r,\gamma_2;g) + N(r,\gamma_3;g) + \frac{1}{2m}N(r,a;g) + \frac{1}{2m}N(r,b;g) + \frac{1}{2m}N(r,c;g) \\
&+ \frac{1}{2m}N(r,0;g') + \frac{9}{2m}\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\
\leq\ & \left(4 + \frac{23}{2m}\right)T(r,f) + \left(4 + \frac{23}{2m}\right)T(r,g) + S(r,f) + S(r,g),
\end{aligned}$$

$$i.e., \quad \left(n - \frac{23}{2m}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

*a contradiction as $n > 5 + \frac{3}{m} > \frac{23}{2m}$ and $m \geq 4$. Then using $(ii)$ of Lemma 2.3 and Lemma 2.8 we can conclude that $f \equiv tg$, where $t^m = 1$.*

**Case 3.3.** *Let $k = 1$. We have*

$$\begin{aligned}
&\overline{N}(r,1;F^m) + \overline{N}(r,1;G^m) - N_E^{1)}(r,1;F^m) \\
\leq\ & \frac{1}{2}N(r,1;F^m) + \frac{1}{2}N(r,1;G^m) + S(r,F^m) + S(r,G^m) \\
\leq\ & \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + S(r,F) + S(r,G). \qquad (44)
\end{aligned}$$

*Suppose that $F^m$ and $G^m$ satisfy $(i)$ of Lemma 2.3. Then from Lemma 2.1 and (44), we get*

$$\begin{aligned}
mT(r,F) + mT(r,G) &= T(r,F^m) + T(r,G^m) \\
&\leq \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + N_2(r,0;F^m) + N_2(r,\infty;F^m) + N_2(r,0;G^m) \\
&\quad + N_2(r,\infty;G^m) + \overline{N}_{(2}(r,1;F^m) + \overline{N}_{(2}(r,1;G^m) + S(r,F) + S(r,G),
\end{aligned}$$

$$\begin{aligned}
i.e., \quad T(r,F) + T(r,G) \leq\ & \frac{2}{m}N_2(r,0;F^m) + \frac{2}{m}N_2(r,\infty;F^m) + \frac{2}{m}N_2(r,0;G^m) + \frac{2}{m}N_2(r,\infty;G^m) \\
&+ \frac{2}{m}N_{(2}(r,1;F^m) + \frac{2}{m}\overline{N}_{(2}(r,1;G^m) + S(r,F) + S(r,G). \qquad (45)
\end{aligned}$$

*Also we see that*

$$\begin{aligned}
\overline{N}_{(2}(r,1;F^m) &\leq N\left(r,\infty;\frac{F^m}{(F^m)'}\right) \\
&= N\left(r,\infty;\frac{(F^m)'}{F^m}\right) + S(r,F) \\
&\leq \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + S(r,F) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,a;f) + \overline{N}(r,b;f) + \overline{N}(r,c;f) + \overline{N}(r,0;f') + S(r,f), \qquad (46)
\end{aligned}$$

*and*

$$\overline{N}_{(2}(r,1;G^m) \leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N}(r,c;g) + \overline{N}(r,0;g') + S(r,g). \qquad (47)$$

*We now discuss following two subcases.*

**Subcase 3.3.1.** *Let $2 \leq m \leq 3$. Then using (26), (27), (36), (37), (45)–(47), we obtain*

$$\begin{aligned}
& (n+4)T(r,f)+(n+4)T(r,g) \\
\leq\ & \left(1+\frac{6}{m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,\gamma_3;f)+\left(1+\frac{2}{m}\right)N(r,a;f) \\
& +\left(1+\frac{2}{m}\right)N(r,b;f)+\left(1+\frac{2}{m}\right)N(r,c;f)+\left(1+\frac{2}{m}\right)N(r,0;f')+\frac{6}{m}\overline{N}(r,\infty;f) \\
& +\left(1+\frac{6}{m}\right)N(r,0;g)+N(r,\gamma_1;g)+N(r,\gamma_2;g)+N(r,\gamma_3;g)+\left(1+\frac{2}{m}\right)N(r,a;g) \\
& +\left(1+\frac{2}{m}\right)N(r,b;g)+\left(1+\frac{2}{m}\right)N(r,c;g)+\left(1+\frac{2}{m}\right)N(r,0;g')+\frac{6}{m}\overline{N}(r,\infty;g) \\
& +S(r,f)+S(r,g) \\
\leq\ & \left(9+\frac{22}{m}\right)T(r,f)+\left(9+\frac{22}{m}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

$$i.e., \quad \left(n-5-\frac{22}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

*which contradicts the fact that $n > 5+\frac{22}{m}$. Thus by Lemma 2.3, we have*

$$F^m = \frac{(B+1)G^m+(A-B-1)}{BG^m+(A-B)},$$

*where $A\ (\neq 0)$, $B$ are two constants. Thus $f \equiv tg$, where $t^m = 1$, by Lemma 2.8.*

**Subcase 3.3.2.** *Let $m \geq 4$. Proceeding similarly as in Subcases 3.1.2 and 3.3.1 we can get*

$$\begin{aligned}
& (n+4)T(r,f)+(n+4)T(r,g) \\
\leq\ & \left(1+\frac{6}{m}\right)N(r,0;f)+N(r,\gamma_1;f)+N(r,\gamma_2;f)+N(r,\gamma_3;f)+\frac{2}{m}N(r,a;f)+\frac{2}{m}N(r,b;f) \\
& +\frac{2}{m}N(r,c;f)+\frac{2}{m}N(r,0;f')+\frac{6}{m}\overline{N}(r,\infty;f)+\left(1+\frac{6}{m}\right)N(r,0;g)+N(r,\gamma_1;g) \\
& +N(r,\gamma_2;g)+N(r,\gamma_3;g)+\frac{2}{m}N(r,a;g)+\frac{2}{m}N(r,b;g)+\frac{2}{m}N(r,c;g) \\
& +\frac{2}{m}N(r,0;g')+\frac{6}{m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\
\leq\ & \left(4+\frac{22}{m}\right)T(r,f)+\left(4+\frac{22}{m}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

$$i.e., \quad \left(n-\frac{22}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

*which contradicts the fact that $n > 5+\frac{3}{m} > \frac{22}{m}$ and $m \geq 4$. Then using (ii) of Lemma 2.3 and Lemma 2.8 we obtain $f \equiv tg$, where $t^m = 1$.*

This completes the proof of Theorem 1.8. □

*Proof.* [Proof of the Theorem 1.9] Proceeding in a similar manner as in the proof of Theorem 1.8 and using Lemmas 2.4, 2.6 and 2.9 we can get the result of the theorem and we omit the details here. □

*Proof.* [Proof of the Theorem 1.10] Proceeding similarly as in the proof of Theorem 1.8 and using Lemmas 2.4, 2.7 and 2.10 we can deduce the conclusion of the theorem. □

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