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Some KKM Theorems in Modular Function Spaces

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Abstract. In this paper, a coincidence theorem is obtained which is generalization of Ky Fan's fixed point theorem in modular function spaces. A modular version of Fan's minimax inequality is proved. Moreover, some best approximation theorems are presented for multi-valued mappings.

1. Introduction

Modular function spaces are natural generalization of spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Calderon-Lozanovskii and many others. The theory of mappings defined on convex subsets of modular function spaces generalized by Khamsi et al. (see e.g. [3–5]). There is a large set of modular space applications in various parts of analysis, probability and mathematical statistics (see e.g. [11–13]).

We need the following definitions in sequel, from [6, 7]:
Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a σ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$, we denote the linear space of all simple functions with supports in $\mathcal{P}$. By $\mathcal{M}_\infty$, we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, +\infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(w) \to f(w)$ for all $w \in \Omega$. By 1_A, we denote the characteristic function of the set A.

Definition 1.1. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*

(i) $\rho(0) = 0$;

(ii) *ρ is monotone, i.e. $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;*

(iii) *ρ is orthogonally subadditive, i.e. $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;*

(iv) *ρ has the Fatou property, i.e. $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;*

(v) *ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.*

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We say that $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. A property holds ρ-almost everywhere if the exceptional set is ρ-null, we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(w)| < \infty \ \rho - a.e.\}.$$

We will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 1.2. *Let ρ be a regular convex function pseudomodular. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.*

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\mathfrak{R}$.

Definition 1.3. *Let ρ be a convex function modular. A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by*

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

The the formula

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$

defines a norm in L_ρ which is frequently called the Luxemburg norm.

The $\|.\|_\rho$-distance, from an f to a set $Y \subset L_\rho$ to be the quantity

$$dist_{\|.\|_\rho}(f, Y) = \inf\{\|f - g\|_\rho : g \in Y\}.$$

From [7], $(L_\rho, \|f\|_\rho)$ is a complete metric space and the norm $\|.\|_\rho$ is monotone with respect to the natural order in $\mathcal{M}$. Therefore we can define the $\|.\|_\rho$-Hausdorff distance by

$$H_{\|.\|_\rho}(X, Y) = \max\left\{\sup\{dist_{\|.\|_\rho}(f, Y) : f \in X\}, \sup\{dist_{\|.\|_\rho}(g, X) : g \in Y\}\right\},$$

for each $X, Y \subseteq L_\rho$.

Definition 1.4. *Let $\rho \in \mathfrak{R}$.*

(i) We say $\{f_n\}$ is ρ-convergent to f and write $f_n \to f$ (ρ) if and only if $\rho(f_n - f) \to 0$.

(ii) A subset $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ (ρ) implies that f belong to B.

(iii) A nonempty subset K of L_ρ is said to be ρ-compact if for any family $\{A_\alpha; A_\alpha \in 2^{L_\rho}, \alpha \in \Gamma\}$ of ρ-closed subsets with $K \cap A_{\alpha_1} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$, for any $\alpha_1, \cdots, \alpha_n \in \Gamma$, we have

$$K \cap \left(\bigcap_{\alpha\in\Gamma} A_\alpha\right) \neq \emptyset.$$

Let $\rho \in \mathfrak{R}$. We have $\rho(f) \leq \liminf \rho(f_n)$, whenever $f_n \to f$ $\rho - a.e.$ This property is equivalent to the Fatou property [6, Theorem 2.1].

The concept of KKM-mapping in modular function spaces, was introduced by Khamsi, Latif and Al-Sulami in 2011 [6]. They proved an analogue of Ky Fan's fixed point theorem in these spaces:

Definition 1.5. *Let $\rho \in \mathfrak{R}$ and let $C \subset L_\rho$ be nonempty. A multi-valued mapping $G : C \multimap L_\rho$ is called a KKM mapping if*

$$conv(\{f_1, \cdots, f_n\}) \subset \bigcup_{1 \le i \le n} G(f_i)$$

for any $f_1, \cdots, f_n \in C$, where the notation $conv(A)$ describes the convex hull of A.

Theorem 1.6. *[6, Theorem 3.2] Let $\rho \in \mathfrak{R}$ and $C \subset L_\rho$ be nonempty and $G : C \multimap L_\rho$ be a KKM mapping such that for any $f \in C$, $G(f)$ is nonempty and ρ-closed. Assume there exists $f_0 \in C$ such that $G(f_0)$ is ρ-compact. Then, we have*

$$\bigcap_{f \in C} G(f) \neq \emptyset.$$

Definition 1.7. *Let $\rho \in \mathfrak{R}$ and let C be nonempty ρ-closed subset of L_ρ. Let $T : G \to L_\rho$ be a map. T is called ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f. Also T will be called strongly ρ-continuous if T is ρ-continuous and*

$$\liminf_{n \to \infty} \rho(g - T(f_n)) = \rho(g - T(f)),$$

for any sequence $\{f_n\} \subset C$ which ρ-converges to f and for any $g \in C$.

In Section 2, we generalized some results of Khamsi et al. in [6]. In the next section, we proved a minimax inequality. Section 4 is devoted to some best approximation theorems for multi-valued mappings.

2. KKM-mapping and Coincidence Theorem

Here, we generalize the Ky Fan's fixed point theorem which established in [6].

Lemma 2.1. *Let $\rho \in \mathfrak{R}$. Let $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $T : K \to L_\rho$ be strongly ρ-continuous and $F : K \to K$ be ρ-continuous. Then, there exists $f_0 \in K$ such that*

$$\rho(F(f_0) - T(f_0)) = \inf_{f \in K} \rho(F(f) - T(f_0)).$$

Proof. Consider the map $G : K \multimap L_\rho$ defined by

$$G(g) = \Big\{ f \in K; \rho(F(f) - T(f)) \le \rho(F(g) - T(f)) \Big\}.$$

Clearly, for each $g \in K$, $G(g) \neq \emptyset$. For any sequence $\{f_n\} \subset G(g)$ which ρ-converges to f, by Fatou property, we have

$$\rho(F(f) - T(f)) \le \liminf_{n \to \infty} \rho(F(f_n) - T(f_n)),$$

but $\{f_n\} \subset G(g)$, so

$$\liminf_{n \to \infty} \rho(F(f_n) - T(f_n)) \le \liminf_{n \to \infty} \rho(F(g) - T(f_n)).$$

Since T is strongly ρ-continuous and F is ρ-continuous

$$\liminf_{n \to \infty} \rho(F(g) - T(f_n)) = \rho(F(g) - T(f)).$$

Therefore

$$\rho(F(f) - T(f)) \le \rho(F(g) - T(f)),$$

namely $f \in G(g)$. Since for any sequence $\{f_n\} \subset G(g)$ which ρ-converges to f, we have $f \in G(g)$, then $G(g)$ is ρ-closed for any $g \in K$. Now, we show that G is a KKM-mapping. If not, then there exists $\{g_1, ..., g_n\} \subset K$ and $f \in conv(\{g_i\})$ such that $f \notin \bigcup_{1 \le i \le n} G(g_i)$.

This implies

$$\rho(F(g_i) - T(f)) \le \rho(F(f) - T(f)), \quad for \quad i = 1, \cdots, n$$

Let $\epsilon > 0$ be such that $\rho(F(g_i) - T(f)) \le \rho(F(f) - T(f)) - \epsilon$, for $i = 1, \cdots, n$. Since ρ is convex, for any $g \in conv(\{g_i\})$, we have

$$\rho(F(g) - T(f)) \le \rho(F(f) - T(f)) - \epsilon.$$

On the other hand $f \in conv(\{g_i\})$, so we get

$$\rho(F(f) - T(f)) \le \rho(F(f) - T(f)) - \epsilon,$$

which is a contradiction. Therefore, G is a KKM-mapping. By the ρ-compactness of K, we deduce that $G(g)$ is a compact for any $g \in K$. Theorem 1.6 implies the existence of $f_0 \in \bigcap_{g \in K} G(g)$. Hence, $\rho(F(f_0) - T(f_0)) \le \rho(F(g) - T(f_0))$ for any $g \in K$. So, we have $\rho(F(f_0) - T(f_0)) = \inf_{g \in K} \rho(F(g) - T(f_0))$. □

Theorem 2.2. *Let $\rho \in \mathfrak{R}$ and $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $T : K \to L_\rho$ be strongly ρ-continuous, $F : K \to K$ be ρ-continuous and $F(K)$ is ρ-compact. Assume that for any $f \in K$, with $F(f) \neq T(f)$, there exists $\alpha \in (0, 1)$ such that*

$$F(K) \bigcap B_\rho\Big(F(f), \alpha\rho(F(f) - T(f))\Big) \bigcap B_\rho\Big(T(f), (1 - \alpha)\rho(F(f) - T(f))\Big) \neq \emptyset.$$

Then, $T(g) = F(g)$ for some $g \in K$.

Proof. From the previous lemma, there exists $f_0 \in K$ such that

$$\rho(F(f_0) - T(f_0)) = \inf_{g \in K} \rho(F(g) - T(f_0)).$$

We claim that $T(f_0) = F(f_0)$. If $T(f_0) \neq F(f_0)$, then by the ρ-compactness of $F(K)$, there exists $\alpha \in (0, 1)$ such that

$$K_0 = F(K) \bigcap B_\rho(F(f_0), \alpha\rho(F(f_0) - T(f_0))) \bigcap B_\rho(T(f_0), (1 - \alpha)\rho(F(f_0) - T(f_0))) \neq \emptyset.$$

Let $F(g) \in K_0$. Then, $\rho(F(g) - T(f_0)) \le (1 - \alpha)\rho(F(f_0) - T(f_0))$, which is a contradiction. □

Corollary 2.3. *Let $\rho \in \mathfrak{R}$ and $K \subset L_\rho$ be nonempty convex and ρ-compact. Let $F : K \to K$ be ρ-continuous and $F(K)$ is ρ-compact. If $T : K \to F(K)$ be strongly ρ-continuous, then $T(g) = F(g)$ for some $g \in K$.*

3. A Minimax Inequality

In this section, a modular version of Fan's minimax inequality [2] is obtained.

Definition 3.1. *Let $\rho \in \mathfrak{R}$, L_ρ be a modular function space and C be a convex subset of L_ρ. A function $f : C \to \mathbb{R}$ is said to be metrically quasi-concave (resp., metrically quasi-convex) if for each $\lambda \in \mathbb{R}$, the set $\{g \in C : f(g) > \lambda\}$ (resp., $\{g \in C : f(g) < \lambda\}$) is convex.*

Lemma 3.2. *Let $\rho \in \mathfrak{R}$. Suppose C is a convex subset of a modular function space L_ρ, and the function $f : C \times C \to \mathbb{R}$ satisfies the following conditions:*

1) for each $g \in C$, the function $f(.,g) : C \to \mathbb{R}$ is metrically quasi-concave (resp., metrically quasi-convex) and

2) there exists $\gamma \in \mathbb{R}$ such that $f(g,g) \leq \gamma$ (resp., $f(g,g) \geq \gamma$) for each $g \in C$.

Then, the mapping $G : C \multimap L_\rho$, which is defined by

$$G(g) = \{h \in C : f(g,h) \leq \gamma\} (resp., G(g) = \{h \in C : f(g,h) \geq \gamma\}),$$

is a KKM-mapping.

Proof. The conclusion is proved for the concave case, the convex case is completely similar. Assume that G is not a KKM-mapping. Then there exists a finite subset $A = \{g_1, \cdots, g_n\}$ of C and a point $g_0 \in conv(A)$ such that $g_0 \notin G(g_i)$ for each $i = 1, \cdots, n$. We set

$$\lambda = \min\{f(g_i, g_0) : i = 1, \cdots, n\} > \gamma,$$

and $B = \{e \in C : f(e, g_0) > \lambda_0\}$, where $\lambda > \lambda_0 > \gamma$. For each i, we have $g_i \in B$. By hypothesis 1), B is convex and hence $conv(A) \subseteq B$. So, $g_0 \in B$, and we have $f(g_0, g_0) > \lambda_0 > \gamma$, which is a contradiction by assumption 2). Thus, G is a KKM-mapping. □

Definition 3.3. *Let $\rho \in \Re$. A real-valued function $f : L_\rho \times L_\rho \to \mathbb{R}$ is said to be ρ-generally lower (resp., upper) semi continuous on L_ρ whenever, for each $g \in L_\rho$, $\{h \in L_\rho : f(g,h) \leq \lambda\}$ (resp., $\{h \in L_\rho : f(g,h) \geq \lambda\}$) is ρ-closed for each $\lambda \in \mathbb{R}$.*

The following is the analogue of Fan's minimax inequality in modular function spaces.

Theorem 3.4. *Let $\rho \in \Re$. Suppose C is a nonempty, ρ-compact and convex subset of a complete modular function space L_ρ and $f : C \times C \to \mathbb{R}$ satisfies the following*

1) f is a ρ-generally lower (resp., upper) semi continuous ;

2) for each $h \in C$, the function $f(.,h) : C \to \mathbb{R}$ is metrically quasi-concave (resp., metrically quasi-convex) and

3) there exists $\gamma \in \mathbb{R}$ such that $f(g,g) \leq \gamma$ (resp., $f(g,g) \geq \gamma$) for each $g \in C$.

Then, there exists an $h_0 \in C$ such that

$$\sup_{g \in C} f(g, h_0) \leq \sup_{g \in C} f(g,g),$$

$$(resp., \inf_{g \in C} f(g, h_0) \geq \inf_{g \in C} f(g,g)).$$

for each $g \in C$.

Proof. By hypothesis 3), $\lambda = \sup_{g \in C} f(g,g) < \infty$. For each $g \in C$, we define the mapping $G : C \multimap C$ by

$$G(g) = \{h \in C : f(g,h) \leq \lambda\},$$

which is ρ-closed by assumption 1). By Lemma 3.2, G is a KKM-mapping. So by using Theorem 1.6, we have

$$\bigcap_{g \in C} G(g) \neq \emptyset.$$

Therefore, there exists an $h_0 \in \bigcap_{g \in C} G(g)$. Thus, $f(g, h_0) \leq \lambda$ for every $g \in C$. Hence,

$$\sup_{g \in C} f(g, h_0) \leq \sup_{g \in C} f(g,g).$$

This completes the proof. □

4. Some Best Approximation Theorems

In this section, we prove some best approximation theorems for multi-valued mappings in modular function spaces.

Definition 4.1. *Let $X, Y \subseteq L_\rho$.*

(i) A map $F : X \multimap Y$ is said to be ρ-upper semi continuous if for each ρ-closed set $B \subseteq Y$, $F^-(B)$ is ρ-closed in X.

(ii) A map $G : D \subseteq X \multimap X$ is called quasi-convex if the set $G^-(C)$ is convex for each convex subset C of X.

First, note that the $\|.\|_\rho$-Hausdorff distance can be rewritten as follows

$$H_{\|.\|_\rho}(X, Y) = \inf\{\epsilon > o : X \subset O_\epsilon(Y) \quad and \quad Y \subset O_\epsilon(X)\},$$

where, for each $A \subset L_\rho$, $O_\epsilon(A) = \{f \in L_\rho : dist_{\|.\|_\rho}(f, A) < \epsilon\}$.
Also, by definitions of ρ-closed and ρ-compact sets in modular function spaces with $\|.\|_\rho$-Hausdorff distance and by [8, Proposition 14.11] we conclude that, if $F(f)$ is ρ-compact for each $f \in X$, then F is ρ-upper semi continuous if and only if for each $f \in X$ and $\epsilon > 0$, there exist $\delta > 0$ such that for each $f' \in B(f, \delta)$, we have $F(f') \subseteq B(F(f), \epsilon)$.

Theorem 4.2. *Let $\rho \in \mathfrak{R}$. Suppose X is a ρ-compact subset of L_ρ and $F, G : X \multimap L_\rho$ are ρ-upper semi continuous maps with nonempty ρ-compact convex values and G is quasi-convex. Then, there exists $f_0 \in X$ such that*

$$H_{\|.\|_\rho}(G(f_0), F(f_0)) = \inf_{f \in X} H_{\|.\|_\rho}(G(f), F(f_0)).$$

Proof. Let $S : X \multimap X$ be defined by

$$S(g) = \Big\{f \in X : H_{\|.\|_\rho}(G(f), F(f)) \le H_{\|.\|_\rho}(G(g), F(f))\Big\}.$$

For each $g \in X$, $S(g) \neq \emptyset$. We show that $S(g)$ is ρ-closed for each $g \in X$. Suppose that $\{g_n\}$ be a sequence in $S(g)$ such that $g_n \longrightarrow g^*(\rho)$. We claim that $g^* \in S(g)$. Let $\epsilon > 0$ be arbitrary. Since F is ρ-upper semi continuous with ρ-compact values, so there exists N_1 such that for each $n \ge N_1$, we have

$$F(g_n) \subseteq \bar{B}(F(g^*), \epsilon).$$

Similarly, there exists N_2 such that for each $n \ge N_2$, we have

$$G(g_n) \subseteq \bar{B}(G(g^*), \epsilon).$$

Let $N = \max\{N_1, N_2\}$. Then, we have

$$\begin{aligned} H_{\|.\|_\rho}(G(g^*), F(g^*)) &\le H_{\|.\|_\rho}(G(g^*), G(g_n)) + H_{\|.\|_\rho}(G(g_n), F(g_n)) \\ &\quad + H_{\|.\|_\rho}(F(g_n), F(g^*)) \\ &\le 2\epsilon + H_{\|.\|_\rho}(G(g_n), F(g_n)) \\ &\le 2\epsilon + H_{\|.\|_\rho}(G(g), F(g_n)) \\ &\le 2\epsilon + H_{\|.\|_\rho}(G(g), F(g^*)) + H_{\|.\|_\rho}(F(g^*), F(g_n)) \\ &\le 3\epsilon + H_{\|.\|_\rho}(G(g), F(g^*)). \end{aligned}$$

Since ϵ was arbitrary, so

$$H_{\|.\|_\rho}(G(g^*), F(g^*)) \le H_{\|.\|_\rho}(G(g), F(g^*)),$$

so $g^* \in S(g)$. Now, we show that for each $\{f_1, \cdots, f_n\} \subset X$, $co(\{f_1, \cdots, f_n\}) \subset S(\{f_1, \cdots, f_n\})$. Assume to the contrary that, if there exists $h \in co(\{f_1, \cdots, f_n\})$ such that $h \notin S(f)$ for each $f \in \{f_1, \cdots, f_n\}$, then $H_{\|.\|_\rho}(G(f), F(h)) < H_{\|.\|_\rho}(G(h), F(h))$, for some $f \in \{f_1, \cdots, f_n\}$. Moreover

$$G(f) \bigcap \Big(\bigcup_{h' \in F(h)} B\Big(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|_\rho}(G(f'), F(h))\Big)\Big) \neq \emptyset,$$

for each $f \in \{f_1, \cdots, f_n\}$. Since $F(h)$ is convex, so

$$\bigcup_{h' \in F(h)} B\Big(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|_\rho}(G(f'), F(h))\Big)$$

is convex. Since G is quasi-convex, then

$$G(h) \bigcap \Big(\bigcup_{h' \in F(h)} B\Big(h', \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|_\rho}(G(f'), F(h))\Big)\Big) \neq \emptyset,$$

and so $H_{\|.\|_\rho}(G(h), F(h)) \leq \max_{f' \in \{f_1, \cdots, f_n\}} H_{\|.\|_\rho}(G(f'), F(h)) < H_{\|.\|_\rho}(G(h), F(h))$. This is a contradiction. Now, by Theorem 1.6, there exists $f_0 \in X$ such that $f_0 \in \bigcap_{f \in X} S(f)$. Hence, $H_{\|.\|_\rho}(G(f_0), F(f_0)) = \inf_{f \in X} H_{\|.\|_\rho}(G(f), F(f_0))$. □

Corollary 4.3. *Let $\rho \in \Re$. Suppose X is a ρ-compact subset of L_ρ and $G : X \multimap X$ is an onto, quasi-convex and ρ-upper semi continuous map with nonempty ρ-compact convex values and $S : X \to X$ is a continuous single valued map. Then, there exists $f_0 \in X$ such that $S(f_0) \in G(f_0)$.*

Corollary 4.4. *Let $\rho \in \Re$. Suppose X is a ρ-compact subset of L_ρ and $G : X \multimap X$ is a quasi-convex and ρ-upper semi continuous map with nonempty ρ-compact convex values. Then, there exists $f_0 \in X$ such that*

$$H_{\|.\|_\rho}(G(f_0), f_0) = \inf_{f \in X} H_{\|.\|_\rho}(G(f), f_0).$$

Corollary 4.5. *Let $\rho \in \Re$. Suppose X is a ρ-compact subset of L_ρ and $G : X \multimap X$ is a ρ-upper semi continuous map with nonempty ρ-compact convex values. If $G(f) \cap X = \emptyset$ for all $f \in \partial X$, then G has a fixed point.*

Proof. If G does not have a fixed point then by Theorem 4.2, there exists $f_0 \in \partial X$ such that

$$0 < H_{\|.\|_\rho}(f_0, G(f_0)) \leq H_{\|.\|_\rho}(f, G(f_0)),$$

for all $f \in X$. Since $f_0 \in \partial X$, we have $G(f_0) \cap X \neq \emptyset$, which is a contradiction. □

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