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Determination of a Time-Dependent Heat Source Under Not Strengthened Regular Boundary and Integral Overdetermination Conditions

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Abstract. We investigate an inverse problem of finding a time-dependent heat source in a parabolic equation with nonlocal boundary and integral overdetermination conditions. The boundary conditions of this problem are regular but not strengthened regular. The principal difference of this problem is: the system of eigenfunctions is not complete. But the system of eigen- and associated functions forming a basis. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the solution are shown by using the generalized Fourier method.

1. Introduction

Let $T > 0$ be a fixed number and denote by $D_T = \{(x,t) : 0 < x < 1; 0 < t < T\}$. Consider the problem of finding a pair of functions $(r(t), u(x,t))$ satisfying the following equations:

$$u_t = u_{xx}(x,t) + r(t)f(x,t), \quad (x,t) \in D_T, \tag{1}$$

$$u(x,0) = \varphi(x), \ 0 \le x \le 1, \tag{2}$$

$$u(0,t) = u(1,t), \quad \alpha u_x(0,t) = u_x(1,t), \quad 0 \le t \le T, \tag{3}$$

$$\int_0^1 u(x,t)dx = E(t), \ 0 \le t \le T. \tag{4}$$

The given problem is an inverse problem. At $\alpha = 0$ the boundary conditions (3) are well-known and called in literature as Samarskii-Ionkin conditions.

The pair $\{r(t), u(x,t)\}$ from the class $C[0,T] \times \left(C^{2,1}(D_T) \bigcap C^{1,0}(D_T)\right)$ for which equations (1)-(4) are satisfied, is called a classical solution of the inverse problem (1)-(4).

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The inverse problem of finding the heat source in a parabolic equation has been investigated in many studies for the cases when the unknown heat source is space-dependent in [4, 6–8, 15, 21, 22, 27] and time-dependent in [7, 14, 18, 24], to name only a few references. The inverse problems in these papers are similar from the mathematical point of view that local boundary and overdetermination conditions are used. The literature on inverse problems for parabolic equations under nonlocal boundary or overdetermination conditions is not so vast, see [5, 12, 14, 18, 24]. The periodic nature of the first equation in the boundary conditions (3) is demonstrated in [5, 20], whilst the relation of (4) to particle diffusion in turbulent plasma, and also to heat propagation in a thin rod is mentioned in [10]. Among the recent works of similar theme, we note [1–3, 9, 13, 16, 23].

The most close to the theme of the present paper is [13]. The existence of classical solution of an inverse problem analogous to our investigated problem has been justified in this paper for the case $\alpha = 0$. The case $\alpha \neq 0$ is considered for the first time in this our work.

In [3] a system of nonlinear impulsive differential equations with two-point and integral boundary conditions is investigated. In [16] the initial-boundary value problem for the heat equation with a dynamic-type boundary condition is considered. In [23] one family of problems simulating the determination of target components and density of sources from given values of the initial and final states is considered. In [1] an overdetermined initial-boundary value problem for a parabolic equation is considered. And stable difference schemes of first and second orders of accuracy are presented for the approximate solution of this problem. In [2] a time-dependent source identification problem for a parabolic equation is investigated. And for the solution of this problem stability inequalities are presented. In [9] the inverse heat source problem of finding the time-dependent source function together with the temperature in cases with Three general nonlocal conditions are considered for the boundary and overdetermination conditions.

2. Existence and Uniqueness of the Solution of the Inverse Problem

The main result on existence and uniqueness of the solution of the inverse problem (1)-(4) is presented as follows.

Theorem 2.1. *Suppose that $\alpha \neq -1$ and the following conditions hold:*

(A1) $E(t) \in C^1[0,T]$;

(A2) $\varphi(x) \in C^4[0,1]$, $\varphi(0) = \varphi(1)$, $\alpha\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \varphi''(1)$ and $\int_0^1 \varphi(x)dx = E(0)$;

(A3) $f(x,t) \in C\left(\overline{D_T}\right)$; $f(x,t) \in C^4[0,1]$ for $\forall t \in [0,T]$; $f(0,t) = f(1,t)$, $\alpha f_x(0,t) = f_x(1,t)$, $f_{xx}(0,t) = f_{xx}(1,t)$, and $\int_0^1 f(x,t)dx \neq 0$ for $\forall t \in [0,T]$.

Then the inverse problem (1)-(4) has a unique solution.

Proof. Usage of the Fourier method for the solution of the direct problem (1) - (3) leads to a spectral problem for the operator l given by the differential expression and the boundary conditions:

$$l(X) \equiv -X''(x) = \lambda X(x),\ 0 < x < 1;\ \alpha X'(0) = X'(1),\ \ X(0) = X(1). \tag{5}$$

If $\alpha \neq -1$ then the boundary conditions in (5) are regular. In the future, we will consider this condition to be satisfied. In this case, the boundary conditions in (5) are regular but not strengthened regular [19].

The case $\alpha = 1$ is simple (see [5]) and we will not go into details. Suppose that $\alpha \neq 1$. In this case the problem (5) has double eigenvalues $\lambda_k = (2k\pi)^2$ (except for the first $\lambda_0 = 0$). Eigenfunctions of the problem are the following:

$$X_0(x) = 2, \lambda_0 = 0;\ \ X_{2k-1}(x) = 4\cos(2\pi kx), \lambda_k = (2k\pi)^2,\ k = 1, 2, ... \tag{6}$$

To avoid the problem of the choice of associated functions [25, 26] for their construction we use the equation:

$$-X''_{2k}(x) = \lambda_k X_{2k}(x) + \sqrt{\lambda_k}X_{2k-1}(x),\ 0 < x < 1;\ \alpha X'_{2k}(0) = X'_{2k}(1),\ \ X_{2k}(0) = X_{2k}(1). \tag{7}$$

Then associated functions have the form:

$$X_{2k}(x) = \frac{2}{1-\alpha}\,(1-(1-\alpha)\,x)\sin(2\pi kx),\ k = 1,2,.... \tag{8}$$

This system of functions $\{X_0(x), X_{2k-1}(x), X_{2k}(x)\}$ form a Riesz basis in $L_2(0,1)$ [17]. The following system:

$$Y_0(x) = \frac{\alpha+(1-\alpha)\,x}{1+\alpha},\ \ Y_{2k-1}(x) = \frac{\alpha+(1-\alpha)\,x}{1+\alpha}\cos(2\pi kx),$$

$$Y_{2k}(x) = 2\frac{1-\alpha}{1+\alpha}\sin(2\pi kx),\ k = 1,2,.... \tag{9}$$

is biorthogonal to this system.

The method of application eigen- and associated functions for the solution of heat conduction problem has been proved in [10, 11]. By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)-(3) for arbitrary $r(t) \in C[0,T]$:

$$\begin{aligned} u(x,t) = &\Big[\varphi_0 + \int_0^t r(\tau) f_0(\tau) d\tau\Big] X_0(x) + \sum_{k=1}^{\infty}\Big[\varphi_{2k} e^{-(2\pi k)^2 t}\Big] X_{2k}(x) + \sum_{k=1}^{\infty}\Big[\int_0^t r(\tau) f_{2k}(\tau) e^{-(2\pi k)^2(t-\tau)} d\tau\Big] X_{2k}(x) \\ &+ \sum_{k=1}^{\infty}\Big[(\varphi_{2k-1} - 4\pi k\varphi_{2k} t) e^{-(2\pi k)^2 t}\Big] X_{2k-1}(x) + \sum_{k=1}^{\infty}\Big[\int_0^t r(\tau) f_{2k-1}(\tau) e^{-(2\pi k)^2(t-\tau)} d\tau\Big] X_{2k-1}(x) \\ &-4\pi\sum_{k=1}^{\infty} k\Big[\int_0^t r(\tau) f_{2k}(\tau)(t-\tau) e^{-(2\pi k)^2(t-\tau)} d\tau\Big] X_{2k-1}(x), \end{aligned} \tag{10}$$

where $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx$ and $f_k(t) = \int_0^1 f(x,t) Y_k(x) dx,\ k = 0,1,2,....$

The assumptions $\varphi(0) = \varphi(1)$, $\alpha\varphi'(0) = \varphi'(1)$, $f(0,t) = f(1,t)$ and $\alpha f_x(0,t) = f_x(1,t)$ are consistent conditions for the representation (10) of the solution $u(x,t)$ to be valid.

Further, under the smoothness assumptions $\varphi(x) \in C^4[0,1], f(x,t) \in C(\overline{D}_T)$ and $f(x,t) \in C^4[0,1], \forall t \in [0,T]$, the series (10) and its x-partial derivative converge uniformly in $\overline{D}_T$ since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{D}_T$.

In addition, the t-partial derivative and the xx-second order partial derivative series are uniformly convergent for $t \geq \varepsilon > 0$ (ε is an arbitrary positive number). Thus, $u(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ and satisfies the conditions (1)-(3). In addition, $u_t(x,t)$ is continuous in $\overline{D}_T$ because the majorizing sum of t-partial derivative series is absolutely convergent under the conditions $\varphi''(0) = \varphi''(1)$ and $f_{xx}(0,t) = f_{xx}(1,t)$ in $\overline{D}_T$.

Eq. (4) can be differentiated under the condition (A_1) to obtain:

$$\int_0^1 u_t(x,t) dx = E'(t). \tag{11}$$

Further, under the consistency assumption $\int_0^1 \varphi(x) dx = E(0)$, the formulas (10) and (11) yield the following Volterra integral equation of the second kind:

$$r(t) = F(t) + \int_0^t K(t,\tau) r(\tau) d\tau,\ t \in [0,T], \tag{12}$$

where

$$F(t) = \frac{E'(t) + 2\varphi_0 - 4\pi\sum_{k=1}^{\infty} k\varphi_{2k} e^{-(2\pi k)^2 t}}{2f_0(t) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k} f_{2k}(t)}, \tag{13}$$

$$K(t,\tau)=\frac{4\pi\sum_{k=1}^{\infty}kf_{2k}(\tau)e^{-(2\pi k)^2(t-\tau)}}{2f_0(t)+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}f_{2k}(t)}. \tag{14}$$

Note that the denominator in expressions (13) and (14) is never equal to zero because of the assumption $\int_0^1 f(x,t)dx\neq 0,\ \forall t=[0,T]$. Under the assumptions $(A_1)-(A_3)$, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions in $[0,T]$ and $[0,T]\times[0,T]$, respectively. We therefore obtain a unique function $r(t)$, continuous on $[0,T]$, which together with the solution of the problem (1)-(3) given by the Fourier series (10), form the unique solution of the inverse problem (1)-(4). □

3. Continuous Dependence of (r,u) Upon the Data

In this section we use the method of [13]. The following result on continuously dependence on the data of the solution of the inverse problem (1)-(4) holds.

Theorem 3.1. *Let $\mathfrak{I}$ be the class of triples in the form of $\Phi=\{f,\varphi,E\}$ which satisfy the assumptions $(A_1)-(A_3)$ of Theorem 2.1 and let*

$$\|f\|_{C^{2,0}(\overline{D}_T)}\leq N_0,\ \ \|\varphi\|_{C^2[0,1]}\leq N_1,$$

$$\|E\|_{C^1[0,T]}\leq N_2,\ \ 0<N_3\leq\min_{(x,t)\in\overline{D}_T}|f(x,t)|,$$

for some positive constants $N_i,\ i=0,1,2,3$.

Then the solution pair (r,u) of the inverse problem (1)-(4) depends continuously upon the data in $\mathfrak{I}$ for small T.

Proof. Let $\Phi=\{f,\varphi,E\}$ and $\overline{\Phi}=\{\bar f,\bar\varphi,\bar E\}$ be two data in $\mathfrak{I}$. Let us denote

$$\|\Phi\|=\|f\|_{C^{2,0}(\overline{D}_T)}+\|\varphi\|_{C^2[0,1]}+\|E\|_{C^1[0,T]},$$

and a similar expression for $\|\overline{\Phi}\|$.

Let (r,u) and $(\bar r,\bar u)$ be solutions of the inverse problem (1)-(4) corresponding to the data Φ and $\overline{\Phi}$, respectively.

It is clear from (12)-(14) that there exist positive constants $N_i,\ i=4,5$ such that

$$\|F\|_{C[0,T]}\leq N_4,\ \ \|K\|_{C([0,T]\times[0,T])}\leq N_5,\ \ \|r\|_{C[0,T]}\leq\frac{N_4}{1-TN_5},$$

where

$$N_4=N_2+\frac{2}{\sqrt6}N_1,\ \ N_5=\frac{2}{\sqrt6}\frac{N_0}{N_3}.$$

It follows from (13) and (14) that

$$\|F-\overline{F}\|_{C[0,T]}\leq N_6\|f-\bar f\|_{C^{2,0}(\overline{D}_T)}+N_7\|\varphi-\overline{\varphi}\|_{C^2[0,1]}+N_8\|E-\overline{E}\|_{C^1[0,T]},$$

$$\|K-\overline{K}\|_{C([0,T]\times[0,T])}\leq N_9\|f-\bar f\|_{C^{2,0}(\overline{D}_T)},$$

where

$$N_6=\frac{1}{N_4^2}\left[\left(\frac{4}{\sqrt6}+\frac{2}{3}\right)N_1+\left(2+\frac{2}{\sqrt6}\right)N_2\right],\ \ N_7=\frac{1}{N_4^2}\left(\frac{4}{\sqrt6}+\frac{2}{3}\right)N_0,$$

$$N_8=\frac{1}{N_4^2}\left(2+\frac{2}{\sqrt6}\right)N_0,\ \ N_9=\frac{1}{N_4^2}\left(\frac{8}{\sqrt6}+\frac{4}{3}\right)N_0.$$

From (12) we also obtain that

$$\|r-\bar{r}\|_{C[0,T]} \le \|F-\bar{F}\|_{C[0,T]} + TN_5\|r-\bar{r}\|_{C[0,T]} + T\frac{N_4}{1-TN_5}\|K-\overline{K}\|_{C([0,]\times[0,T])},$$

or

$$(1-TN_5)\|r-\bar{r}\|_{C[0,T]} \le N_{10}\|\Phi-\overline{\Phi}\|,$$

where $N_{10} = \max\{N_6 + T\frac{N_4N_9}{1-TN_5}, N_7, N_8\}$. The inequality $TN_5 < 1$ holds for small T. Finally, we obtain

$$\|r-\bar{r}\|_{C[0,T]} \le N_{11}\|\Phi-\overline{\Phi}\|, \;\; N_{11} = \frac{N_{10}}{1-TN_5}.$$

A similar estimate can also be obtained for the difference $u-\bar{u}$ from (10):

$$\|u-\bar{u}\|_{C(\overline{D}_T)} \le N_{12}\|\Phi-\overline{\Phi}\|.$$

□

4. Conclusions

The inverse problem which requires determining a time-dependent heat source in the parabolic heat equation under nonlocal boundary and integral overdetermination conditions has been investigated. Theoretically, the existence and uniqueness of the solution have been established, and also the local stability holds for small times.

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