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Core Partial Order in Rings with Involution

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Abstract. Let R be a unital ring with involution. Several characterizations and properties of core partial order in R are given. In particular, we investigate the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ for two core invertible elements $a, b \in R$. Some relationships between core partial order and other partial orders are obtained.

1. Introduction

The core inverse of a complex matrix was introduced by Baksalary and Trenkler [1]. Let $\mathbf{M}_n(\mathbb{C})$ be the ring of all $n \times n$ complex matrices and let $A \in \mathbf{M}_n(\mathbb{C})$. A matrix $X \in \mathbf{M}_n(\mathbb{C})$ is called a core inverse of A, if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the column space of A, and P_A is the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix X exists, then it is unique and denoted by $A^{\#}$. The core partial order for complex matrices was also introduced in [1]. Let $\mathbb{C}_n^{\mathrm{CM}} = \{A \in \mathbf{M}_n(\mathbb{C}) \mid \mathrm{rank}(A) = \mathrm{rank}(A^2)\}$, $A \in \mathbb{C}_n^{\mathrm{CM}}$ and $B \in \mathbf{M}_n(\mathbb{C})$. The binary relation $\overset{\#}{\leq}$ is defined as follows:

$$A \overset{\#}{\leq} B \quad \Leftrightarrow \quad A^{\#}A = A^{\#}B \text{ and } AA^{\#} = BA^{\#}.$$

In [1, Theorem 6], it was proved that core partial order is a matrix partial order. Baksalary and Trenkler gave several characterizations and various relationships between the matrix core partial order and other matrix partial orders by using the decomposition of Hartwig and Spindelböck [4].

Throughout this paper, R is a $*$-ring, i.e., R is an associative ring with identity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$ for all $a, b \in R$. In [11], Rakić and Djordjević generalized the core partial order from the setting of $\mathbf{M}_n(\mathbb{C})$ to that of an arbitrary $*$-ring. They gave various equivalent conditions of core partial order and investigated relationships between the core partial order and other partial orders in the general setting. Motivated by [1, 6, 7, 10, 11], in this paper, we give some new equivalent conditions and properties for core partial order in $*$-rings. Moreover, some new relationships between core partial order and other partial orders are obtained. As an application, we prove the reverse law for two core invertible elements under the core partial order.

Let us recall some notations. An element $a \in R$ is Moore–Penrose invertible if

$$axa = a, \; xax = x, \; (ax)^* = ax \text{ and } (xa)^* = xa$$

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for some $x \in R$. Such x is unique and called the Moore–Penrose inverse of a and denoted by $a^{\dagger}$. The set of all Moore–Penrose invertible elements in R will be denoted by $R^{\dagger}$. As usual, we write $a\{1\}$ for the set of all inner inverses of $a \in R$, i.e., $a\{1\} = \{x \in R \mid axa = a\}$. An element $a \in R$ is said to be group invertible if there exists $x \in R$ such that

$$axa = a,\ xax = x \text{ and } ax = xa.$$

In this case, x is also unique and called the group inverse of a and denoted by $a^{\#}$. The set of all group invertible elements in R will be denoted by $R^{\#}$. An element $a \in R$ is called an EP element if $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$. The set of all EP elements in R will be denoted by R^{EP}. Note that an EP matrix A in $\mathbf{M}_n(\mathbb{C})$ is also known as a range-Hermite matrix, (that is $\mathcal{R}(A) = \mathcal{R}(A^*)$), where A^* is the conjugate transpose of A.

Following Rakić, Dinčić and Djordjević [12], an element $a \in R$ is said to be core invertible if

$$axa = a,\ xR = aR \text{ and } Rx = Ra^*$$

for some $x \in R$. In this case, x is called the core inverse of a and denoted by $a^{\text{#}\!\!\!\bigcirc}$, without ambiguity since such x is unique. The set of all core invertible elements in R will be denoted by $R^{\text{#}\!\!\!\bigcirc}$.

For $a, b \in R$, the minus partial order $a \overset{-}{\leq} b$, star partial order $a \overset{*}{\leq} b$, sharp partial order $a \overset{\#}{\leq} b$ and core partial order $a \overset{\text{#}\!\!\!\bigcirc}{\leq} b$ are defined, respectively, as follows:

- $a \overset{-}{\leq} b$ if and only if $a^-a = a^-b$ and $aa^- = ba^-$ for some $a^- \in a\{1\}$;
- $a \overset{*}{\leq} b$ if and only if $a^*a = a^*b$ and $aa^* = ba^*$;
- $a \overset{\#}{\leq} b$ if and only if $a^{\#}a = a^{\#}b$ and $aa^{\#} = ba^{\#}$;
- $a \overset{\text{#}\!\!\!\bigcirc}{\leq} b$ if and only if $a^{\text{#}\!\!\!\bigcirc}a = a^{\text{#}\!\!\!\bigcirc}b$ and $aa^{\text{#}\!\!\!\bigcirc} = ba^{\text{#}\!\!\!\bigcirc}$.

We refer the reader to [2, 3, 8, 11] for more details on these partial orders.

2. Equivalent Conditions and Properties of Core Partial Order

In this section, some new characterizations of the core partial order in a $*$-ring R are obtained. Let us start with two auxiliary lemmas, which can be found in [8, Lemma 2.2] and [11, Lemma 2.3 and Theorem 2.6].

Lemma 2.1. *Let $a \in R^{\#}$ and $b \in R$. Then*

(1) *$a^{\#}a = a^{\#}b$ if and only if $a^2 = ab$;*

(2) *$aa^{\#} = ba^{\#}$ if and only if $a^2 = ba$;*

(3) *$a \overset{\#}{\leq} b$ if and only if $a^2 = ab = ba$;*

(4) *$a \overset{\#}{\leq} b$ if and only if $a = pb = bp$ for some idempotent $p \in R$.*

Lemma 2.2. *Let $a \in R^{\text{#}\!\!\!\bigcirc}$ and $b \in R$. Then*

(1) *$a^{\text{#}\!\!\!\bigcirc}a = a^{\text{#}\!\!\!\bigcirc}b$ if and only if $a^*a = a^*b$;*

(2) *$aa^{\text{#}\!\!\!\bigcirc} = ba^{\text{#}\!\!\!\bigcirc}$ if and only if $a^2 = ba$ if and only if $aa^{\#} = ba^{\#}$.*

We will use the following notations $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^{\circ}a = \{x \in R \mid xa = 0\}$ and $a^{\circ} = \{x \in R \mid ax = 0\}$. In [5, Lemma 8], Lebtahi et al. proved that $a \overset{-}{\leq} b$ if and only if there exists $c \in \{x \in R \mid axa = a,\ xax = x\}$ such that $b - a \in {}^{\circ}c \cap c^{\circ}$. For the core partial order, we have the following result.

Theorem 2.3. *Let $a \in R^{\circledast}$ and $b \in R$. Then the following conditions are equivalent:*

(1) $a \overset{\circledast}{\leq} b$;

(2) $ba^{\circledast}b = a$ *and* $a^{\circledast}ba^{\circledast} = a^{\circledast}$;

(3) $aa^{\circledast}b = a = ba^{\circledast}a$;

(4) $b - a \in {}^{\circ}a \cap (a^*)^{\circ}$;

(5) $b - a \in (1 - aa^{\circledast})R \cap R(1 - aa^{\circledast})$;

(6) $b - a \in {}^{\circ}(aa^{\circledast}) \cap (aa^{\circledast})^{\circ}$.

Proof. (1)⇔(2) Suppose that $a \overset{\circledast}{\leq} b$. Then $ba^{\circledast}b = aa^{\circledast}b = aa^{\circledast}a = a$ and $a^{\circledast}ba^{\circledast} = a^{\circledast}aa^{\circledast} = a^{\circledast}$. Conversely, if $ba^{\circledast}b = a$ and $a^{\circledast}ba^{\circledast} = a^{\circledast}$, then $aa^{\circledast} = ba^{\circledast}ba^{\circledast} = ba^{\circledast}$ and $a^{\circledast}a = a^{\circledast}ba^{\circledast}b = a^{\circledast}b$.

(1)⇔(3) Suppose that $a \overset{\circledast}{\leq} b$. Then $a^{\circledast}a = a^{\circledast}b$ and $aa^{\circledast} = ba^{\circledast}$. Thus $aa^{\circledast}b = aa^{\circledast}a = a$ and $ba^{\circledast}a = aa^{\circledast}a = a$. Conversely, if $aa^{\circledast}b = a = ba^{\circledast}a$, then pre-multiplication by $a^{\circledast}$ on $aa^{\circledast}b = a$ yields $a^{\circledast}b = a^{\circledast}a$. Similarly we have $ba^{\circledast} = aa^{\circledast}$, thus $a \overset{\circledast}{\leq} b$.

(1) ⇔ (4) follows by Lemma 2.2.

(4) ⇔ (5) Since $a \in R^{\circledast}$, we have ${}^{\circ}a = R(1 - aa^{\circledast})$ and $(a^*)^{\circ} = (1 - (a^*)^{\circledast}a^*)R = (1 - (a^{\circledast})^*a^*)R = (1 - (aa^{\circledast})^*)R = (1 - aa^{\circledast})R$.

(5) ⇔ (6) Since $(aa^{\circledast})^2 = aa^{\circledast}$, it is easy to see that $(1 - aa^{\circledast})R = (aa^{\circledast})^{\circ}$ and $R(1 - aa^{\circledast}) = {}^{\circ}(aa^{\circledast})$. □

Given two idempotents p and q in R, it is well known that every element $a \in R$ can be written as

$$a = paq + pa(1 - q) + (1 - p)aq + (1 - p)a(1 - q)$$

with respect to p and q. In this case, it is convenient to write a as a matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{p \times q},$$

where $a_{11} = paq$, $a_{12} = pa(1 - q)$, $a_{21} = (1 - p)aq$ and $a_{22} = (1 - p)a(1 - q)$. Thus, the operations of addition and multiplication in R are compatible with the usual matrix operations.

In [11, Theorem 2.6], Rakić and Djordjević proved that $a \overset{\circledast}{\leq} b$ if and only if there exist self-adjoint idempotent $p \in R$ and idempotent $q \in R$ such that $a = pb = bq$ and $qa = a$. We now provide some new characterizations for the core partial order in terms of self-adjoint idempotents.

Theorem 2.4. *Let $a \in R^{\circledast}$ and $b \in R$. Then the following conditions are equivalent:*

(1) $a \overset{\circledast}{\leq} b$;

(2) *there exists a self-adjoint idempotent $p \in R$ such that $a = pb$, $ap = bp$ and $aR = pR$;*

(3) *there exists a self-adjoint idempotent $p \in R$ such that $a = pb$ and $ap = bp$;*

(4) *there exists a self-adjoint idempotent $p \in R$ such that a and b can be written as*

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}_{p \times p} \quad \text{and} \quad b = \begin{pmatrix} a_1 & a_2 \\ 0 & b_4 \end{pmatrix}_{p \times p}.$$

Proof. (1)⇒(2) Let $p = aa^{\circledast}$, then $p^2 = p = p^*$, $pb = aa^{\circledast}b = aa^{\circledast}a = a$ and $ap = a^2a^{\circledast} = aa^{\circledast}a^2a^{\circledast} = ba^{\circledast}a^2a^{\circledast} = baa^{\circledast} = bp$. Moreover, we have $aR = pR$ in view of $p = aa^{\circledast} \in aR$ and $a = aa^{\circledast}a = pa \in pR$.

(2)⇒(3) is trivial.

(3)⇒(1) Suppose that $a = pb$ and $ap = bp$, where $p^2 = p = p^*$. Then $a^2 = apb = bpb = ba$ and $a^*a = (pb)^*pb = b^*p^*pb = b^*p^*b = (pb)^*b = a^*b$. Thus $a \overset{\circledast}{\leq} b$ by Lemma 2.2.

(3)⇒(4) Suppose that $a = pb$ and $ap = bp$. Then it is straightforward to check that $pap = pbp$, $pa(1-p) = pb(1-p)$, $(1-p)ap = (1-p)bp = 0$ and $(1-p)a(1-p) = 0$. Now, let $a_1 = pap$, $a_2 = pa(1-p)$ and $b_4 = (1-p)b(1-p)$. Then (4) follows.

(4)⇒(3) Note that p can written as $p = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}$. Therefore, it is easy to see that $a = pb$ and $ap = bp$. □

The following characterizations of the minus partial order will be used in the proof of Theorem 2.6, which plays an important role in the sequel.

Lemma 2.5. [9, Lemma 3.4] *Given $a, b \in R$ such that $a\{1\}$ and $b\{1\}$ are nonempty. The following conditions are equivalent:*

(1) $a \overset{-}{\leq} b$;

(2) $a = bb^-a = ab^-b = ab^-a$ *for some* $b^- \in b\{1\}$;

(3) $a = bb^-a = ab^-b = ab^-a$ *for any* $b^- \in b\{1\}$.

Theorem 2.6. *Let a, $b \in R^{\circledast}$ with $a \overset{\circledast}{\leq} b$. Then:*

(1) $ba^{\circledast} = ab^{\circledast}$, $a^{\circledast}b = b^{\circledast}a$;

(2) $b^{\circledast}ba^{\circledast} = a^{\circledast}bb^{\circledast} = a^{\circledast}ba^{\circledast} = a^{\circledast}$;

(3) $b^{\circledast}aa^{\circledast} = a^{\circledast}ab^{\circledast} = b^{\circledast}ab^{\circledast} = a^{\circledast}$.

Proof. Suppose that $a \overset{\circledast}{\leq} b$. Then $a \overset{-}{\leq} b$ since $a^{\circledast} \in a\{1\}$. Consequently $a = bb^{\circledast}a = bb^{\#}a$ by Lemma 2.5.

(1) We have $ba^{\circledast} = aa^{\circledast} = bb^{\circledast}aa^{\circledast} = (bb^{\circledast}aa^{\circledast})^* = aa^{\circledast}bb^{\circledast} = aa^{\circledast}ab^{\circledast} = ab^{\circledast}$ and $a^{\circledast}b = b^{\circledast}ab^{\circledast}b = b^{\circledast}ba^{\circledast}b = b^{\circledast}aa^{\circledast}b = b^{\circledast}aa^{\circledast}a = b^{\circledast}a$.

(2) It is obvious that $b^{\circledast}ba^{\circledast} = b^{\circledast}ab^{\circledast} = a^{\circledast}$, $a^{\circledast}bb^{\circledast} = b^{\circledast}ab^{\circledast} = a^{\circledast}$ and $a^{\circledast}ba^{\circledast} = a^{\circledast}aa^{\circledast} = a^{\circledast}$.

(3) is similar to (2), we have $b^{\circledast}aa^{\circledast} = b^{\circledast}ba^{\circledast} = a^{\circledast}$, $a^{\circledast}ab^{\circledast} = a^{\circledast}bb^{\circledast} = a^{\circledast}$ and $b^{\circledast}ab^{\circledast} = b^{\circledast}ba^{\circledast} = a^{\circledast}$. □

Remark 2.7. In [6, Theorem 2.4], it is claimed that the following are equivalent for two complex matrices A and B of index 1 with the same order:

(1) $A^{\circledast}BA^{\circledast} = A^{\circledast}$;

(2) $A^{\dagger}BA^{\#} = A^{\circledast}$.

While the implication(2)⇒(1) is always valid, the converse is not true in general. In fact, let $A = B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbf{M}_2(\mathbb{C})$, we have $A^{\#} = A$, $A^{\dagger} = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}$ and $A^{\circledast} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Whence $A^{\circledast}BA^{\circledast} = A^{\circledast}AA^{\circledast} = A^{\circledast}$. However, $A^{\dagger}BA^{\#} \neq A^{\circledast}$. Note that (1)⇒(2) holds in case A is an EP matrix.

Proposition 2.8. *Let $a, b \in R^{\circledast}$. Then $a \overset{\circledast}{\leq} b$ if and only if $a^{\circledast}b = b^{\circledast}a$, $ba^{\circledast} = ab^{\circledast}$ and $ab^{\circledast}a = a$ hold.*

Proof. If $a \overset{\circledast}{\leq} b$, then we have $a^{\circledast}b = b^{\circledast}a$ and $ba^{\circledast} = ab^{\circledast}$ by Theorem 2.6. Hence $ab^{\circledast}a = ba^{\circledast}a = aa^{\circledast}a = a$. Conversely, this follows $a^{\circledast}a = a^{\circledast}ab^{\circledast}a = a^{\circledast}aa^{\circledast}b = a^{\circledast}b$ and $aa^{\circledast} = ab^{\circledast}aa^{\circledast} = ba^{\circledast}aa^{\circledast} = ba^{\circledast}$. □

In [7, Theorem 2.5] Malik et al. investigated the reverse order law for two core invertible complex matrices under the matrix core partial order. In view of [14, Theorem 3.1], the equations $axa = a$ and $xax = x$ in [12, Theorem 2,14] are redundant.

Lemma 2.9. [14, Theorem 3.1] *Let $a, x \in R$, then x is the core inverse of a if and only if the following three equalities hold*

$$(ax)^* = ax, \quad xa^2 = a \quad \text{and} \quad ax^2 = x.$$

Theorem 2.10. *Let $a,\ b \in R^{\#\!\!\!\bigcirc}$ with $a \overset{\#\!\!\!\bigcirc}{\leq} b$. Then:*

(1) $(ab)^{\#\!\!\!\bigcirc} = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = (a^{\#\!\!\!\bigcirc})^2 = (a^2)^{\#\!\!\!\bigcirc} = (ba)^{\#\!\!\!\bigcirc}$;

(2) $ab \in R^{\mathrm{EP}}$ *whenever* $a \in R^{\mathrm{EP}}$.

Proof. (1) Suppose that $a \overset{\#\!\!\!\bigcirc}{\leq} b$. Then $a^{\#\!\!\!\bigcirc}b = b^{\#\!\!\!\bigcirc}a$ by Proposition 2.8. Thus, $b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = b^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}ba^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = (a^{\#\!\!\!\bigcirc})^2 = (a^2)^{\#\!\!\!\bigcirc} = (ba)^{\#\!\!\!\bigcirc}$. Let $x = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}$. Then

$$abx = abb^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = aba^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = aaa^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = aa^{\#\!\!\!\bigcirc} = (aa^{\#\!\!\!\bigcirc})^* = (abb^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc})^*;$$
$$x(ab)^2 = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}(ab)^2 = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}a(ba)b = a^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}aa^2b = a^{\#\!\!\!\bigcirc}a^2b = ab;$$
$$abx^2 = ab(b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc})^2 = a(ba^{\#\!\!\!\bigcirc})a^{\#\!\!\!\bigcirc}(a^{\#\!\!\!\bigcirc})^2 = (a^{\#\!\!\!\bigcirc})^2 = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}.$$

Thus $(ab)^{\#\!\!\!\bigcirc} = b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}$ by Lemma 2.9.

(2) Suppose that $a \in R^{\mathrm{EP}}$. Then $a^{\#\!\!\!\bigcirc}a = aa^{\#\!\!\!\bigcirc}$. Consequently,

$$b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}ab = b^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc}b = a^{\#\!\!\!\bigcirc}ba^{\#\!\!\!\bigcirc}b = a^{\#\!\!\!\bigcirc}b = a^{\#\!\!\!\bigcirc}a;$$
$$abb^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc} = abb^{\#\!\!\!\bigcirc}a(a^{\#\!\!\!\bigcirc})^2 = aba^{\#\!\!\!\bigcirc}b(a^{\#\!\!\!\bigcirc})^2 = aaa^{\#\!\!\!\bigcirc}a(a^{\#\!\!\!\bigcirc})^2 = aa^{\#\!\!\!\bigcirc}.$$

By (1), we have $b^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}ab = (ab)^{\#\!\!\!\bigcirc}ab = ab(ab)^{\#\!\!\!\bigcirc} = abb^{\#\!\!\!\bigcirc}a^{\#\!\!\!\bigcirc}$. Therefore, $ab \in R^{\mathrm{EP}}$. $\square$

3. Relationships between the Core Partial Order and Other Partial Orders

In this section, we consider the relationships between core partial order and other partial orders. Recall that the left star partial order $a \ast\!\leq b$ in R is defined by: $a^*a = a^*b$ and $aR \subseteq bR$. The right sharp partial order $a \leq_{\#} b$ in $R^{\#}$ is defined by: $aa^{\#} = ba^{\#}$ and $Ra \subseteq Rb$. Let us start with a auxiliary lemma.

Lemma 3.1. [1] *Let $a \in R^{\#\!\!\!\bigcirc}$ and $b \in R$. Then $a \overset{\#\!\!\!\bigcirc}{\leq} b$ if and only if $a \ast\!\leq b$ and $a \leq_{\#} b$.*

In [11, Theorem 4.10], Rakić and Djordjević gave the relationship between the core partial order and the minus partial order for $a, b \in R^{\#\!\!\!\bigcirc}$. For instance, it is proved that $a \overset{\#\!\!\!\bigcirc}{\leq} b$ if and only if $a \overset{-}{\leq} b$ and $b^{\#\!\!\!\bigcirc}ab^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}$. By Lemma 3.1, the core partial order implies the left star partial order and the right sharp partial order. Motivated by [11, Theorem 4.10], we have the following theorem.

Theorem 3.2. *Let $a,\ b \in R^{\#\!\!\!\bigcirc}$. Then the following are equivalent:*

(1) $a \overset{\#\!\!\!\bigcirc}{\leq} b$;

(2) $a \ast\!\leq b$ *and* $ba^{\#\!\!\!\bigcirc}b = a$;

(3) $a \ast\!\leq b$ *and* $b^{\#\!\!\!\bigcirc}aa^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}$;

(4) $a \ast\!\leq b$ *and* $b^{\#\!\!\!\bigcirc}ab^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}$;

(5) $a \leq_{\#} b$ *and* $ba^{\#\!\!\!\bigcirc}b = a$;

(6) $a \leq_{\#} b$ *and* $a^{\#\!\!\!\bigcirc}ab^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc}$.

Proof. (1)⇒(2)-(6) is obvious by Theorem 2.3, Theorem 2.6 and Lemma 3.1.

(2)⇒(1) Suppose that $a \ast\!\leq b$ and $ba^{\circledast}b = a$. Then $a^*a = a^*b$ and $aR \subseteq bR$. Note that $a^*a = a^*b$ if and only if $a^{\circledast}a = a^{\circledast}b$ by Lemma 2.2. So we have $aa^{\circledast} = ba^{\circledast}ba^{\circledast} = ba^{\circledast}aa^{\circledast} = ba^{\circledast}$.

(3)⇒(1) Suppose that $a \ast\!\leq b$. Then we have $a = bs$ for some $s \in R$ and hence $a = bs = bb^{\circledast}bs = bb^{\circledast}a$. Now, it follows that $aa^{\circledast} = bb^{\circledast}aa^{\circledast} = ba^{\circledast}$.

(4)⇒(1) Suppose that $a \ast\!\leq b$ and $b^{\circledast}ab^{\circledast} = a^{\circledast}$. Then $a^*a = a^*b$. By Lemma 2.2, we have $a^{\circledast}a = a^{\circledast}b$. Meanwhile, we have $a = bb^{\circledast}a$, which gives

$$ba^{\circledast} = b(b^{\circledast}ab^{\circledast}) = ab^{\circledast}.$$

Pre-multiplying $b^{\circledast}ab^{\circledast} = a^{\circledast}$ by a and post-multiplying $b^{\circledast}ab^{\circledast} = a^{\circledast}$ by $bb^{\circledast}$ yield

$$aa^{\circledast}bb^{\circledast} = ab^{\circledast}ab^{\circledast}bb^{\circledast} = aa^{\circledast}.$$

Since $a^{\circledast}a = a^{\circledast}b$, one can see that $aa^{\circledast} = aa^{\circledast}bb^{\circledast} = aa^{\circledast}ab^{\circledast} = ab^{\circledast}$. Thus, by the equality $ba^{\circledast} = ab^{\circledast}$ and the definition of core partial order, we have $a \overset{\circledast}{\leq} b$.

(5)⇒(1) Suppose that $a \leq_{\#} b$ and $ba^{\circledast}b = a$. Then $aa^{\#} = ba^{\#}$ and $Ra \subseteq Rb$. By Lemma 2.2, we know that $aa^{\#} = ba^{\#}$ if and only if $aa^{\circledast} = ba^{\circledast}$. Thus $a^{\circledast}a = a^{\circledast}ba^{\circledast}b = a^{\circledast}aa^{\circledast}b = a^{\circledast}b$.

(6)⇒(1) In view of (5)⇒(1), we only need to prove $a^{\circledast}a = a^{\circledast}b$. Since $Ra \subseteq Rb$ is equivalent to $a = ab^{\circledast}b$, we have $a^{\circledast}a = a^{\circledast}ab^{\circledast}b = a^{\circledast}b$. □

Recall that the right star partial order $a \leq\ast\, b$ is defined as: $aa^* = ba^*$ and $Ra \subseteq Rb$.

Remark 3.3. In view of (1)⇔(4) in Theorem 3.2, one may ask whether the condition $a \ast\!\leq b$ can be replaced by $a \leq\ast\, b$ or not. In [6, Theorem 2.9], it is claimed that $A \overset{\circledast}{\leq} B$ if and only if $A \leq\ast\, B$ and $B^{\circledast}AB^{\circledast} = A^{\circledast}$, where A is a square complex matrix of index 1 and B is an EP matrix having same order as A. However, it is not true. In fact, let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathbf{M}_2(\mathbb{C})$, then A is core invertible, B is an EP matrix and the condition $A \overset{\circledast}{\leq} B$ is satisfied, but $AA^* \neq BA^*$.

The equivalence of (2)-(4) in the following proposition for the complex matrices has been proved by Malik et al. in [7, Lemma 19].

Proposition 3.4. *Let $a \in R^{\circledast}$, $b \in R$ with $a \overset{\circledast}{\leq} b$. Then the following conditions are equivalent:*

(1) $a \overset{\#}{\leq} b$;

(2) $ab = ba$;

(3) $a^2 \overset{\circledast}{\leq} b^2$;

(4) $a^k \overset{\circledast}{\leq} b^k$, *for any* $k \geq 2$.

Proof. By Lemma 2.2, we have $a \overset{\circledast}{\leq} b$ if and only if both $a^*a = a^*b$ and $a^2 = ba$ hold.

(1) ⇒ (2) is obvious by Lemma 2.1.

(2) ⇒ (4) If $ab = ba$, then $ab = ba = a^2$ by Lemma 2.2. If $k \geq 2$, we first show that $ab^{k-1} = a^k$. When $k = 2$, $ab = ba = a^2$. Suppose that $a^{k-1} = ab^{k-2}$, then $a^k = aa^{k-1} = abb^{k-2} = ab^{k-1}$. Thus we have $ab^{k-1} = a^k$ by induction. We will show that $(a^k)^{\circledast}a^k = (a^k)^{\circledast}b^k$. Indeed, $(a^k)^{\circledast}b^k = (a^{\circledast})^k b^k = (a^{\circledast})^{k-1}a^{\circledast}bb^{k-1} = (a^{\circledast})^{k-1}a^{\circledast}ab^{k-1} = (a^{\circledast})^k ab^{k-1} = (a^k)^{\circledast}ab^{k-1} = (a^k)^{\circledast}a^k$. Similarly, $b^k(a^k)^{\circledast} = a^k(a^k)^{\circledast}$.

(4) ⇒ (3) is trivial.

(3) ⇒ (1) If $a^2 \overset{\circledast}{\leq} b^2$, then $(a^2)^{\circledast}a^2 = (a^2)^{\circledast}b^2$. And

$$(a^2)^{\circledast}a^2 = (a^{\circledast})^2a^2 = a^{\circledast}a^{\circledast}a^2 = a^{\circledast}a = a^{\#}a,$$

$$(a^2)^{\textcircled{\#}}b^2 = (a^{\textcircled{\#}})^2b^2 = a^{\textcircled{\#}}a^{\textcircled{\#}}bb = a^{\textcircled{\#}}a^{\textcircled{\#}}ab = a^{\#}b,$$

thus $a^{\#}a = a^{\#}b$. Hence $a^2 = aaa^{\#}a = aaa^{\#}b = ab = ba$ by $ba = a^2$. $\square$

In [1, Theorem 7], Baksalary and Trenker proved that for complex matrices A and B, if A is an EP matrix, then $A \overset{\textcircled{\#}}{\leq} B$ if and only if $A \overset{*}{\leq} B$. In [6, Theorem 3.3], Mailk proved that for complex matrices A and B, if A is an EP matrix, then $A \overset{\textcircled{\#}}{\leq} B$ if and only if $A \overset{\#}{\leq} B$. It is easy to check that the following proposition is valid for elements in rings by [12, Theorem 3.1].

Proposition 3.5. *Let $a \in R^{\mathrm{EP}}$ and $b \in R$. Then the following are equivalent:*

(1) $a \overset{\textcircled{\#}}{\leq} b$;

(2) $a \overset{\#}{\leq} b$;

(3) $a \overset{*}{\leq} b$.

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58(6) (2010) 681–697.
[2] M.P. Drazin, Natural structures on semigroups with involution, Bull. Amer. Math. Soc. 84(1) (1978) 139–141.
[3] R.E. Hartwig, How to order regular elements ?, Math. Japon. 25 (1980) 1–13.
[4] R.E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commute, Linear Multilinear Algebra 14 (1983) 241–256.
[5] L. Lebtahi, P. Patrićio, N. Thome, The diamond partial order in rings, Linear Multilinear Algebra 62(3) (2013) 386–395.
[6] S.B. Malik, Some more properties of core partial order, Appl. Math. Comput. 221 (2013) 192–201.
[7] S.B. Malik, L. Rueda, N. Thome, Further properties on the core partial order and other matrix partial orders, Linear Multilinear Algebra, 62(12) (2014) 1629–1648.
[8] S.K. Mitra, On group inverses and the sharp order, Linear Algebra Appl. 92 (1987) 17–37.
[9] D.S. Rakić, Decomposition of a ring incluced by minus partial order, Electron. J. Linear Algebra 23 (2012) 1040–1059.
[10] D.S. Rakić, Generalization of sharp and core partial order using annihilators, Banach J. Math. Anal. 9(3) (2015) 228–242.
[11] D.S. Rakić, D.S. Djordjević, Star, sharp, core and dual core partial order in rings with involution, Appl. Math. Comput. 259 (2015) 800–818.
[12] D.S. Rakić, N. Č. Dinčić, D.S. Djordjević, Group, Moore–Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115–133.
[13] H.X. Wang, X.J. Liu, Characterizations of the core inverse and the partial ordering, Linear Multilinear Algebra 63(9) (2015) 1829–1836.
[14] S.Z. Xu, J.L. Chen, X.X. Zhang, New characterizations for core inverses in rings with involution, Front. Math. China. 12(1) (2017) 231–246.