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On the Differentiation of the Functional in Distributed Optimization Problems with Imperfect Contact

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Abstract. We investigate issues of numerical solving of optimal control problems for second order elliptic equations with non-self-adjoint operators - convection-diffusion problems. Control processes are described by semi-linear convection-diffusion equation with discontinuous data and solutions (states) subject to the boundary interface conditions of imperfect type (i.e., problems with a jump of the coefficients and the solution on the interface; the jump of the solution is proportional to the normal component of the flux). Controls are involved in the coefficients of diffusion and convective transfer. We prove differentiability and Lipshitz continuity of the cost functional, depending on a state of the system and a control. The calculation of the gradients uses the numerical solutions of direct problems for the state and adjoint problems.

1. Introduction

The main object of study in this paper is a semilinear elliptic equation of the second order with non-self-adjoint operators, known as a stationary convection-diffusion equation, with discontinuous data and solutions (states) (DCS) subject to the boundary interface conditions of imperfect type (i.e., problems with a jump of the coefficients and the solution on the interface; the jump of the solution is proportional to the normal component of the flux, see [3], [4]). Convection-diffusion problems are typical for mathematical models of liquid and gas mechanics, since heat and impurities transfer can occur not only due to diffusion, but also to the motion of the medium (see [5]). Convective-diffusion process can play a decisive role in modeling of a wide variety of processes, in particular, of environmental problems associated with the description of the impurities distribution processes in the atmosphere and water reservoirs, and modeling of groundwater pollution. Basic models of many processes in gas- and fluid dynamics are boundary value problems for stationary and non-stationary convection-diffusion equations - second-order elliptic or parabolic equations with minor terms. Currently, the most profound results in the theory of numerical solution to problem for PDEs and optimization problems are obtained for processes with self-adjoint operators.

Gradient methods are very popular and reliable tool of solving minimization problems, and in particular, optimal control problems (see, for example, [6], Chapter 8). And they require Lipschitz derivative of the objective functional. Note that the subject of this article is related to [2]. The principal difference is that controls are involved in the coefficients of diffusion and convective transfer. We prove differentiability and

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Lipshitz continuity of the cost functional, depending on a state of the system and a control, for the problem of optimal control of a semilinear stationary convection-diffusion equation with DCS subject to the boundary interface conditions of imperfect type. Effective procedures for calculating gradients of minimized functionals using the solutions of direct problems for the state and adjoint problems are obtained.

2. Statement of the Problem

Let $\Omega = \left\{r = (r_1, r_2) \in \mathbb{R}^2 : 0 \le r_\alpha \le l_\alpha, \alpha = 1, 2\right\}$ be a rectangle in $\mathbb{R}^2$ with a boundary $\partial\Omega = \Gamma$. Suppose that the domain Ω is splitted by an "internal contact boundary" $\overline{S} = \left\{r_1 = \xi,\ 0 \le r_2 \le l_2\right\}$, where $0 < \xi < l_1$, into subdomains $\Omega_1 \equiv \Omega^- = \left\{0 < r_1 < \xi, \quad 0 < r_2 < l_2\right\}$ and $\Omega_2 \equiv \Omega^+ = \left\{\xi < r_1 < l_1, \quad 0 < r_2 < l_2\right\}$ with boundaries $\partial\Omega_1 \equiv \partial\Omega^-$ and $\partial\Omega_2 \equiv \partial\Omega^+$. Thus, Ω is the union of Ω_1 and Ω_2 and the interior points of the interface $\overline{S}$ between Ω_1 and Ω_2, while $\partial\Omega$ is the outer boundary of Ω. Let $\overline{\Gamma}_k$ denote the boundaries of Ω_k without S, $k = 1, 2$. Therefore $\partial\Omega_k = \overline{\Gamma}_k \cup S$, where Γ_k, $k = 1, 2$ are open nonempty subsets of $\partial\Omega_k$, $k = 1, 2$; and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega = \Gamma$. Let n_α, $\alpha = 1, 2$ denote the outward normal to the boundary $\partial\Omega_\alpha$ of Ω_α, $\alpha = 1, 2$. Let $n = n(x)$ be a unit normal to S at a point $x \in S$, directed, for example, so that n is the outward normal on S with respect to Ω_1; i.e., n is directed inside Ω_2. While formulating boundary value problems for states of control processes below, we assume that S is a straight line along which the coefficients and solutions of boundary value problems are discontinuous, while in domains Ω_1 and Ω_2 they possess certain smoothness.

We consider the following problem: Find a function $u(r)$, defined on $\overline{\Omega}$, satisfying in each subdomain Ω_1 and Ω_2 the equation

$$Lu(r) = -\sum_{\alpha=1}^{2} \frac{\partial}{\partial r_\alpha}\left(k(r)\frac{\partial u}{\partial r_\alpha}\right) + \sum_{\alpha=1}^{2} \vartheta^{(\alpha)} \frac{\partial u}{\partial r_\alpha} + d(r)q(u) = f(r), \quad r = (r_1, r_2) \in \Omega_1 \cup \Omega_2, \tag{1}$$

and the conditions

$$\begin{gathered} u(r) = 0, \quad r \in \partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2, \\ \left[k(r)\frac{\partial u}{\partial r_1}\right] = 0, \quad G(r) = \left(k_1(r)\frac{\partial u_1}{\partial r_1}\right) = \theta(r_2)[u], \quad x \in S, \end{gathered} \tag{2}$$

where $u(r) = \begin{cases} u_1(r), & r \in \Omega_1; \\ u_2(r), & r \in \Omega_2, \end{cases} \qquad q(\xi) = \begin{cases} q_1(\xi_1), & \xi_1 \in \mathbb{R}; \\ q_2(\xi_2), & \xi_2 \in \mathbb{R}, \end{cases}$

$$k(r), d(r), f(r), \vartheta^{(\alpha)}(r) = \begin{cases} k_1(r), q_1(r), f_1(x), \vartheta_1^{(\alpha)}(r), & r \in \Omega_1; \\ k_2(r), q_2(r), f_2(r), \vartheta_2^{(\alpha)}(r), & r \in \Omega_2, \end{cases} \quad \alpha = 1, 2. \tag{3}$$

Here $[u] = u_2(r) - u_1(r) = u^+(r) - u^-(r)$ is the jump of the function $u(r)$ on S; $d(r)$, $f(r)$) are given functions defined independently in Ω_1 and Ω_2, and having a first kind jump at S; $\vartheta_2^{(p)}(r)$, $p = 1, 2$ are given functions defined in Ω_2; $q_\alpha(\xi_\alpha)$, $\alpha = 1, 2$ are given functions defined for $\xi_\alpha \in \mathbb{R}$, $\alpha = 1, 2$; $\theta(r_2)$ is a given function on S; and $g(r) = (g_1(r), g_2(r), g_3(r), g_4(r)) = \left(k_1(r), k_2(r), \vartheta_1^{(1)}(r), \vartheta_1^{(2)}(r)\right)$ is a control-vector.

The given functions are assumed to satisfy the following conditions: $d(r) \in L_\infty(\Omega_1) \times L_\infty(\Omega_2)$, $f(r) \in L_2(\Omega_1) \times L_2(\Omega_2)$, $\theta(r_2) \in L_\infty(S)$, $\vartheta_2^{(p)}(r) \in L_2(\Omega_2)$, $p = 1, 2$; $0 \le d_0 \le d(r) \le \overline{d}_0$, $r \in \Omega_1 \cup \Omega_2$; $0 < \theta_0 \le \theta(r_2) \le \overline{\theta}_0$, $r_2 \in S$, $\zeta_{p+2} \le \vartheta_2^{(p)}(r) \le \overline{\zeta}_{p+2}$, $p = 1, 2$, $r \in \Omega_2$, d_0, $\overline{d}_0$, θ_0, $\overline{\theta}_0$, ζ_{p+2}, $\overline{\zeta}_{p+2}$, – are given constants; functions $q_\alpha(\xi_\alpha)$, $\alpha = 1, 2$ defined on $\mathbb{R}$ with values on $\mathbb{R}$ satisfy the conditions $q_\alpha(0) = 0$, $0 \le q_0 \le (q_\alpha(\xi_1) - q_\alpha(\xi_2))/(\xi_1 - \xi_2) \le L_q < \infty$ for all $\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \ne \xi_2$, $L_q = Const$.

We introduce the set of admissible controls $U = \prod_{k=1}^{4} U_k \subset W_\infty^1(\Omega_1) \times W_\infty^1(\Omega_2) \times L_\infty(\Omega_1) \times L_\infty(\Omega_1) = B$,

$$g_p(r) \in U_p = \left\{ g_p = k_p \in W^1_\infty(\Omega_p) = B_p : 0 < \nu_p \le g_p(r) \le \overline{\nu}_p, \left| \frac{\partial g_p(r)}{\partial r_1} \right| \le R^{(1)}_p, \left| \frac{\partial g_p(r)}{\partial r_2} \right| \le R^{(2)}_p \text{ a.e. on } \Omega_p \right\},$$
$$p = 1,2, \quad g_3(r) \in U_3 = \left\{ g_3(r) = \vartheta^{(1)}_1(r) \in L_\infty(\Omega_1) = B_3 : \zeta_1 \le g_3(r) \le \overline{\zeta}_1, \text{ a.e. on } \Omega_1 \right\}, \tag{4}$$
$$g_4(r) \in U_4 = \left\{ g_4(r) = \vartheta^{(2)}_1(r) \in L_\infty(\Omega_1) = B_4 : \zeta_2 \le g_4(r) \le \overline{\zeta}_2, \text{ a.e. on } \Omega_1 \right\},$$

where $B_p = W^1_\infty(\Omega_p)$, $p = 1,2$, are the sets of controls $g_p(r) = k_p(r)$, $p = 1,2$, defined in Ω_1 and Ω_2, and $B_p = L_\infty(\Omega_1)$, $p = 3,4$, – are the sets of controls $g_{p+2}(r) = \vartheta^{(p)}_1(r)$, $p = 1,2$, defined in Ω_1 and Ω_2, respectively, where ν_p, $\overline{\nu}_p$, $R^{(p)}_1$, $R^{(p)}_2$, ζ_p, $\overline{\zeta}_p$, $p = 1,2$, are given numbers. We assume that: $-m_1 \le \zeta_1 \le \overline{\zeta}_1 \le m_1$, $-p_1 \le \zeta_2 \le \overline{\zeta}_2 \le p_1$, $-m_2 \le \zeta_3 \le \overline{\zeta}_3 \le m_2$, $-p_2 \le \zeta_4 \le \overline{\zeta}_4 \le p_2$, $m_\alpha, p_\alpha = Const > 0$, $\alpha = 1,2$,

$$\delta_\alpha = \max_{\substack{\epsilon_1,\epsilon_2 > 0 \\ \epsilon_1 + \epsilon_2 \le \nu_\alpha}} \left\{ \frac{\nu_\alpha - (\epsilon_1 + \epsilon_2)}{C^2_{\Omega_\alpha}} + \lambda - \frac{m^2_\alpha}{4\epsilon_1} - \frac{p^2_\alpha}{4\epsilon_2} \right\} > 0, \ \alpha = 1,2,$$

here $C^2_{\Omega_1} = \left(\frac{8}{\xi^2_1} + \frac{8}{l^2_2} \right)^{-1}$, $C^2_{\Omega_2} = \left(\frac{8}{(l_1 - \xi_1)^2} + \frac{8}{l^2_2} \right)^{-1}$; λ is any of the following constants: 1) $\lambda = q_0\, d_0$, $\quad d_0 \ge 0$; 2) $\lambda = d_0$ is an arbitrary constant as $q(u) = u$; 3) $\lambda = -L_q\zeta_0$, where $\zeta_0 = \max\left\{ |d_0|, \left|\overline{d}_0\right| \right\}$.

We introduce the cost functional $J : U \to \mathbb{R}^1$ as

$$g \to J(g) = \int\limits_{\Omega_1} |u(r_1, r_2; g) - u^{(1)}_0(r)|^2 d\Omega_1 = I(u(r; g)), \tag{5}$$

where $u^{(1)}_0(r) \in W^1_2(\Omega_1)$ is a given function.

The problem of optimal control is to find a control $g \in U$, that minimizes the functional $g \to J(g)$ on set $U \subset B$, more precisely, we need to minimize functional (5) on the solutions $u(r) = u(r; g)$ to problem (1)-(5), associated with all admissible controls $g(r) = \left(k_1(r), k_2(r), \vartheta^{(1)}_1(r), \vartheta^{(2)}_1(r) \right) \in U$.

Under a solution to direct problem (1)-(5) for each fixed control $g(r) \in U$ we understand a function $u(r) \equiv u(r; g) \in \overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$, satisfying for all $v \in \overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$ the identity

$$\int\limits_{\Omega_1 \cup \Omega_2} \left[\sum_{\alpha=1}^{2} \left(k(r) \frac{\partial u}{\partial r_\alpha} \frac{\partial v}{\partial r_\alpha} + \sum_{\alpha=1}^{2} \vartheta^{(\alpha)} \frac{\partial u}{\partial r_\alpha} v + d(r) q(u) v \right] d\Omega_0 + \int\limits_S \theta(x)[u][v]\, dS = \int\limits_{\Omega_1 \cup \Omega_2} f(r) v\, d\Omega_0.$$

For the definition of spaces $\overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$ see work [1].

3. Difference Approximation of Optimal Control Problems

Optimization problems (5), (1)-(4) are associated with the following difference approximations: minimize the grid functional

$$J_h(\Phi_h) = \sum_{x \in \bar{\omega}^{(1)}} |y(x; \Phi_h) - u^{(1)}_{0h}|^2 \hbar_1 \hbar_2 = \|y(x; \Phi_h) - u^{(1)}_{0h}\|^2_{L_2(\bar{\omega}^{(1)})}, \tag{6}$$

provided that the grid function $y(\Phi_h) \in \overset{\circ}{V}_{\gamma^{(1)},\gamma^{(2)}}(\bar{\omega}^{(1,2)})$, which is the solution of the difference boundary value problem for problem (1)-(4), satisfies, for any grid function $v \in \overset{\circ}{V}_{\gamma^{(1)},\gamma^{(2)}}(\bar{\omega}^{(1,2)})$, the summation identity

$$\sum_{\alpha=1}^{2} \Big\{ \sum_{\omega^{(\alpha)+}_1} \sum_{\omega_2} b^{(\alpha)}_{\alpha h}(\Phi_{\alpha h}(x_1, x_2)) y_{\alpha \bar{x}_1} v_{\alpha \bar{x}_1} h_1 h_2 + \Big(\sum_{\omega^{(\alpha)}_1} \sum_{\omega^+_2} \tilde{b}^{(\alpha)}_{\alpha h}(\Phi_{\alpha h}(x_1, x_2)) y_{\alpha \bar{x}_1} v_{\alpha \bar{x}_1} h_1 h_2 +$$

$$
\begin{gathered}
+\frac{1}{2}\sum_{\omega_2^+}\tilde{b}^{(\alpha)}_{\alpha h}(\Phi_{\alpha h}(\xi,x_2))y_{\alpha\bar{x}_2}(\xi,x_2)v_{\alpha\bar{x}_2}(\xi,x_2)h_1h_2\Big)\Big\}+\sum_{\omega_2}\theta_h(x_2)[y(\xi,x_2)]\,[v(\xi,x_2)]\,h_2+\\
+\sum_{\omega^{(1)}}\sum_{\alpha=1}^{2}\Phi_{\alpha+2,h}(x)y_{1\overset{0}{x}_\alpha}(x)v_1(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\sum_{\alpha=1}^{2}\Phi_{\alpha+2,h}(\xi,x_2)y_{1\overset{0}{x}_\alpha}(\xi,x_2)v_1(\xi,x_2)h_1h_2+\\
+\sum_{\omega^{(2)}}\sum_{\alpha=1}^{2}\vartheta^{(\alpha)}_{2h}(x)y_{2\overset{0}{x}_\alpha}(x)v_2(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}\sum_{\alpha=1}^{2}\vartheta^{(\alpha)}_{2h}(\xi,x_2)y_{2\overset{0}{x}_\alpha}(\xi,x_2)v_2(\xi,x_2)h_1h_2+\\
+\sum_{\alpha=1}^{2}\Big(\sum_{\omega^{(\alpha)}}d_{\alpha h}(x)q_\alpha(y_\alpha(x))v_\alpha(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}d_{\alpha h}(\xi,x_2)q_\alpha(y_\alpha(\xi,x_2))v_\alpha(\xi,x_2)h_1h_2\Big)=\\
=\sum_{\alpha=1}^{2}\Big(\sum_{\omega^{(\alpha)}}f_{\alpha h}(x)v_\alpha(x)h_1h_2+\frac{1}{2}\sum_{\omega_2}f_{\alpha h}(\xi,x_2)v_\alpha(\xi,x_2)h_1h_2\Big),
\end{gathered}
\tag{7}
$$

while the grid controls $\Phi_h=\{\Phi_{1h},\Phi_{2h},\Phi_{3h},\Phi_{4h}\}$ are such that

$$
\begin{gathered}
U_h=\prod_{k=1}^{4}U_{kh}\subset W^1_\infty(\bar{\omega}^{(1)})\times W^1_\infty(\bar{\omega}^{(2)})\times L_\infty(\bar{\omega}^{(1)})\times L_\infty(\bar{\omega}^{(1)})=B_h,\\
\Phi_{ph}(x)\in U_{ph}=\Big\{\Phi_{ph}(x)\in W^1_\infty(\bar{\omega}^{(p)})=B_{ph}:0<\nu_p\le\Phi_{ph}(x)\le\bar{\nu}_p,x\in\bar{\omega}^{(p)},\\
|\Phi_{phx_1}(x)|\le R^{(1)}_p,x\in\omega_1^{(p)-}\times\bar{\omega}_2,|\Phi_{phx_2}(x)|\le R^{(2)}_p,x\in\omega_1^{(p)}\times\bar{\omega}_2^-\Big\},\quad p=1,2,\\
\Phi_{ph}(x)\in U_{ph}=\Big\{\Phi_{ph}(x)\in L_\infty(\bar{\omega}^{(1)})=B_{ph}:\zeta_{p-2}\le\Phi_{ph}(x)\le\bar{\zeta}_{p-2},x\in\bar{\omega}^{(1)}\Big\},\quad p=3,4,
\end{gathered}
\tag{8}
$$

where $B_{1h}=W^1_\infty(\bar{\omega}^{(1)})$, $B_{2h}=W^1_\infty(\bar{\omega}^{(2)})$ are the sets of grid controls Φ_{1h}, Φ_{2h}, defined on the grids $\bar{\omega}^{(1)}$, $\bar{\omega}^{(2)}$, equipped with the norms

$$
\|\Phi_{\alpha h}(x)\|_{W^1_\infty(\bar{\omega}^{(\alpha)})}=\max_{\bar{\omega}^{(\alpha)}}|\Phi_{\alpha h}(x)|+\max_{\omega_1^{(\alpha)-}\times\bar{\omega}_2}|\Phi_{\alpha hx_1}(x)|+\max_{\bar{\omega}_1^{(\alpha)}\times\omega_2^-}|\Phi_{\alpha hx_2}(x)|,\quad\alpha=1,2,
\tag{9}
$$

respectively. Here

$$
\begin{gathered}
b^{(1)}_{1h}(\Phi_{1h}(x_1,x_2))=\frac{\Phi^{(-1_2)}_{1h}(x)+\Phi^{(-1_1,-1_2)}_{1h}(x)+\Phi^{(+1_2)}_{1h}(x)+\Phi^{(-1_1,+1_2)}_{1h}(x)}{4},\ \tilde{b}^{(1)}_{1h}(\Phi_{1h}(x_1,x_2))=\frac{\Phi_{1h}(x)+\Phi^{(-1_2)}_{1h}(x)}{2},\\
b^{(2)}_{2h}(\Phi_{2h}(x_1,x_2))=\frac{\Phi^{(-1_2)}_{2h}(x)+\Phi^{(-1_1,-1_2)}_{2h}(x)+\Phi^{(+1_2)}_{2h}(x)+\Phi^{(-1_1,+1_2)}_{2h}(x)}{4},\ \tilde{b}^{(2)}_{2h}(\Phi_{2h}(x_1,x_2))=\frac{\Phi_{2h}(x)+\Phi^{(-1_2)}_{2h}(x)}{2},
\end{gathered}
$$

$\Phi^{(-1_1,-1_2)}_{1h}(x)=\Phi_{1h}(x_1-h_1,x_2-h_2)$, $\Phi^{(-1_2)}_{1h}(x)=\Phi_{1h}(x_1,x_2-h_2)$, $\Phi^{(-1_1,+1_2)}_{1h}(x)=\Phi_{1h}(x_1-h_1,x_2+h_2)$, $\Phi^{(+1_2)}_{1h}(x)=\Phi_{1h}(x_1,x_2+h_2)$, $\Phi^{(-1_1,-1_2)}_{2h}(x)=\Phi_{2h}(x_1-h_1,x_2-h_2)$, $\Phi^{(-1_2)}_{2h}(x)=\Phi_{2h}(x_1,x_2-h_2)$, $\Phi^{(-1_1,+1_2)}_{2h}(x)=\Phi_{2h}(x_1-h_1,x_2+h_2)$, $\Phi^{(+1_2)}_{2h}(x)=\Phi_{1h}(x_1,x_2+h_2)$, and $\vartheta^{(\alpha)}_{2h}(x)$, $d_{\alpha h}(x)$, $\alpha=1,2$, $\theta_h(x_2)$, $f_{\alpha h}(x)$, $\alpha=1,2$, $u^{(1)}_{0h}(x)$ are grid approximations of the functions $\vartheta^{(\alpha)}(r)$, $d_\alpha(r)$, $\alpha=1,2$, $\theta(r_2)$, $f_\alpha(r)$, $\alpha=1,2$, $u^{(1)}_0(r)$, defined via Steklov averages (see [1]). For the definition of grids $\overline{\omega}^{(1,2)}$, $\omega^{(\alpha)}\cup\gamma_S$, $\alpha=1,2$ and grid spaces $\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$ see work [1].

4. Differentiability of the Grid Functional $J_h(\Phi_h)$

Theorem 4.1. *Assume that the function $q(s):\mathbb{R}\to\mathbb{R}$ satisfies the conditions: $q(0)=0$, $q(s)$ is differentiable in s, and $0<q_0\le q'_s(s)<L_q<\infty$, $\left|q'_s(s_1)-q'_s(s_2)\right|\le\bar{L}_q|s_1-s_2|$, for all $s_1,s_2\in\mathbb{R}$, $L_q,\overline{L}_q=Const>0$. Suppose that $d\in L_\infty(\Omega_1)\times L_\infty(\Omega_2)$, $f\in L_2(\Omega_1)\times L_2(\Omega_2)$, $\theta(r_2)\in L_\infty(S)$, $\vartheta^{(p)}_2\in L_2(\Omega_2)$, $p=1,2$, and $u_0\in W^1_2(\Omega_1)$. Then the grid functional $J_h(\Phi_h)$ is Fréchet-differentiable at Φ_h on U_h, in the space $\widetilde{B}_h=L_\infty(\bar{\omega}^{(1)})\times L_\infty(\bar{\omega}^{(2)})\times\Big(L_2(\omega^{(1)}\cup\gamma_S)\Big)^2$, and a functional gradient $J'_h(\Phi_h)$ at the point $(\Phi_h)=(\Phi_{1h},\Phi_{2h},\Phi_{3h},\Phi_{4h})$ is given by*

$$
<J'_h(\Phi_h),\Delta\Phi_h>=\sum_{\alpha=1}^{2}\Big(\frac{\partial J_h}{\partial\Phi_{\alpha h}},\Delta\Phi_{\alpha h}\Big)_{L_2(\bar{\omega}^{(\alpha)})}+\sum_{\beta=3}^{4}\Big(\frac{\partial J_h}{\partial\Phi_{\beta h}},\Delta\Phi_{\beta h}\Big)_{L_2(\omega^{(1)}\cup\gamma_S)},
$$

$$
\frac{\partial J_h}{\partial \Phi_{\alpha h}} = \begin{cases}
\Big(y^{(+1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_2)}_{\alpha\bar{x}_1}(x) + y^{(+1_1,+1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_1,+1_2)}_{\alpha\bar{x}_1}(x) + y^{(-1_2)}_{\alpha\bar{x}_1}(x)\psi^{(-1_2)}_{\alpha\bar{x}_1}(x) + y^{(+1_1,-1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_1,-1_2)}_{\alpha\bar{x}_1}(x)\Big)/4+ \\
+\Big(y_{\alpha\bar{x}_2}(x)\psi_{\alpha\bar{x}_2}(x) + y^{(+1_2)}_{\alpha\bar{x}_2}(x)\psi^{(+1_2)}_{\alpha\bar{x}_2}(x)\Big)/2, \ \textit{for } x\in\omega^{(\alpha)}; \\
\Big(y^{(+1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_2)}_{\alpha\bar{x}_1}(x) + y^{(+1_1,+1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_1,+1_2)}_{\alpha\bar{x}_1}(x)\Big)/2 + y^{(+1_2)}_{\alpha\bar{x}_2}(x)\psi^{(+1_2)}_{\alpha\bar{x}_2}(x), \ \textit{for } x\in\omega^{(\alpha)}_1\times\{0\}; \\
\Big(y^{(-1_2)}_{\alpha\bar{x}_1}(x)\psi^{(-1_2)}_{\alpha\bar{x}_1}(x) + y^{(+1_1,-1_2)}_{\alpha\bar{x}_1}(x)\psi^{(+1_1,-1_2)}_{\alpha\bar{x}_1}(x)\Big)/2 + y_{\alpha\bar{x}_2}(x)\psi_{\alpha\bar{x}_2}(x), \quad x\in\omega^{(\alpha)}_1\times\{l_2\},\ \alpha=1,2; \\
\Big(y^{(+1_2)}_{1\bar{x}_1}(x)\psi^{(+1_2)}_{1\bar{x}_1}(x) + y^{(-1_2)}_{1\bar{x}_1}(x)\psi^{(-1_2)}_{1\bar{x}_1}(x)\Big)/2 + \Big(y^{(+1_2)}_{1\bar{x}_2}(x)\psi^{(+1_2)}_{1\bar{x}_2}(x) + y_{1\bar{x}_2}(x)\psi_{1\bar{x}_2}(x)\Big)/2, \ x\in\{\xi\}\times\omega_2; \\
\Big(y^{(+1_1,+1_2)}_{1\bar{x}_1}(x)\psi^{(+1_1,+1_2)}_{1\bar{x}_1}(x) + y^{(+1_1,-1_2)}_{1\bar{x}_1}(x)\psi^{(+1_1,-1_2)}_{1\bar{x}_1}(x)\Big)/2, \ \textit{for } x\in\{0\}\times\omega_2; \\
y^{(+1_1,+1_2)}_{1\bar{x}_1}(x)\psi^{(+1_1,+1_2)}_{1\bar{x}_1}(x), \ \textit{for } x=(0,0); \quad y^{(+1_2)}_{1\bar{x}_1}(x)\psi^{(+1_2)}_{1\bar{x}_1}(x) + y^{(+1_2)}_{1\bar{x}_2}(x)\psi^{(+1_2)}_{1\bar{x}_2}(x), \ \textit{for } x=(\xi,0); \\
y^{(+1_1,-1_2)}_{1\bar{x}_1}(x)\psi^{(+1_1,-1_2)}_{1\bar{x}_1}(x), \ \textit{for } x=(0,l_2); \quad y^{(-1_2)}_{1\bar{x}_1}(x)\psi^{(-1_2)}_{1\bar{x}_1}(x) + y_{1\bar{x}_2}(x)\psi_{1\bar{x}_2}(x), \ \textit{for } x=(\xi,l_2); \\
\dfrac{y^{(+1_1,+1_2)}_{2\bar{x}_1}\psi^{(+1_1,+1_2)}_{2\bar{x}_1} + y^{(+1_1,-1_2)}_{2\bar{x}_1}\psi^{(+1_1,-1_2)}_{2\bar{x}_1}}{2} + \dfrac{y^{(+1_2)}_{2\bar{x}_2}(x)\psi^{(+1_2)}_{2\bar{x}_2}(x) + y_{2\bar{x}_2}(x)\psi_{2\bar{x}_2}(x)}{2}, \ \textit{for } x\in\{\xi\}\times\omega_2; \\
\Big(y^{(+1_2)}_{2\bar{x}_1}(x)\psi^{(+1_2)}_{2\bar{x}_1}(x) + y^{(-1_2)}_{2\bar{x}_1}(x)\psi^{(-1_2)}_{2\bar{x}_1}(x)\Big)/2, \ \textit{for } x\in\{l_2\}\times\omega_2; \\
y^{(+1_1,+1_2)}_{2\bar{x}_1}(x)\psi^{(+1_1,+1_2)}_{2\bar{x}_1}(x) + y^{(+1_2)}_{2\bar{x}_2}(x)\psi^{(+1_2)}_{2\bar{x}_2}(x), \ \textit{for } x=(\xi,0); \quad y^{(+1_2)}_{2\bar{x}_1}(x)\psi^{(+1_2)}_{2\bar{x}_1}(x), \ \textit{for } x=(l_1,0); \\
y^{(+1_1,-1_2)}_{2\bar{x}_1}(x)\psi^{(+1_1,-1_2)}_{2\bar{x}_1}(x) + y_{2\bar{x}_2}(x)\psi_{2\bar{x}_2}(x), \ \textit{for } x=(\xi,l_2); \quad y^{(-1_2)}_{2\bar{x}_1}(x)\psi^{(-1_2)}_{2\bar{x}_1}(x), \ \textit{for } x=(l_1,l_2);
\end{cases} \tag{10}
$$

$$
\frac{\partial J_h}{\partial \Phi_{3h}} = y_{1\overset{\circ}{x}_1}(x)\psi_1(x), \qquad \frac{\partial J_h}{\partial \Phi_{4h}} = y_{1\overset{\circ}{x}_2}(x)\psi_1(x), \quad x\in\omega^{(1)}\cup\gamma_S,
$$

using the numerical solution $y(x;\Phi_h)$ of the grid state problem (7)*, and a solution $\psi(x;\Phi_h)$ of an adjoint problem:*

$$
\begin{gathered}
-\Big(b^{(1)}_{1h}(\Phi_{1h})\,\psi_{1\bar{x}_1}\Big)_{x_1} - \Big(\tilde{b}^{(1)}_{1h}(\Phi_{1h})\psi_{1\bar{x}_2}\Big)_{x_2} + \sum_{\alpha=1}^{2}\Phi_{\alpha+2,h}\,\psi_{1\overset{\circ}{x}_\alpha} + d_{1h}\,q_{1y_1}\,\psi_1 = -2\Big(y_1(x) - u^{(1)}_{0h}(x)\Big), \quad x\in\omega^{(1)}, \\
\psi_1(x) = 0, \ x\in\gamma^{(1)} = \partial\omega^{(1)}\setminus\gamma_S; \\
-\Big(b^{(2)}_{2h}(\Phi_{2h})\psi_{2\bar{x}_1}\Big)_{x_1} - \Big(\tilde{b}^{(2)}_{2h}(\Phi_{2h})\psi_{2\bar{x}_2}\Big)_{x_2} + \sum_{\alpha=1}^{2}\vartheta^{(\alpha)}_{2h}\psi_{2\overset{\circ}{x}_\alpha} + d_{2h}(x)q_{2y_2}\psi_2(x) = 0, \ x\in\omega^{(2)}, \\
\psi_2(x) = 0, \qquad x\in\gamma^{(2)} = \partial\omega^{(2)}\setminus\gamma_S; \\
\frac{2}{h_1}\Big[b^{(1)}_{1h}(\Phi_{1h}(\xi_1,x_2))\psi_{1\bar{x}_1}(\xi_1,x_2) + \theta_h(x_2)\psi_1(\xi,x_2)\Big] + \sum_{\alpha=1}^{2}\Phi_{\alpha+2,h}(\xi,x_2)\psi_{1\overset{\circ}{x}_\alpha}(\xi,x_2) - \\
-\Big(\tilde{b}^{(1)}_{1h}(\Phi_{1h}(\xi,x_2))\psi_{1\bar{x}_2}(\xi,x_2)\Big)_{x_2} + d_{1h}(\xi,x_2)q_{1y_1}\psi_1(\xi,x_2) = -2\Big(y_1(\xi,x_2) - u^{(1)}_{0h}(\xi,x_2)\Big) + \frac{2}{h_1}\theta_h(x_2)\psi_2(\xi,x_2), \\
-\frac{2}{h_1}\Big[b^{(2)}_{2h}(\Phi_{2h}(\xi_1+h_1,x_2))\psi_{2x_1}(\xi,x_2) - \theta_h(x_2)\psi_2(\xi,x_2)\Big] + \sum_{\alpha=1}^{2}\vartheta^{(\alpha)}_{2h}(\xi,x_2)\psi_{2\overset{\circ}{x}_\alpha}(\xi,x_2) - \\
-\Big(\tilde{b}^{(2)}_{2h}(\Phi_{2h}(\xi,x_2))\psi_{2\bar{x}_2}(\xi,x_2)\Big)_{x_2} + d_{2h}(\xi,x_2)q_{2y_2}\psi_2(\xi,x_2) = \frac{2}{h_1}\theta_h(x_2)\psi_1(\xi,x_2), \ x\in\gamma_S.
\end{gathered} \tag{11}
$$

Proof. Let Φ_h and $\Phi_h + \Delta\Phi_h$ be arbitrary controls in U_h, and let $y(\Phi_h)$ and $y(\Phi_h + \Delta\Phi_h)$ be the solutions of the state problems in optimization problem (7), corresponding the controls Φ_h and $\Phi_h + \Delta\Phi_h$, and let $J_h(\Phi_h)$ and $J_h(\Phi_h+\Delta\Phi_h)$ be the corresponding values of the grid functional J_h. We define $\Delta y(x) = y(x;\Phi_h+\Delta\Phi_h)-y(x;\Phi_h)$, $\Delta J_h(\Phi_h) = J_h(\Phi_h + \Delta\Phi_h) - J_h(\Phi_h)$.

First of all, we note that the increment Δy satisfies for any $v \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}\,(\overline{\omega}^{(1,2)})$ the problem:

$$\begin{aligned}
&\sum_{\alpha=1}^{2}\Big(\sum_{\omega_1^{(\alpha)+}}\sum_{\omega_2} b_{\alpha h}^{(\alpha)}(\Phi_{\alpha h}+\Delta\Phi_{\alpha h})(\Delta y_\alpha)_{\bar{x}_1} v_{\alpha\bar{x}_1} h_1h_2 + \sum_{\omega_1^{(\alpha)}}\sum_{\omega_2^+} \tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h}+\Delta\Phi_{\alpha h})(\Delta y_\alpha)_{\bar{x}_2} v_{\alpha\bar{x}_2} h_1h_2+\\
&+\frac{1}{2}\sum_{\omega_2^+}\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h}+\Delta\Phi_{\alpha h})(\xi,x_2)(\Delta y_\alpha)_{\bar{x}_2}(\xi,x_2)v_{\alpha\bar{x}_2}(\xi,x_2)h_1h_2\Big)+\\
&+\sum_{\omega_2}\theta_h(x_2)\Big[\Delta y\Big]\Big[v\Big](\xi,x_2)h_2 + \sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\Big(\Phi_{\alpha+2,h}+\Delta\Phi_{\alpha+2,h}\Big)\Delta y_{1\overset{\circ}{x}_\alpha}(x)\,v_1(x)h_1h_2+\\
&+\sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\Big(\Phi_{\alpha+2,h}(\xi,x_2)+\Delta\Phi_{\alpha+2,h}(\xi,x_2)\Big)\Delta y_{1\overset{\circ}{x}_\alpha}(\xi,x_2)\,v_1(\xi,x_2)h_1h_2+\\
&+\sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\vartheta_{2h}^{(\alpha)}(x)\,\Delta y_{2\overset{\circ}{x}_\alpha}(x)\,v_2(x)h_1h_2 + \sum_{\alpha=1}^{2}\sum_{\omega_2}\vartheta_{2h}^{(\alpha)}(\xi,x_2)\,(\Delta y_2(\xi,x_2))_{\overset{\circ}{x}_\alpha}\,v_2(\xi,x_2)h_1h_2+\\
&+\sum_{\alpha=1}^{2}\Big(\sum_{\omega^{(\alpha)}}d_{\alpha h}(x)\Big[q_\alpha(y_\alpha(x;\Phi_h+\Delta\Phi_h))-q_\alpha(y_\alpha(x;\Phi_h))\Big]v_\alpha(x)h_1h_2+\\
&+\frac{1}{2}\sum_{\omega_2}d_{\alpha h}(\xi,x_2)\Big[q_\alpha(y_\alpha(\xi,x_2;\Phi_h+\Delta\Phi_h))-q_\alpha(y_\alpha(\xi,x_2;\Phi_h))\Big]v_\alpha(\xi,x_2)h_1h_2\Big)=\\
&=-\sum_{\alpha=1}^{2}\Big(\sum_{\omega_1^{(\alpha)+}}\sum_{\omega_2}\Delta b_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\,y_{\alpha\bar{x}_1}(\Phi_h)v_{\alpha\bar{x}_1}h_1h_2 - \sum_{\omega_1^{(\alpha)}}\sum_{\omega_2^+}\Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\,y_{\alpha\bar{x}_2}(\Phi_h)v_{\alpha\bar{x}_2}h_1h_2-\\
&-\frac{1}{2}\sum_{\omega_2^+}\Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})(\xi,x_2)\,y_{\alpha\bar{x}_2}(\xi,x_2)v_{\alpha\bar{x}_2}(\xi,x_2)h_1h_2\Big)-\\
&-\sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\Delta\Phi_{\alpha+2,h}\,y_{1\overset{\circ}{x}_\alpha}(\Phi_h)\,v_1(x)h_1h_2 - \sum_{\alpha=1}^{2}\sum_{\omega_2}\Delta\Phi_{\alpha+2,h}(\xi,x_2)\,y_{1\overset{\circ}{x}_\alpha}(\xi,x_2;\Phi_h)\,v_1(\xi,x_2)h_1h_2.
\end{aligned} \tag{12}$$

Then the increment of the functional $J_h(\Phi_h)$ can be represented as:

$$\Delta J_h(\Phi_h) = J_h(\Phi_h+\Delta\Phi_h) - J_h(\Phi_h) = 2\sum_{\overline{\omega}^{(1)}}\Big(y(\Phi_h)-u_{0h}^{(1)}(x)\Big)\,\Delta y\,\hbar_1\hbar_2 + \sum_{\overline{\omega}^{(1)}}\Big(\Delta y\Big)^2\hbar_1\hbar_2. \tag{13}$$

To transform the functional increment (13) let us introduce the function $\psi(\Phi_h)$ as the solution of the auxiliary boundary value problem (adjoint problem) (11). The solution of adjoint problem (11) is a function $\psi(\Phi_h) \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}\,(\overline{\omega}^{(1,2)})$, satisfying for any grid function $v(x) \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}\,(\overline{\omega}^{(1,2)})$ the summation identity:

$$\begin{aligned}
&\sum_{\alpha=1}^{2}\Big(\sum_{\omega_1^{(\alpha)+}}\sum_{\omega_2} b_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\psi_{\alpha\bar{x}_1}v_{\alpha\bar{x}_1}h_1h_2 + \sum_{\omega_1^{(\alpha)}}\sum_{\omega_2^+}\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\psi_{\alpha\bar{x}_2}v_{\alpha\bar{x}_2}h_1h_2+\\
&+\frac{1}{2}\sum_{\omega_2^+}\tilde{b}_{\alpha h}^{(\alpha)}(\xi,x_2;\Phi_{\alpha h})\psi_{\alpha\bar{x}_2}(\xi,x_2)v_{\alpha\bar{x}_2}(\xi,x_2)h_1h_2\Big) + \sum_{\omega_2}\theta_h(x)\Big[\psi(\xi,x_2)\Big]\Big[v(\xi,x_2)\Big]h_2+\\
&+\sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\Phi_{\alpha+2,h}(x)\psi_{1\overset{\circ}{x}_\alpha}(x)\,v_1(x)h_1h_2 + \sum_{\alpha=1}^{2}\sum_{\omega_2}\Phi_{\alpha+2,h}(\xi,x_2)\psi_{1\overset{\circ}{x}_\alpha}(\xi,x_2)\,v_1(\xi,x_2)h_1h_2+\\
&+\sum_{\alpha=1}^{2}\sum_{\omega^{(2)}}\vartheta_{2h}^{(\alpha)}(x)\psi_{2\overset{\circ}{x}_\alpha}(x)\,v_2(x)h_1h_2 + \sum_{\alpha=1}^{2}\sum_{\omega_2}\vartheta_{2h}^{(\alpha)}(\xi,x_2)\psi_{2\overset{\circ}{x}_\alpha}(\xi,x_2)\,v_2(\xi,x_2)h_1h_2+
\end{aligned} \tag{14}$$

$$+\sum_{\alpha=1}^{2}\sum_{\omega^{(\alpha)}} d_{\alpha h}(x)q_{\alpha y_\alpha}\cdot\psi_\alpha(x)v_\alpha(x)h_1h_2+\frac{1}{2}\sum_{\alpha=1}^{2}\sum_{\omega_2} d_{\alpha h}(\xi,x_2)q_{\alpha y_\alpha}\cdot\psi_\alpha(\xi,x_2)v_\alpha(\xi,x_2)h_1h_2=$$
$$=-2\sum_{\omega^{(1)}}\Big(y(x)-u_{0h}^{(1)}(x)\Big)v_1(x)h_1h_2-\sum_{\omega_2}\Big(y(\xi,x_2)-u_{0h}^{(1)}(\xi,x_2)\Big)v_1(\xi,x_2)h_1h_2,\ \forall v(x)\in\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)}).$$

We set $v=\psi$ in (12) and $v=\Delta y$ in (14), and subtract one from the other and paste the results into (13). Then the increment of the grid functional can be represented as

$$\Delta J_h(\Phi_h)=J_h(\Phi_h+\Delta\Phi_h)-J_h(\Phi_h)=\sum_{\alpha=1}^{2}\Big(\sum_{\omega_1^{(\alpha)+}}\sum_{\omega_2}\Delta b_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\,y_{\alpha\bar{x}_1}\psi_{\alpha\bar{x}_1}h_1h_2+$$
$$+\sum_{\omega_1^{(\alpha)}}\sum_{\omega_2^+}\Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\,y_{\alpha\bar{x}_2}\psi_{\alpha\bar{x}_2}h_1h_2+\frac{1}{2}\sum_{\omega_2^+}\Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})(\xi,x_2)\,y_{\alpha\bar{x}_2}(\xi,x_2)\psi_{\alpha\bar{x}_2}(\xi,x_2)h_1h_2\Big)+ \tag{15}$$
$$+\sum_{\alpha=1}^{2}\sum_{\omega^{(1)}}\Delta\Phi_{\alpha+2,h}\,y_{1\overset{\circ}{x}_\alpha}\,\psi_1(x)h_1h_2+\sum_{\alpha=1}^{2}\sum_{\omega_2}\Delta\Phi_{\alpha+2,h}(\xi,x_2)\,y_{1\overset{\circ}{x}_\alpha}(\xi,x_2)\,\psi_1(\xi,x_2)h_1h_2+R_h,$$

where $R_h=\sum_{k=1}^{11}R_{hk},\quad R_{h1}=\sum_{\omega_1^{(1)+}\times\omega_2}\Big(\Delta y_1\Big)^2\hbar_1h_2;$

$$\begin{aligned}
R_{h2}&=\sum_{\omega_1^{(1)}\times\omega_2} d_{1h}(x)\psi_1(x)\Big[q_1(y_1+\Delta y_1)-q_1(y_1)-q_{1y_1}\Delta y_1\Big]h_1h_2;\\
R_{h3}&=\sum_{\omega_1^{(2)}\times\omega_2} d_{2h}(x)\psi_2(x)\Big[q_2(y_2+\Delta y_2)-q_2(y_2)-q_{2y_2}\Delta y_2\Big]h_1h_2;\\
R_{h4}&=\sum_{\omega_2} d_{1h}(\xi,x_2)\psi_1(\xi,x_2)\Big[\frac{1}{2}\Big(q_1(y_1+\Delta y_1)-q_1(y_1)\Big)-q_{1y_1}\Delta y_1(\xi,x_2)\Big]h_1h_2;\\
R_{h5}&=\sum_{\omega_2} d_{2h}(\xi,x_2)\psi_2(\xi,x_2)\Big[\frac{1}{2}\Big(q_2(y_2+\Delta y_2)-q_2(y_2)\Big)-q_{2y_2}\Delta y_2(\xi,x_2)\Big]h_1h_2;\\
R_{h6}&=\sum_{\omega_1^{(1)+}}\sum_{\omega_2} b_{1h}^{(1)}(\Delta\Phi_{1h})\,(\Delta y_1)_{\bar{x}_1}\cdot\psi_{1\bar{x}_1}h_1h_2;\ R_{h7}=\sum_{\omega_1^{(1)}}\sum_{\omega_2^+}\tilde{b}_{1h}^{(1)}(\Delta\Phi_{1h})\,(\Delta y_1)_{\bar{x}_2}\cdot\psi_{1\bar{x}_2}h_1h_2;\\
R_{h8}&=\frac{1}{2}\sum_{\omega_2}\tilde{b}_{1h}^{(1)}(\Delta\Phi_{1h})\,(\Delta y_1)_{\bar{x}_2}(\xi,x_2)\cdot\psi_{1\bar{x}_2}(\xi,x_2)h_1h_2;\\
R_{h9}&=\sum_{\omega_1^{(2)+}}\sum_{\omega_2} b_{2h}^{(2)}(\Delta\Phi_{2h})\,(\Delta y_2)_{\bar{x}_1}\cdot\psi_{2\bar{x}_1}h_1h_2;\ R_{h10}=\sum_{\omega_1^{(2)}}\sum_{\omega_2^+}\tilde{b}_{2h}^{(2)}(\Delta\Phi_{2h})\,(\Delta y_2)_{\bar{x}_2}\cdot\psi_{2\bar{x}_2}h_1h_2;\\
R_{h11}&=\frac{1}{2}\sum_{\omega_2}\tilde{b}_{2h}^{(2)}(\Delta\Phi_{2h})\,(\Delta y_2)_{\bar{x}_2}(\xi,x_2)\cdot\psi_{2\bar{x}_2}(\xi,x_2)h_1h_2.
\end{aligned} \tag{16}$$

By using inequality $\|y(x;\Phi_h)\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\le M\sum_{k=1}^{2}\|f_{kh}\|_{L_2(\omega^{(k)}\cup\gamma_S)}$, $\forall\Phi_h\in U_h$, we get estimates for the increment Δy and the solution ψ of the auxiliary problem (14):

$$\begin{aligned}
&\Big\|\Delta y\Big\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\le C_0\Big(\sum_{\alpha=1}^{2}\Big\|\Delta\Phi_{\alpha h}\Big\|_{L_\infty(\bar{\omega}^{(\alpha)})}+\sum_{\beta=1}^{2}\Big\|\Delta\Phi_{\beta+2,h}\Big\|_{L_2(\omega^{(1)}\cup\gamma_S)}\Big)=C_0\Big\|\Delta\Phi_h\Big\|_{\overline{B}_h},\\
&\Big\|\psi\Big(\Phi_h\Big)\Big\|_{\overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})}\le\widetilde{M}=Const,\qquad\forall\Phi_h\in U_h.
\end{aligned} \tag{17}$$

Since under the conditions of the theorem the function $q(y)$ satisfies additional restriction

$$\Big|q_i(y_i+\Delta y_i)-q_i(y_i)-q_i'(y_i)\Delta y_i\Big|\le\frac{\overline{L}_q}{2}\Big|\Delta y_i\Big|^2,\quad i=1,2,$$

we have

$$|R_{h2}| \le \sum_{\omega_1^{(1)}\times\omega_2} \Big| d_{1h}(x)\psi_1(x)\Big[q_1(y_1+\Delta y_1) - q_1(y_1) - q_{1y_1}\Big]\Delta y_1\Big| h_1 h_2 \le$$

$$\le \frac{\overline{L}_q}{2}\,\overline{d}_0 \sum_{\omega_1^{(1)}\times\omega_2} \big|\Delta y_1\big|^2 \big|\psi_1\big| h_1 h_2 \le C\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})}\big\|\psi\big\|_{W_2^1(\overline{\omega}^{(1)})};$$

$$|R_{h3}| \le C\big\|\Delta y_2\big\|^2_{W_2^1(\overline{\omega}^{(2)})}\big\|\Delta\psi_2\big\|_{W_2^1(\overline{\omega}^{(2)})};$$

$$\big|R_{h4}\big| = \Big|\sum_{\omega_2} d_{1h}(\xi,x_2)\psi_1(\xi,x_2)\Big[\frac{1}{2}\Big(q_1(y_1+\Delta y_1) - q_1(y_1)\Big) - q_{1y_1}\Delta y_1(\xi,x_2)\Big]h_1 h_2\Big| \le$$

$$\le C\big\|\Delta\psi_1\big\|_{W_2^1(\overline{\omega}^{(1)})}\Big\{\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})} + \big\|\Delta y_1\big\|_{W_2^1(\overline{\omega}^{(1)})}\Big\};$$

$$\big|R_{h5}\big| \le C\big\|\Delta\psi_2\big\|_{W_2^1(\overline{\omega}^{(2)})}\Big\{\big\|\Delta y_2\big\|^2_{W_2^1(\overline{\omega}^{(2)})} + \big\|\Delta y_2\big\|_{W_2^1(\overline{\omega}^{(2)})}\Big\};$$

$$\big|R_{h1}\big| \le \sum_{\omega_1^+\times\omega_2} \big|\Delta y_1\big|^2 \hbar_1 h_2 \le C\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})}.$$

Besides, we have

$$\big|R_{h6} + R_{h7} + R_{h8}\big| \le C\big\|\Delta\Phi_{1h}\big\|_{L_\infty(\overline{\omega}^{(1)})}\Big\{\big\|\Delta y_1\big\|^2_{W_2^1(\overline{\omega}^{(1)})}\big\|\psi_1\big\|_{W_2^1(\overline{\omega}^{(1)})}\Big\};$$

$$\big|R_{h9} + R_{h10} + R_{h11}\big| \le C\big\|\Delta\Phi_{2h}\big\|_{L_\infty(\overline{\omega}^{(2)})}\Big\{\big\|\Delta y_2\big\|^2_{W_2^1(\overline{\omega}^{(2)})}\big\|\psi_2\big\|_{W_2^1(\overline{\omega}^{(2)})}\Big\}.$$

Therefore, it follows that

$$\Delta J_h(\Phi_h) = \sum_{\alpha=1}^{2}\Big(\sum_{\omega_1^{(\alpha)+}}\sum_{\omega_2} \Delta b_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\, y_{\alpha\bar{x}_1}\psi_{\alpha\bar{x}_1} h_1 h_2 + \sum_{\omega_1^{(\alpha)}}\sum_{\omega_2^+} \Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})\, y_{\alpha\bar{x}_2}\psi_{\alpha\bar{x}_2} h_1 h_2 +$$

$$+\frac{1}{2}\sum_{\omega_2^+} \Delta\tilde{b}_{\alpha h}^{(\alpha)}(\Phi_{\alpha h})(\xi,x_2)\, y_{\alpha\bar{x}_2}(\xi,x_2)\psi_{\alpha\bar{x}_2}(\xi,x_2) h_1 h_2\Big) + \sum_{\alpha=1}^{2}\sum_{\omega^{(1)}} \Delta\Phi_{\alpha+2,h}\, y_{1\mathring{x}_\alpha}\, \psi_1(x) h_1 h_2 +$$

$$+\sum_{\alpha=1}^{2}\sum_{\omega_2} \Delta\Phi_{\alpha+2,h}(\xi,x_2)\, y_{1\mathring{x}_\alpha}(\xi,x_2)\, \psi_1(\xi,x_2) h_1 h_2 + o\Big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\Big).$$

Then

$$\Delta J_h(\Phi_h) =< J_h'(\Phi_h), \Delta\Phi_h > + o\Big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\Big), \tag{18}$$

where $< J_h'(\Phi_h), \Delta\Phi_h >= \sum_{\alpha=1}^{2}\Big(\frac{\partial J_h}{\partial\Phi_{\alpha h}}, \Delta\Phi_{\alpha h}\Big)_{L_2(\bar{\omega}^{(\alpha)})} + \sum_{\beta=3}^{4}\Big(\frac{\partial J_h}{\partial\Phi_{\beta h}}, \Delta\Phi_{\beta h}\Big)_{L_2(\omega^{(1)}\cup\gamma_S)}.$

Thus, in formula (18) for the increment of the functional the first component is a bounded linear functional on $\widetilde{B}_h$ with respect to $\Phi_{\beta h}$, $\beta = \overline{1,4}$, while the last one is of the order $o\Big(\big\|\Delta\Phi_h\big\|_{\widetilde{B}_h}\Big)$. It means that the functional $J_h(\Phi_h)$ is differentiable on the set U_h, in the space $\widetilde{B}_h$. And the functional gradient $J_h(\Phi_h)$ at point $\Phi_h \in U_h$ is given by (10). The theorem is proved. □

Theorem 4.2. *Let the conditions of the above theorem hold. Then the grid functional $J_h(\Phi_h)$ belongs to $C^{1,1}(\widetilde{B}_h)$, that is we have the estimate*

$$\big\|J_h'(\Phi_h + \Delta\Phi_h) - J_h'(\Phi_h)\big\| \le C\big\|\Phi_h\big\|_{\widetilde{B}_h}.$$

5. Concluding Remarks

The above results can be directly applied for solving the difference optimal control problems (6)-(8). In particularly, one can implement difference analogues of gradient methods (see [6]) for solving optimal control problems (6)-(8), using the calculated gradients of minimized grid functionals. The calculation of the gradients uses the numerical solutions of direct problems for the state and adjoint problems, obtained on the basis of the iterative method (see [2]).

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