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Dunkl Generalization of q-Szász-Mirakjan Operators which Preserve x^2

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Abstract. In the present paper we construct q-Szász-Mirakjan operators generated by Dunkl generalization of the exponential function which preserve x^2. We obtain some approximation results via universal Korovkin's type theorem for these operators and study convergence properties by using the modulus of continuity. Furthermore, we obtain a Voronovskaja type theorem for these operators.

1. Introduction and Preliminaries

Bernstein [3] introduced a sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),\ x \in [0,1], \tag{1}$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.
Szász [16] introduced the operators

$$S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),\ f \in C[0,\infty), x \geq 0. \tag{2}$$

It has been observed that a sequence of linear positive operators preserves constant as well as linear functions i.e. $L_n(e_i,x) = e_i(x)$ for $e_i(x) = x^i$ $(i = 0,1)$. These conditions hold for Bernstein polynomials, Szász-Mirakjan operators, Baskakov operators, Phillips operators and so on. For each of the above operators $L_n(e_2,x) \neq e_2(x)$. King [8] gave the modification of the well known Bernstein polynomials in order to preserve e_0 and e_2. He considered $r_n^*(x)$ as

$$r_n^*(x) = \begin{cases} -\frac{1}{2(n-1)} + \sqrt{(\frac{n}{n-1})x^2 + \frac{1}{4(n-1)^2}} & \text{if } n = 2,3,\cdots, \\ x^2 & \text{if } n = 1. \end{cases} \tag{3}$$

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The modified form of Bernstein operators becomes

$$V_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right), \tag{4}$$

with $0 \leq r_n^*(x) \leq 1, \quad n = 1, 2, \cdots, \quad 0 \leq x \leq 1$. Obviously, $\lim_{n\to\infty} r_n^*(x) = x$. Also,

$$V_n(e_0, x) = 1, V_n(e_1, x) = r_n^*(x), V_n(e_2, x) = x^2.$$

Approximation results on Szász-Mirakjan operators, q-Szász-Mirakjan operators and q-Stancu-Beta operators preserving e_2 have been studied in [6], [9] and [11], respectively. The purpose of this paper is to construct and investigated Dunkl analogue of q-Szász-Mirakjan operators which preserve the functions e_0 and e_2. Recently, work on Dunkl analogues has been done in [10], [13], [14] and [12].

We recall some definitions and notations of q-calculus which plays an important role in approximation theory (see [2]).

Let $k \in \mathbb{N}_0$ and $q \in (0, 1)$. Then q-integer $[k]_q$ is defined as

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ k & \text{if } q = 1. \end{cases}$$

The q-factorial $[k]_q!$ is defined as

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0. \end{cases}$$

and for $k \in \mathbb{N}$, q-binomial coefficient $\begin{bmatrix} k \\ r \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} k \\ r \end{bmatrix}_q = \begin{cases} \frac{[k]_q!}{[r]_q![k-r]_q!} & \text{if } 1 \leq r \leq k, \\ 1 & \text{if } r = 0, \\ 0 & \text{if } r > k. \end{cases}$$

There are two q-analogues of the exponential function e^x (see [7])

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}, \quad |q| < 1,$$

where

$$(1-x)_q^{\infty} = \prod_{j=0}^{\infty} (1 - q^j x).$$

It is obvious that for $q = 1$, q-calculus reduces to ordinary calculus.

Sucu [15] defined Dunkl analogue of Szász operator by

$$S_n(f;x) = \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} f\left(\frac{k+2\mu\theta_k}{n}\right) \frac{(nx)^k}{\gamma_\mu(k)}, \tag{5}$$

where $\mu \geq 0$, $n \in \mathbb{N}$, $x \geq 0$, $f \in C[0,\infty)$ and

$$e_\mu(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_\mu(k)}.$$

Here

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma(k+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma(k+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}.$$

Recursion relation for γ_μ is given by

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \quad k \in \mathbb{N}_0,$$

where

$$\theta_k = \begin{cases} 0 & \text{if } k \ \textit{is even}, \\ 1 & \text{if } k \ \textit{is odd}. \end{cases}$$

Cheikh et al. [5] defined the Dunkl analogue of classical q-Hermite polynomials and gave definitions of the q-Dunkl analogue of exponential functions, explicit formula and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$, respectively.

$$e_{\mu,q}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_{\mu,q}(k)}, \quad x \in [0,\infty), \tag{6}$$

and

$$E_{\mu,q}(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{\gamma_{\mu,q}(k)}, \quad x \in [0,\infty), \tag{7}$$

An explicit formula for $\gamma_{\mu,q}(k)$ is given by

$$\gamma_{\mu,q}(k) = \frac{(q^{2\mu+1},q^2)_{[\frac{k+1}{2}]}(q^2,q^2)_{[\frac{k}{2}]}}{(1-q)^k}, \qquad k \in \mathbb{N}_0, \tag{8}$$

where

$$(x,q)_k = \prod_{n=0}^{k-1}(1-q^n x).$$

Some of the special cases of $\gamma_{\mu,q}(k)$ are as follows

$$\gamma_{\mu,q}(0) = 1, \qquad \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q}, \quad \gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right),$$

$$\gamma_{\mu,q}(3)=\left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^{2}}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right),$$

$$\gamma_{\mu,q}(4)=\left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^{2}}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right)\left(\frac{1-q^{4}}{1-q}\right).$$

Recursion relation for $\gamma_{\mu,q}$ is given by

$$\gamma_{\mu,q}(k+1)=[k+1+2\mu\theta_{k+1}]_q\gamma_{\mu,q}(k),\quad k\in\mathbb{N}_0, \tag{9}$$

where

$$\theta_k=\begin{cases}0 & \text{if } k \ \textit{is even},\\ 1 & \text{if } k \ \textit{is odd}.\end{cases}$$

Let $B_m[0,\infty)$ be the set of all functions f satisfying the condition $|f(x)|\leq M_f(1+x^m)$, where $x\in[0,\infty)$, $m>0$ and M_f is a constant depending on f. We write

$$C_m[0,\infty)=B_m[0,\infty)\cap C[0,\infty),$$

and

$$C_m^*[0,\infty)=\left\{f\in C_m[0,\infty):\exists\lim_{x\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\}.$$

These spaces are endowed with the norm

$$\|f\|_m=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}<\infty.$$

Let $f\in C[0,\infty)$. The modulus of continuity $\omega(f,\delta)$ is defined by

$$\omega(f,\delta)=\sup_{|t-x|\leq\delta,\ \ x,t\in[0,\infty)}|f(t)-f(x)|. \tag{10}$$

If λ is any positive real number then

$$\omega(f,\lambda\delta)\leq(1+\lambda)\omega(f,\delta). \tag{11}$$

If f is uniformly continuous on $(0,\infty)$ then it is necessary and sufficient that

$$\lim_{\delta\to 0}\omega(f,\delta)=0.$$

2. Auxiliary Results

We define a q-Dunkl analogue of Szász-Mirakjan operator as follows:

$$D_{n,q}(f;x)=\frac{1}{E_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}f\left(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q}\right)q^{\frac{k(k-1)}{2}}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}, \tag{12}$$

where $\mu>\frac{1}{2}$, $n\in\mathbb{N}$, $x\geq 0$, $0<q<1$ and $f\in C[0,\infty)$.

Lemma 2.1. *For $\mu>\frac{1}{2}$, $0<q<1$, and $m\in\mathbb{N}$, we have a recurrence relation given by*

$$\sum_{j=0}^{m}\binom{m}{j}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{m-j}\frac{x}{q^j}D_{n,q}(e_j;x)\leq D_{n,q}(e_{m+1};x)$$

$$\leq\sum_{j=0}^{m}\binom{m}{j}\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{m-j}\frac{x}{q^j}D_{n,q}(e_j;x).$$

Proof.

$$\begin{aligned} D_{n,q}(e_{m+1};x) &= \frac{1}{E_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\Big(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q}\Big)^{m+1} q^{\frac{k(k-1)}{2}}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \\ &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\Big(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q}\Big)^{m} q^{\frac{(k-1)(k-2)}{2}}\frac{([n]_q x)^{k-1}}{\gamma_{\mu,q}(k-1)} \end{aligned}$$

Using $[k+1+2\mu\theta_{k+1}]_q = [k+2\mu\theta_k]_q + q^{2\mu\theta_k+k}[2\mu(-1)^k+1]_q$,

$$\begin{aligned} &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\Big(\frac{[k-1+2\mu\theta_{k-1}]_q + q^{2\mu\theta_{k-1}+k-1}[2\mu(-1)^{k-1}+1]_q}{q^{k-1}[n]_q}\Big)^{m} \\ &\quad\times q^{\frac{(k-1)(k-2)}{2}}\frac{([n]_q x)^{k-1}}{\gamma_{\mu,q}(k-1)} \\ &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\frac{([k-1+2\mu\theta_{k-1}]_q)^j(q^{2\mu\theta_{k-1}+k-1}[2\mu(-1)^{k-1}+1]_q)^{m-j}}{q^{(k-1)m}[n]_q^m} \\ &\quad\times\ q^{\frac{(k-1)(k-2)}{2}}\frac{([n]_q x)^{k-1}}{\gamma_{\mu,q}(k-1)} \\ &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{k=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\frac{([k-1+2\mu\theta_{k-1}]_q)^j(q^{2\mu\theta_{k-1}}[2\mu(-1)^{k-1}+1]_q)^{m-j}}{q^{(k-1)j}[n]_q^m} \\ &\quad\times\ q^{\frac{(k-1)(k-2)}{2}}\frac{([n]_q x)^{k-1}}{\gamma_{\mu,q}(k-1)} \\ &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{j=0}^{m}\binom{m}{j}\frac{1}{q^j[n]_q^{m-j}}\sum_{k=0}^{\infty}(q^{2\mu\theta_k}[2\mu(-1)^k+1]_q)^{m-j}\Big(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q}\Big)^j \\ &\quad\times q^{\frac{k(k-1)}{2}}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \end{aligned}$$

Separating even and odd terms,

$$\begin{aligned} &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{j=0}^{m}\binom{m}{j}\frac{1}{q^j[n]_q^{m-j}}\sum_{k=0}^{\infty}\Big((q^{2\mu\theta_{2k}}[2\mu(-1)^{2k}+1]_q)^{m-j}\Big(\frac{[2k+2\mu\theta_{2k}]_q}{q^{2k-1}[n]_q}\Big)^j \\ &\quad\times q^{\frac{2k(2k-1)}{2}}\frac{([n]_q x)^{2k}}{\gamma_{\mu,q}(2k)} + (q^{2\mu\theta_{2k+1}}[2\mu(-1)^{2k+1}+1]_q)^{m-j}\Big(\frac{[2k+1+2\mu\theta_{2k+1}]_q}{q^{2k}[n]_q}\Big)^j \\ &\quad\times q^{\frac{2k(2k+1)}{2}}\frac{([n]_q x)^{2k+1}}{\gamma_{\mu,q}(2k+1)}\Big) \\ &= \frac{x}{E_{\mu,q}([n]_q x)}\sum_{j=0}^{m}\binom{m}{j}\frac{1}{q^j[n]_q^{m-j}}\sum_{k=0}^{\infty}\Big(([1+2\mu]_q)^{m-j}\Big(\frac{[2k+2\mu\theta_{2k}]_q}{q^{2k-1}[n]_q}\Big)^j q^{\frac{2k(2k-1)}{2}} \\ &\quad\times\frac{([n]_q x)^{2k}}{\gamma_{\mu,q}(2k)} + (q^{2\mu}[1-2\mu]_q)^{m-j}\Big(\frac{[2k+1+2\mu\theta_{2k+1}]_q}{q^{2k}[n]_q}\Big)^j q^{\frac{2k(2k+1)}{2}}\frac{([n]_q x)^{2k+1}}{\gamma_{\mu,q}(2k+1)}\Big) \end{aligned}$$

Using $[1-2\mu]_q \leq [1+2\mu]_q$,

$$\leq \frac{x}{E_{\mu,q}([n]_q x)} \sum_{j=0}^{m} \binom{m}{j} \frac{([1+2\mu]_q)^{m-j}}{q^j [n]_q^{m-j}} \sum_{k=0}^{\infty} \Big(\Big(\frac{[2k+2\mu\theta_{2k}]_q}{q^{2k-1}[n]_q} \Big)^j q^{\frac{2k(2k-1)}{2}} \frac{([n]_q x)^{2k}}{\gamma_{\mu,q}(2k)}$$
$$+ \Big(\frac{[2k+1+2\mu\theta_{2k+1}]_q}{q^{2k}[n]_q} \Big)^j q^{\frac{2k(2k+1)}{2}} \frac{([n]_q x)^{2k+1}}{\gamma_{\mu,q}(2k+1)} \Big)$$

$$\leq \frac{x}{E_{\mu,q}([n]_q x)} \sum_{j=0}^{m} \binom{m}{j} \frac{([1+2\mu]_q)^{m-j}}{q^j [n]_q^{m-j}} \sum_{k=0}^{\infty} \Big(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q} \Big)^j q^{\frac{k(k-1)}{2}} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}$$
$$\leq \sum_{j=0}^{m} \binom{m}{j} \frac{x([1+2\mu]_q)^{m-j}}{q^j [n]_q^{m-j}} D_{n,q}(e_j;x).$$

Similarly, on the other hand we have

$$D_{n,q}(e_{m+1};x) \geq \sum_{j=0}^{m} \binom{m}{j} \frac{x(q^{2\mu}[1-2\mu]_q)^{m-j}}{q^j [n]_q^{m-j}} D_{n,q}(e_j;x).$$

□

Lemma 2.2. *Let $D_{n,q}(.;.)$ be the operator given by* (12)*. Then we have the following identities and inequalities:*

1. $D_{n,q}(e_0;x) = 1$,
2. $D_{n,q}(e_1;x) = x$,
3. $\frac{x^2}{q} + \frac{q^{2\mu}[1-2\mu]_q x}{[n]_q} \leq D_{n,q}(e_2;x) \leq \frac{x^2}{q} + \frac{[1+2\mu]_q x}{[n]_q}$,
4. $\frac{x^3}{q^3} + \frac{(2q+1)x^2}{q^2} \frac{q^{2\mu}[1-2\mu]_q}{[n]_q} + \Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q} \Big)^2 x \leq D_{n,q}(e_3;x) \leq \frac{x^3}{q^3} + \frac{(2q+1)x^2}{q^2} \frac{[1+2\mu]_q}{[n]_q} + \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^2 x$,
5. $\frac{x^4}{q^6} + \frac{(3q^2+2q+1)x^3}{q^5} \frac{q^{2\mu}[1-2\mu]_q}{[n]_q} + \frac{(3q^2+3q+1)x^2}{q^3} \Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q} \Big)^2 + \Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q} \Big)^3 x \leq D_{n,q}(e_4;x) \leq \frac{x^4}{q^6} + \frac{(3q^2+2q+1)x^3}{q^5} \frac{[1+2\mu]_q}{[n]_q} + \frac{(3q^2+3q+1)x^2}{q^3} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^2 + \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^3 x$.

Proof. Proof is based on Lemma 2.1. We calculate only $D_{n,q}(e_3;x)$ and $D_{n,q}(e_4;x)$.

$$\begin{aligned} D_{n,q}(e_3;x) &\leq \sum_{j=0}^{2} \binom{2}{j} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^{2-j} \frac{x}{q^j} D_{n,q}(e_j;x) \\ &\leq \binom{2}{0} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^2 x D_{n,q}(e_0;x) + \binom{2}{1} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big) \frac{x}{q} D_{n,q}(e_1;x) \\ &\quad + \binom{2}{2} \frac{x}{q^2} D_{n,q}(e_2;x) \\ &\leq \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^2 x + \frac{2x^2}{q} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big) + \frac{x}{q^2} \Big(\frac{x^2}{q} + \frac{[1+2\mu]_q x}{[n]_q} \Big) \\ &\leq \frac{x^3}{q^3} + \frac{(2q+1)x^2}{q^2} \Big(\frac{[1+2\mu]_q}{[n]_q} \Big) + \Big(\frac{[1+2\mu]_q}{[n]_q} \Big)^2 x. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
D_{n,q}(e_3;x) &\geq \sum_{j=0}^{2}\binom{2}{j}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2-j}\frac{x}{q^j}D_{n,q}(e_j;x)\\
&\geq \binom{2}{0}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2}xD_{n,q}(e_0;x)+\binom{2}{1}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)\\
&\quad\times\frac{x}{q}D_{n,q}(e_1;x)+\binom{2}{2}\frac{x}{q^2}D_{n,q}(e_2;x)\\
&\geq \left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2}x+\frac{2x^2}{q}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)+\frac{x}{q^2}\left(\frac{x^2}{q}+\frac{q^{2\mu}[1-2\mu]_q x}{[n]_q}\right)\\
&\geq \frac{x^3}{q^3}+\frac{(2q+1)x^2}{q^2}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)+\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2}x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_{n,q}(e_4;x) &\leq \sum_{j=0}^{3}\binom{3}{j}\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{3-j}\frac{x}{q^j}D_{n,q}(e_j;x)\\
&\leq \binom{3}{0}\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{3}xD_{n,q}(e_0;x)+\binom{3}{1}\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{2}\frac{x}{q}D_{n,q}(e_1;x)\\
&\quad+\binom{3}{2}\left(\frac{[1+2\mu]_q}{[n]_q}\right)\frac{x}{q^2}D_{n,q}(e_2;x)+\binom{3}{3}\frac{x}{q^3}D_{n,q}(e_3;x)\\
&\leq \left(\frac{[1+2\mu]_q}{[n]_q}\right)^{3}x+\frac{3x^2}{q}\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{2}+\frac{3x}{q^2}\left(\frac{[1+2\mu]_q}{[n]_q}\right)\left(\frac{x^2}{q}+\frac{[1+2\mu]_q x}{[n]_q}\right)\\
&\quad+\frac{x}{q^3}\left(\frac{x^3}{q^3}+\frac{(2q+1)x^2}{q^2}\left(\frac{[1+2\mu]_q}{[n]_q}\right)+x\left(\frac{[1+2\mu]_q}{[n]_q}\right)^{2}\right)\\
&\leq \frac{x^4}{q^6}+\frac{(3q^2+2q+1)x^3}{q^5}\left(\frac{[1+2\mu]_q}{[n]_q}\right)+\frac{(3q^2+3q+1)x^2}{q^3}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2}\\
&\quad+\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{3}x.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D_{n,q}(e_4;x) &\geq \sum_{j=0}^{3}\binom{3}{j}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{3-j}\frac{x}{q^j}D_{n,q}(e_j;x)\\
&\geq \binom{3}{0}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{3}xD_{n,q}(e_0;x)+\binom{3}{1}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)^{2}\frac{x}{q}D_{n,q}(e_1;x)\\
&\quad+\binom{3}{2}\left(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\right)\frac{x}{q^2}D_{n,q}(e_2;x)+\binom{3}{3}\frac{x}{q^3}D_{n,q}(e_3;x)
\end{aligned}$$

$$
\begin{aligned}
&\geq \Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)^3 x + \frac{3x^2}{q}\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)^2 + \frac{3x}{q^2}\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)\\
&\quad \times\Big(\frac{x^2}{q} + \frac{q^{2\mu}[1-2\mu]_q x}{[n]_q}\Big) + \frac{x}{q^3}\Big(\frac{x^3}{q^3} + \frac{(2q+1)x^2}{q^2}\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)\\
&\quad + x\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)^2\Big)\\
&\geq \frac{x^4}{q^6} + \frac{(3q^2+2q+1)x^3}{q^5}\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big) + \frac{(3q^2+3q+1)x^2}{q^3}\Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)^2\\
&\quad + \Big(\frac{q^{2\mu}[1-2\mu]_q}{[n]_q}\Big)^3 x.
\end{aligned}
$$

□

Now, we want to transform the operators defined by (12) in order to preserve the quadratic function e_2. We define the functions

$$
v_{n,q}(x) = \frac{-q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2}}{2[n]_q}, \; x \geq 0, \tag{13}
$$

and the linear positive operators

$$
D^*_{n,q}(f;x) = D_{n,q}(f;v_{n,q}(x)) = \frac{1}{E_{\mu,q}([n]_q v_{n,q}(x))}\sum_{k=0}^{\infty} f\Big(\frac{[k+2\mu\theta_k]_q}{q^{k-1}[n]_q}\Big) q^{\frac{k(k-1)}{2}}\frac{([n]_q v_{n,q}(x))^k}{\gamma_{\mu,q}(k)}. \tag{14}
$$

Lemma 2.3. *The operators defined by* (14) *satisfy the following identities:*

1. $D^*_{n,q}(e_0;x) = 1$,
2. $D^*_{n,q}(e_1;x) = v_{n,q}(x)$,
3. $D^*_{n,q}(e_2;x) = x^2$,
4. $D^*_{n,q}((e_1 - e_0 x)^2;x) = 2x(x - v_{n,q}(x))$.

Proof. Using Lemma 2.2 and (14), we have

$$
\begin{aligned}
D^*_{n,q}(e_0;x) &= 1\\
D^*_{n,q}(e_1;x) &= v_{n,q}(x)\\
\frac{(v_{n,q}(x))^2}{q} + \frac{q^{2\mu}[1-2\mu]_q v_{n,q}(x)}{[n]_q} &\leq D^*_{n,q}(e_2;x) \leq \frac{(v_{n,q}(x))^2}{q} + \frac{[1+2\mu]_q v_{n,q}(x)}{[n]_q}
\end{aligned}
$$

Now,

$$
\begin{aligned}
D^*_{n,q}(e_2;x) &\geq \frac{(v_{n,q}(x))^2}{q} + \frac{q^{2\mu}[1-2\mu]_q v_{n,q}(x)}{[n]_q}\\
&\geq \frac{\Big(-q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2}\Big)^2}{4q[n]_q^2}
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{q^{2\mu}[1-2\mu]_q\Big(-q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2x^2}\Big)}{2[n]_q^2} \\
\geq \;& \frac{2q^{4\mu+1}[1-2\mu]_q^2 + 4[n]_q^2x^2 - 2q^{2\mu}[1-2\mu]_q\sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2x^2}}{4[n]_q^2} \\
& + \frac{-2q^{4\mu+1}[1-2\mu]_q^2 + 2q^{2\mu}[1-2\mu]_q\sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2x^2}}{4[n]_q^2} \\
\geq \;& \frac{4[n]_q^2x^2}{4[n]_q^2} = x^2.
\end{aligned}
$$

Similarly, on the other hand

$$
\begin{aligned}
D_{n,q}^*(e_2;x) &\leq \frac{(v_{n,q}(x))^2}{q} + \frac{[1+2\mu]_q v_{n,q}(x)}{[n]_q} \\
&\leq x^2.
\end{aligned}
$$

Hence, we have

$$D_{n,q}^*(e_2;x) = x^2.$$

Now, we have to prove (4). Since

$$D_{n,q}^*((e_1-e_0x)^2;x) = D_{n,q}((e_1-e_0x)^2;v_{n,q}(x))$$

By linearity of $D_{n,q}$ and from (1), (2) and (3), we have

$$
\begin{aligned}
D_{n,q}^*((e_1-e_0x)^2;x) &= D_{n,q}(e_2;v_{n,q}(x)) - 2xD_{n,q}(e_1;v_{n,q}(x)) + x^2D_{n,q}(e_0;v_{n,q}(x)) \\
&= x^2 - 2xv_{n,q}(x) + x^2 \\
&= 2x(x - v_{n,q}(x)).
\end{aligned}
$$

□

Lemma 2.4. *For $\mu > \frac{1}{2}$, $n \in \mathbb{N}$ and $q \in (0,1)$ let $v_{n,q}$ be defined by* (13). *Then the following statements hold:*

1. $v_{n,q}(0) = 0$,
2. $0 \leq v_{n,q}(x) \leq x$,
3. $x - v_{n,q}(x)$ *is strictly increasing in x and*

$$x - v_{n,q}(x) \leq \frac{1}{[n]_q}\frac{(1-q^n)x + [1+2\mu]_q}{1+\sqrt{q}},$$

$$D_{n,q}^*((e_1-e_0x)^2;x) \leq \frac{2}{[n]_q}\frac{(1-q^n)x^2 + [1+2\mu]_qx}{1+\sqrt{q}}.$$

Proof. To prove (3) we can consider the function $h : [0,\infty) \to \mathbb{R}$ defined by

$$h(x) = x - v_{n,q}(x)$$

Clearly, h is strictly increasing.

$$
\begin{aligned}
0 \leq h(x) &= x - \left(\frac{-q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2}}{2[n]_q} \right) \\
&= \frac{1}{2[n]_q} \left(2[n]_q x + q^{2\mu+1}[1-2\mu]_q - \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2} \right) \\
&= \frac{1}{2[n]_q} \frac{(2[n]_q x + q^{2\mu+1}[1-2\mu]_q)^2 - \left(\sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2} \right)^2}{2[n]_q x + q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2}} \\
&= \frac{1}{2[n]_q} \frac{4[n]_q(1-q^n)x^2 + 4[n]_q q^{2\mu+1}[1-2\mu]_q x}{2[n]_q x + q^{2\mu+1}[1-2\mu]_q + \sqrt{(q^{2\mu+1}[1-2\mu]_q)^2 + 4q[n]_q^2 x^2}} \\
&\leq \frac{1}{2[n]_q} \frac{4[n]_q(1-q^n)x^2 + 4[n]_q q^{2\mu+1}[1-2\mu]_q x}{2[n]_q x(1+\sqrt{q})} \\
&\leq \frac{1}{[n]_q} \frac{(1-q^n)x + q^{2\mu+1}[1-2\mu]_q}{(1+\sqrt{q})} \\
&\leq \frac{1}{[n]_q} \frac{(1-q^n)x + [1+2\mu]_q}{(1+\sqrt{q})}.
\end{aligned}
$$

□

3. Convergence of Modified q-Dunkl Szász Operators

In order to obtain the convergence results for the operators D^*_{n,q_n}, we take $q = (q_n)$ where $q_n \in (0,1)$ such that

$$
q_n \to 1, \ \ q_n^n \to a \ \ as \ \ n \to \infty. \tag{15}
$$

Lemma 3.1. *Let $q = (q_n)$ with $q_n \in (0,1)$ for all $n \in \mathbb{N}$ satisfy* (15)*. Then for every $x \in [0,\infty)$ we have,*

$$
\lim_{n\to\infty} [n]_{q_n} D^*_{n,q_n}(e_1 - e_0 x; x) = -\frac{(1-a)x + (1+2\mu)}{2}, \tag{16}
$$

$$
\lim_{n\to\infty} [n]_{q_n} D^*_{n,q_n}((e_1 - e_0 x)^2; x) = (1-a)x^2 + (1+2\mu)x, \tag{17}
$$

$$
3x^2(1-2\mu)^2 + 6(1-a)(1-2\mu)x^3 + 3(1-a)^2x^4 \leq \lim_{n\to\infty} [n]^2_{q_n} D^*_{n,q_n}((e_1 - e_0 x)^4; x)
$$

$$
\leq 3x^2(1+2\mu)^2 + 6(1-a)(1+2\mu)x^3 + 3(1-a)^2x^4. \tag{18}
$$

Proof. The proof is based on the following limit

$$
\lim_{n\to\infty} [n]_{q_n}(x - v_{n,q_n}(x)) = \frac{(1-a)x + (1+2\mu)}{2}.
$$

Using Lemma 2.3, proof of (16) and (17) are obvious. To show (18), we give an explicit formula for $D^*_{n,q_n}((e_1 - e_0 x)^4; x)$. With the help of Lemma 2.3, we have

$$
D^*_{n,q_n}((e_1 - e_0 x)^4; x) = D^*_{n,q_n}(e_4; x) - 4x D^*_{n,q_n}(e_3; x) + 6x^2 D^*_{n,q_n}(e_2; x) - 4x^3 D^*_{n,q_n}(e_1; x) + x^4.
$$

$$
\begin{aligned}
&\leq \frac{(v_{n,q_n}(x))^4}{q_n^6} + (3q_n^2+2q_n+1)\frac{(v_{n,q_n}(x))^3}{q_n^5}\frac{[1+2\mu]_{q_n}}{[n]_{q_n}} \\
&\quad + (3q_n^2+3q_n+1)\frac{(v_{n,q_n}(x))^2}{q_n^3}\Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)^2 + \Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)^3 v_{n,q_n}(x) \\
&\quad - 4x\Big(\frac{(v_{n,q_n}(x))^3}{q_n^3} + (2q_n+1)\frac{(v_{n,q_n}(x))^2}{q_n^2}\frac{[1+2\mu]_{q_n}}{[n]_{q_n}} + \Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)^2 v_{n,q_n}(x)\Big) \\
&\quad + 6x^2\Big(\frac{(v_{n,q_n}(x))^2}{q_n} + \frac{[1+2\mu]_{q_n}}{[n]_{q_n}}v_{n,q_n}(x)\Big) - 4x^3 v_{n,q_n}(x) + x^4 \\
&\leq \Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)^3 v_{n,q_n}(x) + \Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)^2\frac{1}{q_n^3}\Big((3q_n^2+3q_n+1)(v_{n,q_n}(x))^2 - 4xq_n^3 v_{n,q_n}(x)\Big) \\
&\quad + \Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)\frac{1}{q_n^5}\Big((3q_n^2+2q_n+1)(v_{n,q_n}(x))^3 - 4xq_n^3(2q_n+1)(v_{n,q_n}(x))^2 + 6x^2q_n^5 v_{n,q_n}(x)\Big) \\
&\quad + \Big(\frac{(v_{n,q_n}(x))^4}{q_n^6} - \frac{4x}{q_n^3}(v_{n,q_n}(x))^3 + \frac{6x^2}{q_n}(v_{n,q_n}(x))^2 - 4x^3 v_{n,q_n}(x) + x^4\Big) \\
&\leq A_{n,q_n}(x) + B_{n,q_n}(x) + C_{n,q_n}(x) + D_{n,q_n}(x).
\end{aligned}
$$

Simple but tedious calculation shows that

$$
\lim_{n\to\infty}[n]_{q_n}^2 D^*_{n,q_n}((e_1-e_0x)^4;x) \leq 0 + 3x^2(1+2\mu)^2 + 6(1+2\mu)(1-a)x^3 + 3(1-a)^2x^4.
$$

Let us show details of third limit,

$$
\begin{aligned}
\lim_{n\to\infty}[n]_{q_n}^2 C_{n,q_n}(x) &\leq \lim_{n\to\infty}[n]_{q_n}^2\Big(\frac{[1+2\mu]_{q_n}}{[n]_{q_n}}\Big)\frac{1}{q_n^5}\Big((3q_n^2+2q_n+1)(v_{n,q_n}(x))^3 \\
&\quad - 4xq_n^3(2q_n+1)(v_{n,q_n}(x))^2 + 6x^2q_n^5 v_{n,q_n}(x)\Big) \\
&\leq \lim_{n\to\infty}\Big(\frac{[n]_{q_n}}{q_n^5}[1+2\mu]_{q_n}(3q_n^2+2q_n+1)(v_{n,q_n}(x)-x)(v_{n,q_n}(x))^2 \\
&\quad + \frac{[n]_{q_n}}{q_n^5}[1+2\mu]_{q_n}(3q_n^2+2q_n+1-8q_n^4-4q_n^3)(v_{n,q_n}(x)-x)xv_{n,q_n}(x) \\
&\quad + \frac{[n]_{q_n}}{q_n^5}[1+2\mu]_{q_n}(q_n-1)(6q_n^4-6q_n^2-2q_n^3-3q_n-1)x^2 v_{n,q_n}(x)\Big) \\
&\to -6(1+2\mu)\frac{(1-a)x+(1+2\mu)}{2}x^2 + 6(1+2\mu)\frac{(1-a)x+(1+2\mu)}{2}x^2 \\
&\quad + 6(1+2\mu)(1-a)x^3 \\
&= 6(1+2\mu)(1-a)x^3.
\end{aligned}
$$

Similarly, on the other hand

$$
\lim_{n\to\infty}[n]_{q_n}^2 D^*_{n,q_n}((e_1-e_0x)^4;x) \geq 0 + 3x^2(1-2\mu)^2 + 6(1-2\mu)(1-a)x^3 + 3(1-a)^2x^4.
$$

□

Theorem 3.2. *Let $q=(q_n)$ with $q_n \in (0,1)$ for all $n \in \mathbb{N}$ satisfy* (15)*. Then the sequence $D^*_{n,q_n}(f;x)$ converges uniformly to f on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. Assume that $\lim_{n\to\infty} q_n = 1$. Fix $A > 0$ and consider the lattice homomorphism $T_A : C[0,\infty) \to C[0,A]$ defined by

$$T_A(f) = f\,|_{[0,A]}$$

Now, we see

$$T_A(D^*_{n,q_n}(e_0;x)) = T_A(1),$$

$$T_A(D^*_{n,q_n}(e_1;x)) = T_A(v_{n,q_n}(x)) \to T_A(x)$$

and

$$T_A(D^*_{n,q_n}(e_2;x)) = T_A(x^2),$$

uniformly on $[0,A]$. We see that $C^*_2[0,\infty)$ is isomorphic to $C[0,1]$ and the set $\{1,t,t^2\}$ is a Korovkin's set in $C^*_2[0,\infty)$ (see [1]). By Korovkin's theorem, $D^*_{n,q_n}(f;x) \to f(x)$ uniformly on $[0,A]$ as $n \to \infty$, provided $f \in C^*_2[0,\infty)$ and $A > 0$. Now we prove converse result by contradiction. Suppose that $\{q_n\}$ does not converge to 1. Then it has to contain a subsequence $\{q_{n_k}\} \subset (0,1)$ such that $q_{n_k} \to \alpha \in [0,1)$, as $k \to \infty$. Therefore

$$\frac{1}{[n_k]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-q_{n_k}^{n_k}} \to 1-\alpha \; as \; k \to \infty.$$

Also,

$$[1-2\mu]_{q_{n_k}} = \frac{1-(q_{n_k})^{1-2\mu}}{1-q_{n_k}} \to \frac{1-\alpha^{1-2\mu}}{1-\alpha} \; as \; k \to \infty,$$

and we have

$$\begin{aligned} D^*_{n,q_{n_k}}(e_1;x) - x &= v_{n,q_{n_k}}(x) - x \\ &= \frac{-q_{n_k}^{2\mu+1}[1-2\mu]_{q_{n_k}} + \sqrt{(q_{n_k}^{2\mu+1}[1-2\mu]_{q_{n_k}})^2 + 4q_{n_k}[n]^2_{q_{n_k}}x^2}}{2[n]_{q_{n_k}}} \\ &\to \frac{-\alpha^{2\mu+1}(\frac{1-\alpha^{1-2\mu}}{1-\alpha}) + \sqrt{(\alpha^{2\mu+1}(\frac{1-\alpha^{1-2\mu}}{1-\alpha}))^2 + 4\alpha(\frac{1}{1-\alpha})^2x^2}}{\frac{2}{1-\alpha}} \\ &\neq 0. \end{aligned}$$

This is a contradiction. Thus $q_n \to 1$ *as* $n \to \infty$. □

The following result is a Dunkl q-analogue of Theorem 1 of [4].

Theorem 3.3. *Let $D^*_{n,q}(.;.)$ be the operators defined by* (14) *and $f \in C^*_m[0,\infty)$. Let*

$$f^*(z) = f(z^2), z \in [0,\infty).$$

Then for all $t > 0$ and $x \in [0,\infty)$, we have

$$|D^*_{n,q_n}(f(t);x) - f(x)| \le 2\omega\left(f^*; \sqrt{\frac{2}{[n]_{q_n}} \frac{(1-q_n^n)x + [1+2\mu]_{q_n}}{(1+\sqrt{q_n})}}\right),$$

where $\omega(f,\delta)$ is the modulus of continuity of the function $f \in C^*_m[0,\infty)$ defined in (10)*. Therefore $D^*_{n,q_n}(f;x)$ converges to f uniformly on $[0,A]$ as $n \to \infty$, whenever f is uniformly continuous.*

Proof. Let $t > 0, x \in [0,\infty)$ and $f \in C^*_m[0,\infty)$ be fixed.
By the definition of f^*, we have

$$D^*_{n,q_n}(f;x) = D^*_{n,q_n}(f^*(\sqrt{\cdot});x)$$

Thus,

$$\begin{aligned}
|D^*_{n,q_n}(f;x) - f(x)| &= |D^*_{n,q_n}(f^*(\sqrt{\cdot});x) - f^*(\sqrt{x})| \\
&= \Big|\sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x)) f^*\Big(\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}}\Big) - f^*(\sqrt{x})\Big| \\
&= \Big|\sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x))\Big(f^*\Big(\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}}\Big) - f^*(\sqrt{x})\Big)\Big| \\
&\leq \sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x))\Big|f^*\Big(\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}}\Big) - f^*(\sqrt{x})\Big| \\
&\leq \sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x))\omega\Big(f^*;\Big|\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}} - \sqrt{x}\Big|\Big) \\
&\leq \sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x))\omega\Big(f^*;\frac{\Big|\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}} - \sqrt{x}\Big|}{D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)} D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)\Big).
\end{aligned}$$

From the inequality (11), we obtain

$$\begin{aligned}
|D^*_{n,q_n}(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x))\Big(1 + \frac{\Big|\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}} - \sqrt{x}\Big|}{D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)}\Big) \\
&\quad \times \omega(f^*; D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)) \\
&\leq \omega(f^*; D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)) \sum_{k=0}^{\infty} s_{n,k}(q_n, v_{n,q_n}(x)) \\
&\quad \times \Big(1 + \frac{\Big|\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}} - \sqrt{x}\Big|}{D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)}\Big) \\
&= 2\omega(f^*; D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x)).
\end{aligned}$$

In order to complete the proof we need to show that for all $t > 0$ and $x \in [0,\infty)$,

$$D^*_{n,q_n}(|\sqrt{\cdot} - \sqrt{x}|;x) \leq \sqrt{\frac{2}{[n]_{q_n}} \frac{(1-q_n^n)x + [1+2\mu]_{q_n}}{(1+\sqrt{q_n})}}.$$

Using Cauchy Schwartz inequality and Lemma 2.4, we have

$$
\begin{aligned}
D^*_{n,q_n}(|\sqrt{.}-\sqrt{x}|;x) &= \sum_{k=0}^{\infty}\left|\sqrt{\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}}-\sqrt{x}\right| s_{n,k}(q_n,v_{n,q_n}(x)) \\
&\leq \frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\left|\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}-x\right| s_{n,k}(q_n,v_{n,q_n}(x)) \\
&\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{\infty}\left|\frac{[k+2\mu\theta_k]_{q_n}}{q_n^{k-1}[n]_{q_n}}-x\right|^2 s_{n,k}(q_n,v_{n,q_n}(x))} \\
&\leq \frac{1}{\sqrt{x}}\sqrt{D^*_{n,q_n}((.-x)^2;x)} \\
&\leq \frac{1}{\sqrt{x}}\sqrt{2x(x-v_{n,q_n}(x))} \\
&\leq \sqrt{2(x-v_{n,q_n}(x))} \\
&\leq \sqrt{\frac{2}{[n]_{q_n}}\frac{(1-q_n^n)x+[1+2\mu]_{q_n}}{(1+\sqrt{q_n})}}.
\end{aligned}
$$

This completes the proof. □

Next, we prove Voronovskaja type result for Dunkl analogue of q-Szász operators.

Theorem 3.4. *Let $q=(q_n)$ with $q_n\in(0,1)$ for all $n\in\mathbb{N}$ satisfy* (15). *For any $f\in C_2^*[0,\infty)$ such that $f',f''\in C_2^*[0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\Big(D^*_{n,q_n}(f;x)-f(x)\Big)=\frac{(1-a)x+(1+2\mu)}{2}\Big(xf''(x)-f'(x)\Big),$$

uniformly on any $[0,A]$, $A>0$.

Proof. Let $f,f',f''\in C_2^*[0,\infty)$ and $x\in[0,\infty)$ be fixed. By Taylor formula, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+(t-x)^2r(t;x), \tag{19}$$

where $r(t;x)$ is the Peano form of the remainder $r(.;x)\in C_2^*[0,\infty)$ and $\lim_{t\to x}r(t;x)=0$. Applying D^*_{n,q_n} to (19), we obtain

$$
\begin{aligned}
D^*_{n,q_n}(f(t);x)-f(x) &= f'(x)D^*_{n,q_n}((e_1-e_0x);x)+\frac{1}{2}f''(x)D^*_{n,q_n}((e_1-e_0x)^2;x) \\
&\quad + D^*_{n,q_n}(r(t;x)(e_1-e_0x)^2;x).
\end{aligned}
$$

By Cauchy-Schwartz inequality, we have

$$D^*_{n,q_n}(r(.;x)(.-x)^2;x)\leq\sqrt{D^*_{n,q_n}(r^2(.;x);x)}\sqrt{D^*_{n,q_n}((.-x)^4;x)}. \tag{20}$$

Observe that $r^2(x,x)=0$ and $r^2(.;x)\in C_2^*[0,\infty)$. Then from Theorem 3.2, we have

$$\lim_{n\to\infty}D^*_{n,q_n}(r^2(.;x);x)=r^2(x,x)=0, \tag{21}$$

uniformly with respect to $x \in [0, A]$. Now from (20), (21) and Lemma 3.1, we have

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}\Big(D^*_{n,q_n}(f(t);x) - f(x)\Big) &= f'(x)\lim_{n\to\infty}[n]_{q_n}D^*_{n,q_n}((e_1 - e_0x);x)\\ &\quad + \frac{1}{2}f''(x)\lim_{n\to\infty}[n]_{q_n}D^*_{n,q_n}((e_1 - e_0x)^2;x)\\ &= f'(x)\Big(-\frac{(1-a)x + (1+2\mu)}{2}\Big)\\ &\quad + f''(x)\Big((1-a)x^2 + (1+2\mu)x\Big)\\ &= \frac{1-a}{2}x\Big(xf''(x) - f'(x)\Big)\\ &\quad + \frac{1+2\mu}{2}\Big(xf''(x) - f'(x)\Big)\\ &= \frac{(1-a)x + (1+2\mu)}{2}\Big(xf''(x) - f'(x)\Big).\end{aligned}$$

This completes the proof. □

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