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The Topology of θ_ω-Open Sets

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Abstract. We define the θ_ω-closure operator as a new topological operator. We show that θ_ω-closure of a subset of a topological space is strictly between its usual closure and its θ-closure. Moreover, we give several sufficient conditions for the equivalence between θ_ω-closure and usual closure operators, and between θ_ω-closure and θ-closure operators. Also, we use the θ_ω-closure operator to introduce θ_ω-open sets as a new class of sets and we prove that this class of sets lies strictly between the class of open sets and the class of θ-open sets. We investigate θ_ω-open sets, in particular, we obtain a product theorem and several mapping theorems. Moreover, we introduce ω-T_2 as a new separation axiom by utilizing ω-open sets, we prove that the class of ω-T_2 is strictly between the class of T_2 topological spaces and the class of T_1 topological spaces. We study relationship between ω-T_2 and ω-regularity. As main results of this paper, we give a characterization of ω-T_2 via θ_ω-closure and we give characterizations of ω-regularity via θ_ω-closure and via θ_ω-open sets.

1. Introduction

Let (X, τ) be a topological space and let $A \subseteq X$. Denote the closure of A by $\overline{A}$. A point $x \in X$ is in θ-closure of A [27] ($x \in Cl_\theta(A)$) if $\overline{U} \cap A \neq \emptyset$ for any $U \in \tau$ and with $x \in U$. A set A is called θ-closed [27] if $Cl_\theta(A) = A$. The complement of a θ-closed set is called a θ-open set. Denote the family of all θ-open sets in (X, τ) by τ_θ. It is known that τ_θ forms a topology on X coarser than the topology τ and $\tau_\theta = \tau$ if and only if (X, τ) is regular. Authors in [6, 7, 17, 18, 21–25, 28] continued the study of θ-closure operator, θ-open sets, and their related topological concepts. Recently, authors in [8–10, 19] have studied several generalizations of θ-open sets. A set A is ω-open set in (X, τ) [20] if for each $x \in A$, there is $U \in \tau$ such that $x \in U$ and $U - A$ is countable, or equivalently, A is ω-open set in (X, τ) [1] if for each $x \in A$, there is $U \in \tau$ and a countable set $C \subseteq X$ such that $x \in U - C \subseteq A$. Denote the family of all ω-open sets in (X, τ) by τ_ω. It is known that τ_ω forms a topology on X finer than τ. ω-open sets played a vital role in general topology research see, [1, 4, 5, 11–16, 29]. Al Ghour in [1], used ω-open sets to define ω-regularity as a generalization of regularity as follows: A topological space (X, τ) is ω-regular if for each closed set F in (X, τ) and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U$ and $F \subseteq V$ with $U \cap V = \emptyset$. The closure of A in the topological space (X, τ_ω) is called the ω-closure of A in (X, τ) and is denoted by $\overline{A}^{\omega}$. In this work, we use the ω-closure operator to define the θ_ω-closure operator in a similar way to that used in the definition of the θ-closure operator as follows: A point $x \in X$ is in θ_ω-closure of A ($x \in Cl_{\theta_\omega}(A)$) if $\overline{U}^{\omega} \cap A \neq \emptyset$ for any $U \in \tau$ with

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$x \in U$. A set A is called θ_ω-closed if $Cl_{\theta_\omega}(A) = A$. The complement of a θ_ω-closed set is called a θ_ω-open set. Denote the family of all θ_ω-open sets in (X, τ) by τ_{θ_ω}. We will show that τ_{θ_ω} forms a topology on X which is strictly between τ_θ and τ. Moreover, $\tau_{\theta_\omega} = \tau$ if and only if (X, τ) is ω-regular. In section 2, we define the θ_ω-closure operator as a new topological operator. We show that the θ_ω-closure of a subset of a topological space is strictly between its usual closure and its θ-closure. Moreover, we give several sufficient conditions for the equivalence between θ_ω-closure and usual closure operators, and between θ_ω-closure and θ-closure operators. Also, we use the θ_ω-closure operator to introduce θ_ω-open sets as a new class of sets and we prove that this class of sets lies strictly between the class of open sets and the class of θ-open sets. We investigate θ_ω-open sets, in particular, we obtain a product theorem and several mapping theorems.

In section 3, we introduce ω-T_2 as a new separation axiom by utilizing ω-open sets, we prove that the class of ω-T_2 is strictly between the class of T_2 topological spaces and the class of T_1 topological spaces. We study relationships between ω-T_2 and ω-regularity. As the main results of this chapter, we give a chaterization of ω-T_2 via θ_ω-closure and we give characterizations of ω-regularity via θ_ω-closure and via θ_ω-open sets.

In this paper, $\mathbb{R}, \mathbb{Q}, \mathbb{Q}^c$ and $\mathbb{N}$ denote, respectively the set of real numbers, the set of rational numbers, the set of irrational numbers and the set of natural numbers.

2. θ_ω-Closure Operator and the Topology of θ_ω-Open Sets

Let us start by the following definition:

Definition 2.1. ([27]) Let (X, τ) be a topological space and let $A \subseteq X$.

a. A point x in X is in the θ-closure of A ($x \in Cl_\theta(A)$) if $\overline{U} \cap A \neq \emptyset$ for any $U \in \tau$ and $x \in U$.
b. A is θ-closed if $Cl_\theta(A) = A$.
c. A is θ-open if the complement of A is θ-closed.
d. The family of all θ-open sets in (X, τ) is denoted by τ_θ.

Theorem 2.2. ([27]) *Let (X, τ) be a topological space. Then*

a. τ_θ forms a topology on X.
b. $\tau_\theta \subseteq \tau$ and $\tau_\theta \neq \tau$ in general.

Definition 2.3. ([20]) Let (X, τ) be a topological space and let $A \subseteq X$.

a. A point x in X is a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable.
b. A set A is ω-closed if it contains all its condensation points.
c. A set A is ω-open if the complement of A is ω-closed.

The family of all ω-open sets in a topological space (X, τ) is denoted by τ_ω. For a subset A of a topological space (X, τ), it is known that $A \in \tau_\omega$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U$ and $U - A$ is countable.

Theorem 2.4. ([2]) *Let (X, τ) be a topological space. Then*

a. τ_ω is a topology on X.
b. $\tau \subseteq \tau_\omega$ and $\tau_\omega \neq \tau$ in general.

Notation 2.5. ([1]) Let (X, τ) be a topological space and let $A \subseteq X$. The closure of A in (X, τ_ω) will be denoted by $\overline{A}^\omega$.

Theorem 2.6. ([1]) *Let (X, τ) be a topological space and let $A \subseteq X$. Then $\overline{A}^\omega \subseteq \overline{A}$ and $\overline{A}^\omega \neq \overline{A}$ in general.*

The following is the main definition of this work:

Definition 2.7. Let (X, τ) be a topological space and let $A \subseteq X$.

a. A point $x \in X$ is in the θ_ω-closure of A ($x \in Cl_{\theta_\omega}(A)$) if $\overline{U}^\omega \cap A \neq \emptyset$ for any $U \in \tau$ with $x \in U$.

b. A set A is called θ_ω-closed if $Cl_{\theta_\omega}(A) = A$.

c. A set A is called θ_ω-open if its complement is θ_ω-closed.

d. The family of all θ_ω-open sets in (X, τ) will be denoted by τ_{θ_ω}.

Theorem 2.8. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*

a. $\overline{A} \subseteq Cl_{\theta_\omega}(A) \subseteq Cl_\theta(A)$.

b. If A is θ-closed, then A is θ_ω-closed.

c. If A is θ_ω-closed, then A is closed.

Proof. (a) To see that $\overline{A} \subseteq Cl_{\theta_\omega}(A)$, let $x \in \overline{A}$ and let $U \in \tau$ with $x \in U$. Since $x \in \overline{A}$, $U \cap A \neq \emptyset$. Since $U \subseteq \overline{U}^\omega$, we have $\overline{U}^\omega \cap A \neq \emptyset$. Therefore, $x \in Cl_{\theta_\omega}(A)$. To see that $Cl_{\theta_\omega}(A) \subseteq Cl_\theta(A)$, let $x \in Cl_{\theta_\omega}(A)$ and let $U \in \tau$ with $x \in U$. Since $x \in Cl_{\theta_\omega}(A)$, $\overline{U}^\omega \cap A \neq \emptyset$. Since $\overline{U}^\omega \subseteq \overline{U}$, it follows that $\overline{U} \cap A \neq \emptyset$. Therefore, $x \in Cl_\theta(A)$.

(b) Suppose that A is θ-closed. Then $Cl_\theta(A) = A$. Thus by (a), $Cl_{\theta_\omega}(A) = A$ and hence A is θ_ω-closed.

(c) Suppose that A is θ_ω-closed. Then $Cl_{\theta_\omega}(A) = A$. Thus by (a), $\overline{A} = A$ and hence A is closed. □

Definition 2.9. Let (X, τ) be a topological space.

a. ([26]) (X, τ) is called locally countable if for each $x \in X$, there is $U \in \tau$ such that $x \in U$ and U is countable.

b. ([2]) (X, τ) is called anti-locally countable if each $U \in \tau - \{\emptyset\}$ is uncountable.

Lemma 2.10. ([1]) *a. If (X, τ) is an anti-locally countable topological space, then for all $A \in \tau_\omega$, $\overline{A}^\omega = \overline{A}$.*

b. If (X, τ) is locally countable, then τ_ω is the discrete topology.

Recall that a topological space (X, τ) is called locally indiscrete if every open set in (X, τ) is closed.

Definition 2.11. A topological space (X, τ) is said to be ω-locally indiscrete if every open set in (X, τ) is ω-closed.

Theorem 2.12. *a. Every locally indiscrete topological space is ω-locally indiscrete.*

b. Every locally countable topological space is ω-locally indiscrete.

Proof. (a) Follows from the fact that every closed set in a topological space is ω-closed.

(b) Let (X, τ) be locally countable. Then by Lemma 2.10 (b), τ_ω is the discrete topology. Thus, every open set in (X, τ) is ω-closed and hence (X, τ) is ω-locally indiscrete. □

The following example will show that the converse of each of the two implications in Theorem 2.12 is not true in general:

Example 2.13. Consider $(\mathbb{R}, \tau)$ where $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}\}$. Then $(\mathbb{R}, \tau)$ is ω-locally indiscrete. On the other hand, since $\mathbb{N}$ is open but not closed, then $(\mathbb{R}, \tau)$ is not locally indiscrete. Also, clearly that $(\mathbb{R}, \tau)$ is not locally countable.

Theorem 2.14. *Let (X, τ) be an ω-locally indiscrete topological space and let $A \subseteq X$. Then*

a. $\overline{A} = Cl_{\theta_\omega}(A)$.

b. If A is closed in (X, τ), then A is θ_ω-closed in (X, τ).

Proof. (a) By Theorem 2.8 (a), $\overline{A} \subseteq Cl_{\theta_\omega}(A)$. To see that $Cl_{\theta_\omega}(A) \subseteq \overline{A}$, let $x \in Cl_{\theta_\omega}(A)$ and $U \in \tau$ such that $x \in U$. Then $\overline{U}^\omega \cap A \neq \emptyset$. Since (X, τ) is ω-locally indiscrete, it follows that $\overline{U}^\omega = U$ and hence $U \cap A \neq \emptyset$. It follows that $x \in \overline{A}$.

(b) Suppose that A is closed in (X, τ), then $A = \overline{A}$. Thus by (a), $A = Cl_{\theta_\omega}(A)$ and hence A is θ_ω-closed in (X, τ). □

Corollary 2.15. *Let (X, τ) be locally indiscrete and let $A \subseteq X$. Then:*
a. $\overline{A} = Cl_{\theta_\omega}(A)$.
b. If A is closed in (X, τ), then A is θ_ω-closed in (X, τ).

Proof. Theorems 2.12 (a) and 2.14. □

Corollary 2.16. *Let (X, τ) be locally countable and let $A \subseteq X$. Then*
a. $\overline{A} = Cl_{\theta_\omega}(A)$.
b. If A is closed in (X, τ), then A is θ_ω-closed in (X, τ).

Proof. Theorems 2.12 (b) and 2.14. □

Theorem 2.17. *Let (X, τ) be an anti-locally countable topological space and let $A \subseteq X$. Then*
a. $Cl_\theta(A) = Cl_{\theta_\omega}(A)$.
b. If A is θ_ω-closed in (X, τ), then A is θ-closed in (X, τ).

Proof. (a) By Theorem 2.8 (a), $Cl_{\theta_\omega}(A) \subseteq Cl_\theta(A)$. To see that $Cl_\theta(A) \subseteq Cl_{\theta_\omega}(A)$ let $x \in Cl_\theta(A)$ and $U \in \tau$ such that $x \in U$. Then $\overline{U} \cap A \neq \emptyset$. Since (X, τ) is anti-locally countable, then by Lemma 2.10 (a), $\overline{U}^\omega = \overline{U}$ and hence $\overline{U}^\omega \cap A \neq \emptyset$. It follows that $x \in Cl_{\theta_\omega}(A)$.

(b) Suppose that A is θ_ω-closed in (X, τ), then $A = Cl_{\theta_\omega}(A)$. Thus by (a), $A = Cl_\theta(A)$ and hence A is θ-closed in (X, τ). □

Theorem 2.18. *Let (X, τ) be a topological space. Then $\tau_\theta \subseteq \tau_{\theta_\omega} \subseteq \tau$.*

Proof. To see that $\tau_\theta \subseteq \tau_{\theta_\omega}$, let $A \in \tau_\theta$. Then $X - A$ is θ-closed and by Theorem 2.8 (b), $X - A$ is θ_ω-closed. Thus $A \in \tau_{\theta_\omega}$. To see that $\tau_{\theta_\omega} \subseteq \tau$, let $A \in \tau_{\theta_\omega}$. Then $X - A$ is θ_ω-closed and by Theorem 2.8 (c), $X - A$ is closed. Thus $A \in \tau$. □

Lemma 2.19. ([27]) *Let (X, τ) be a topological space. Then for each $A \in \tau$, $Cl_\theta(A) = \overline{A}$.*

Theorem 2.20. *Let (X, τ) be a topological space.*
a. If $A \subseteq B \subseteq X$, then $Cl_{\theta_\omega}(A) \subseteq Cl_{\theta_\omega}(B)$.
b. For each subsets $A, B \subseteq X$, $Cl_{\theta_\omega}(A \cup B) = Cl_{\theta_\omega}(A) \cup Cl_{\theta_\omega}(B)$.
c. For each subset $A \subseteq X$, $Cl_{\theta_\omega}(A)$ is closed in (X, τ).
d. For each $A \in \tau_\omega$, $Cl_{\theta_\omega}(A) = \overline{A}$.
e. For each $A \in \tau$, $Cl_\theta(A) = Cl_{\theta_\omega}(A) = \overline{A}$.

Proof. (a) Let $x \in Cl_{\theta_\omega}(A)$ and $U \in \tau$ with $x \in U$. Since $x \in Cl_{\theta_\omega}(A)$, $\overline{U}^\omega \cap A \neq \emptyset$. Since $A \subseteq B$, $\overline{U}^\omega \cap B \neq \emptyset$. This implies that $x \in Cl_{\theta_\omega}(B)$.

(b) By (a), we have $Cl_{\theta_\omega}(A) \cup Cl_{\theta_\omega}(B) \subseteq Cl_{\theta_\omega}(A \cup B)$. Let $x \notin Cl_{\theta_\omega}(A) \cup Cl_{\theta_\omega}(B)$. Then there are $U, V \in \tau$ such that $x \in U \cap V$, $\overline{U}^\omega \cap A = \emptyset$ and $\overline{V}^\omega \cap B = \emptyset$. Thus, we have $x \in U \cap V \in \tau$ and

$$\begin{aligned} \overline{U \cap V}^\omega \cap (A \cup B) &= \left(\overline{U \cap V}^\omega \cap A\right) \cup \left(\overline{U \cap V}^\omega \cap B\right) \\ &\subseteq \left(\overline{U}^\omega \cap A\right) \cup \left(\overline{V}^\omega \cap B\right) \\ &= \emptyset \cup \emptyset \\ &= \emptyset. \end{aligned}$$

It follows that $x \notin Cl_{\theta_\omega}(A \cup B)$.

(c) We show that $X - Cl_{\theta_\omega}(A) \in \tau$. Let $x \in X - Cl_{\theta_\omega}(A)$. Then there is $U \in \tau$ such that $x \in U$ and $\overline{U}^\omega \cap A = \emptyset$. Thus, $U \cap Cl_{\theta_\omega}(A) = \emptyset$. It follows that $X - Cl_{\theta_\omega}(A) \in \tau$.

(d) By Theorem 2.8 (a), $\overline{A} \subseteq Cl_{\theta_\omega}(A)$. Conversely, suppose to the contrary that there is $x \in Cl_{\theta_\omega}(A) \cap \left(X - \overline{A}\right)$. Since $X - \overline{A} \in \tau$, we must have $\overline{X - \overline{A}}^\omega \cap A \neq \emptyset$. Choose $y \in \overline{X - \overline{A}}^\omega \cap A$. Since $A \in \tau_\omega$, then $\left(X - \overline{A}\right) \cap A \neq \emptyset$, a contradiction.

(e) Follows from (d) and Lemma 2.19. □

Theorem 2.21. *Let (X, τ) be a topological space. Then*
a. $\emptyset$ and X are θ_ω-closed sets.
b. Finite union of θ_ω-closed sets is θ_ω-closed.
c. Arbitrary intersection of θ_ω-closed sets is θ_ω-closed.

Proof. (a) Follows from Theorems 2.2 (a) and 2.8 (b).

(b) It is sufficient to see that the union of two θ_ω-closed sets is θ_ω-closed. Let A and B be any two θ_ω-closed sets in (X, τ). Then $Cl_{\theta_\omega}(A) = A$ and $Cl_{\theta_\omega}(B) = B$. By Theorem 2.20 (b),

$$\begin{aligned} Cl_{\theta_\omega}(A \cup B) &= Cl_{\theta_\omega}(A) \cup Cl_{\theta_\omega}(B) \\ &= A \cup B. \end{aligned}$$

It follows that $A \cup B$ is θ_ω-closed.

(c) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of θ_ω-closed sets in (X, τ). Then for all $\alpha \in \Delta, Cl_{\theta_\omega}(A_\alpha) = A_\alpha$. We show that $Cl_{\theta_\omega}(\cap\{A_\alpha : \alpha \in \Delta\}) \subseteq \cap\{A_\alpha : \alpha \in \Delta\}$. Let $x \in Cl_{\theta_\omega}(\cap\{A_\alpha : \alpha \in \Delta\})$ and let $U \in \tau$ such that $x \in U$. Then $\overline{U}^\omega \cap (\cap\{A_\alpha : \alpha \in \Delta\}) \neq \emptyset$. Therefore, $\overline{U}^\omega \cap A_\alpha \neq \emptyset$ for all $\alpha \in \Delta$. It follows that $x \in \cap\{Cl_{\theta_\omega}(A_\alpha) : \alpha \in \Delta\} = \cap\{A_\alpha : \alpha \in \Delta\}$. □

Theorem 2.22. *Let (X, τ) be a topological space. Then τ_{θ_ω} is a topology on X.*

Proof. (1) The fact that $\emptyset, X \in \tau_{\theta_\omega}$ follows from Theorem 2.21.

(2) Let $A, B \in \tau_{\theta_\omega}$. Then $X - A$ and $X - B$ are θ_ω-closed sets. By Theorem 2.21 (b),

$$X - (A \cap B) = (X - A) \cup (X - B)$$

is θ_ω-closed sets. Hence $A \cap B \in \tau_{\theta_\omega}$.

(3) Let $\{A_\alpha : \alpha \in \Delta\} \subseteq \tau_{\theta_\omega}$. Then $\{X - A_\alpha : \alpha \in \Delta\}$ is a family of θ_ω-closed sets in (X, τ). Thus by Theorem 2.21 (c),

$$X - \cup\{A_\alpha : \alpha \in \Delta\} = \cap\{X - A_\alpha : \alpha \in \Delta\}$$

is θ_ω-closed set. Hence $\cup\{A_\alpha : \alpha \in \Delta\} \in \tau_{\theta_\omega}$. □

Theorem 2.23. *Let (X, τ) be a topological space and $A \subseteq X$. Then $A \in \tau_{\theta_\omega}$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U}^\omega \subseteq A$.*

Proof. Suppose that $A \in \tau_{\theta_\omega}$ and let $x \in A$. Then $X - A$ is θ_ω-closed and $x \notin X - A$. Thus, $x \notin Cl_{\theta_\omega}(X - A)$ and hence there is $U \in \tau$ such that $x \in U$ and $\overline{U}^\omega \cap (X - A) = \emptyset$. Therefore, we have $x \in U \subseteq \overline{U}^\omega \subseteq A$.

Conversely, suppose for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U}^\omega \subseteq A$ and suppose on that contrary that $A \notin \tau_{\theta_\omega}$. Then $X - A$ is not θ_ω-closed and $Cl_{\theta_\omega}(X - A) \neq X - A$. Choose $x \in Cl_{\theta_\omega}(X - A) - (X - A)$. Since $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq \overline{U}^\omega \subseteq A$. Thus we have $x \in U \in \tau$ and hence $\overline{U}^\omega \cap (X - A) = \emptyset$. Therefore $x \notin Cl_{\theta_\omega}(X - A)$, a contradiction. □

Corollary 2.24. *Every open ω-closed set in a topological space is θ_ω-open.*

Proof. Let (X, τ) be a topological space and let A be open and ω-closed set in (X, τ). Let $x \in A$. Since A is ω-closed, then $\overline{A}^\omega = A$. Take $U = A$. Then $U \in \tau$ and $x \in U = \overline{U}^\omega = A \subseteq A$. Thus by Theorem 2.23, it follows that A is θ_ω-open. □

Corollary 2.25. *Every countable open set in a topological space is θ_ω-open.*

Proof. Follows directly form Corollary 2.24 since countable sets in a topological space are ω-closed. □

The following example shows that θ_ω-open sets are strictly between θ-open sets and open sets:

Example 2.26. Consider $(\mathbb{R}, \tau)$ where $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$. Then
a. $\tau_{\theta_\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}$.
b. $\tau_\theta = \{\emptyset, \mathbb{R}\}$.

Proof. (a) Note that $\tau_\omega = \tau_{coc} \cup \{A : A \subseteq \mathbb{N}\}$ where τ_{coc} is the cocountable topology on $\mathbb{R}$. Then a subset $B \subseteq \mathbb{R}$ is closed in $(\mathbb{R}, \tau_\omega)$ if and only if either B is countable or $B = \mathbb{R} - A$ where $A \subseteq \mathbb{N}$. So $\overline{\mathbb{Q}^c}^\omega = \mathbb{R} - \mathbb{N}$. If $\mathbb{Q}^c \in \tau_{\theta_\omega}$, then there is $U \in \tau$ such that $\sqrt{2} \in U \subseteq \overline{U}^\omega \subseteq \mathbb{Q}^c$. Since $\sqrt{2} \in U \in \tau$, then $\mathbb{Q}^c \subseteq U$ and so $\mathbb{R} - \mathbb{N} = \overline{\mathbb{Q}^c}^\omega \subseteq \overline{U}^\omega \subseteq \mathbb{Q}^c$ which is impossible, it follows that $\mathbb{Q}^c \in \tau - \tau_{\theta_\omega}$. If $\mathbb{N} \cup \mathbb{Q}^c \in \tau_{\theta_\omega}$, then there is $U \in \tau$ such that $\sqrt{2} \in U \subseteq \overline{U}^\omega \subseteq \mathbb{N} \cup \mathbb{Q}^c$. Since $\sqrt{2} \in U \in \tau$, then $\mathbb{Q}^c \subseteq U$ and so $\mathbb{R} - \mathbb{N} = \overline{\mathbb{Q}^c}^\omega \subseteq \overline{U}^\omega \subseteq \mathbb{N} \cup \mathbb{Q}^c$ which is impossible and it follows that $\mathbb{N} \cup \mathbb{Q}^c \in \tau - \tau_{\theta_\omega}$. By Corollary 2.25, $\mathbb{N} \in \tau_{\theta_\omega}$. This ends the proof that $\tau_{\theta_\omega} = \{\emptyset, \mathbb{R}, \mathbb{N}\}$.

(b) By Theorem 2.18, $\tau_\theta \subseteq \tau_{\theta_\omega}$. So to see that $\tau_\theta = \{\emptyset, \mathbb{R}\}$, it is sufficient to show that $\mathbb{N} \notin \tau_\theta$. If $\mathbb{N} \in \tau_\theta$, then there is $U \in \tau$ such that $1 \in U \subseteq \overline{U} \subseteq \mathbb{N}$. Since $1 \in U \in \tau$, we have $U = \mathbb{N}$ and so $\overline{\mathbb{N}} = \mathbb{N}$, but $\overline{\mathbb{N}} = \mathbb{Q}$. Therefore, $\mathbb{N} \notin \tau_\theta$. □

If (X, τ) and (Y, σ) are two topological spaces, then $\tau \times \sigma$ will denote the product topology on $X \times Y$, also π_x and π_y will denote the projection functions on X and Y, respectively.

Lemma 2.27. ([3]) *Let (X, τ) and (Y, σ) be two topological spaces.*
(a) $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.
(b) If $A \subseteq X$ and $B \subseteq Y$, then $\overline{A}^\omega \times \overline{B}^\omega \subseteq \overline{A \times B}^\omega$.

Theorem 2.28. *Let (X, τ) and (Y, σ) be two topological spaces. If $G \in (\tau \times \sigma)_{\theta_\omega}$, then $\pi_x(G) \in \tau_{\theta_\omega}$ and $\pi_y(G) \in \sigma_{\theta_\omega}$.*

Proof. Let $x \in \pi_x(G)$. Choose $y \in Y$ such that $(x, y) \in G$. Since $G \in (\tau \times \sigma)_{\theta_\omega}$, there is $H \in \tau \times \sigma$ such that $(x, y) \in H \subseteq \overline{H}^\omega \subseteq G$. Choose $U \in \tau$ and $V \in \sigma$ such that $(x, y) \in U \times V \subseteq H$. Thus, by Lemma 2.27 (b),

$$(x, y) \in U \times V \subseteq \overline{U}^\omega \times \overline{V}^\omega \subseteq \overline{U \times V}^\omega \subseteq \overline{H}^\omega \subseteq G$$

and hence

$$x \in U \subseteq \overline{U}^\omega \subseteq \pi_x(G).$$

It follows that $\pi_x(G) \in \tau_{\theta_\omega}$. Similarly, we can show that $\pi_y(G) \in \sigma_{\theta_\omega}$. □

If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a closed function, then $f : (X, \tau) \longrightarrow (Y, \sigma_\omega)$ is closed, but the converse is not true in general as the following example shows:

Example 2.29. Define $f : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_u)$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

For every closed subset C of $(\mathbb{R}, \tau_u)$, $f(C) \subseteq \mathbb{Q}$, which shows that f is ω-closed. Since $\mathbb{R}$ is closed in $(\mathbb{R}, \tau_u)$ but $f(\mathbb{R}) = \mathbb{Q}$ is not closed in $(\mathbb{R}, \tau_u)$, then f is not closed.

Theorem 2.30. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is open and $f : (X, \tau) \longrightarrow (Y, \sigma_\omega)$ is closed, then $f : (X, \tau_\theta) \longrightarrow (Y, \sigma_{\theta_\omega})$ is open.*

Proof. Let $A \in \tau_\theta$ and let $y \in f(A)$. Choose $x \in A$ such that $y = f(x)$. Choose $V \in \tau$ such that $x \in V \subseteq \overline{V} \subseteq A$. Thus, $f(x) = y \in f(V) \subseteq f\left(\overline{V}\right) \subseteq f(A)$. Since f is open, then $f(V) \in \sigma$. Since f is ω-closed, then $f\left(\overline{V}\right)$ is ω-closed and so $\overline{f(V)}^\omega \subseteq f\left(\overline{V}\right) \subseteq f(A)$. It follows that $f(A) \in \sigma_{\theta_\omega}$. □

Theorem 2.31. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is open and $f : (X, \tau_\omega) \longrightarrow (Y, \sigma_\omega)$ is closed, then $f : (X, \tau_{\theta_\omega}) \longrightarrow (Y, \sigma_{\theta_\omega})$ is open.*

Proof. Let $A \in \tau_{\theta_\omega}$ and let $y \in f(A)$. Choose $x \in A$ such that $y = f(x)$. Choose $V \in \tau$ such that $x \in V \subseteq \overline{V}^{\omega} \subseteq A$. Thus, $f(x) = y \in f(V) \subseteq f\left(\overline{V}^{\omega}\right) \subseteq f(A)$. Since f is open, then $f(V) \in \sigma$. Since f is ω-closed, then $f\left(\overline{V}\right)$ is ω-closed and so $\overline{f(V)}^{\omega} \subseteq f\left(\overline{V}\right) \subseteq f(A)$. It follows that $f(A) \in \sigma_{\theta_\omega}$. $\square$

Theorem 2.32. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be function. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $f : (X, \tau_\omega) \longrightarrow (Y, \sigma_\omega)$ are both continuous, then $f : (X, \tau_{\theta_\omega}) \longrightarrow (Y, \sigma_{\theta_\omega})$ is continuous.*

Proof. Let $B \in \sigma_{\theta_\omega}$ and let $x \in f^{-1}(B)$. Then $f(x) \in B$ and so there is $V \in \sigma$ such that $f(x) \in V \subseteq \overline{V}^{\omega} \subseteq B$. Thus, $x \in f^{-1}(V) \subseteq f^{-1}\left(\overline{V}^{\omega}\right) \subseteq f^{-1}(B)$. Since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is continuous, then $f^{-1}(V) \in \tau$. Since $f : (X, \tau_\omega) \longrightarrow (Y, \sigma_\omega)$ is continuous, then $f^{-1}\left(\overline{V}^{\omega}\right)$ is ω-closed and so $\overline{f^{-1}(V)}^{\omega} \subseteq f^{-1}\left(\overline{V}^{\omega}\right) \subseteq f^{-1}(B)$. It follows that $f^{-1}(B) \in \tau_{\theta_\omega}$. $\square$

3. Separation Axioms

Definition 3.1. A topological space (X, τ) is said to be ω-T_2 if for any pair (x, y) of distinct points in X there exist $U \in \tau, V \in \tau_\omega$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 3.2. *A topological space (X, τ) is ω-T_2 if and only if for each $x \in X, Cl_{\theta_\omega}(\{x\}) = \{x\}$.*

Proof. Suppose that (X, τ) is ω-T_2 and suppose on the contrary that for some $x \in X, Cl_{\theta_\omega}(\{x\}) \neq \{x\}$. Choose $y \in Cl_{\theta_\omega}(\{x\}) - \{x\}$. Then there exist $U \in \tau_\omega$ and $V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $y \in V \in \tau$ and $y \in Cl_{\theta_\omega}(\{x\})$, then $\overline{V}^{\omega} \cap \{x\} \neq \emptyset$. Thus we have $x \in U \in \tau_\omega$ and $x \in \overline{V}^{\omega}$ and hence $U \cap V \neq \emptyset$, a contradiction.

Conversely, suppose for each $x \in X, Cl_{\theta_\omega}(\{x\}) = \{x\}$. Let $x, y \in X$ with $x \neq y$. By assumption, $Cl_{\theta_\omega}(\{y\}) = \{y\}$ and so we have $x \notin Cl_{\theta_\omega}(\{y\})$. Thus there is $U \in \tau$ such that $x \in U$ and $\overline{U}^{\omega} \cap \{y\} = \emptyset$. Take $V = X - \overline{U}^{\omega}$. Then we have $y \in V \in \tau_\omega$ and $U \cap V = \emptyset$. This ends the proof that (X, τ) is ω-T_2. $\square$

Theorem 3.3. *If (X, τ) is an ω-T_2 topological space, then (X, τ_ω) is T_2.*

Proof. Obvious. $\square$

The converse of Theorem 3.3 is not true in general as the following example clarifies:

Example 3.4. Consider (X, τ) where X is any countable set which contains at least two distinct points and τ is the indiscrete topology. It is obvious that τ_ω is the discrete topology and so (X, τ_ω) is T_2. Choose $x, y \in X$ such that $x \neq y$. If $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U, y \in V$. Then $U = X$ and $U \cap V \neq \emptyset$. It follows that (X, τ) is not ω-T_2.

Theorem 3.5. *Every ω-T_2 topological space is T_1.*

Proof. Let (X, τ) be ω-T_2. We show for each $x \in X, \overline{\{x\}} \subseteq \{x\}$. Let $x \in X$. Since (X, τ) is ω-T_2, then by Theorem 3.2, $Cl_{\theta_\omega}(\{x\}) = \{x\}$. By Theorem 2.8 (a), we have $\overline{\{x\}} \subseteq Cl_{\theta_\omega}(\{x\}) = \{x\}$. $\square$

The following example shows that the converse of Theorem 3.5 is not true in general:

Example 3.6. Consider $(\mathbb{R}, \tau)$ where τ is the cofinite topology. It is clear that $(\mathbb{R}, \tau)$ is T_1. It is not difficult to check that τ_ω is the cocountable topology. Thus $(\mathbb{R}, \tau_\omega)$ is not T_2 and by Theorem 3.3, $(\mathbb{R}, \tau)$ not ω-T_2.

Theorem 3.7. *Every locally countable T_1 topological space is ω-T_2.*

Proof. Let (X, τ) be locally countable and T_1. Let $x, y \in X$ with $x \neq y$. Since (X, τ) is locally countable, then τ_ω is the discrete topology and so $\{y\} \in \tau_\omega$. On the other hand since (X, τ) is T_1, then $\{y\}$ is closed in (X, τ) and $X - \{y\} \in \tau$. Take $U = X - \{y\}$ and $V = \{y\}$. Then $U \in \tau, V \in \tau_\omega, x \in U, y \in V$ and $U \cap V = \emptyset$. This shows that (X, τ) is ω-T_2. □

Theorem 3.8. *Every T_2 topological space is ω-T_2.*

Proof. Obvious. □

The following example shows that the converse of Theorem 3.8 is not true in general:

Example 3.9. Consider $(\mathbb{N}, \tau)$ where τ is the cofinite topology. It is clear that $(\mathbb{N}, \tau)$ is T_1 and locally countable and thus by Theorem 3.7, it is ω-T_2. On the other hand, it is well known that $(\mathbb{N}, \tau)$ is not T_2.

Definition 3.10. ([1]) A topological space (X, τ) is called ω-regular if for each closed set F in (X, τ) and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

Theorem 3.11. ([1]) *A topological space (X, τ) is ω-regular if and only if for each $U \in \tau$ and each $x \in U$ there is $V \in \tau$ such that $x \in V \subseteq \overline{V}^{\omega} \subseteq U$.*

Theorem 3.12. ([27]) *A topological space (X, τ) is regular if and only if $\tau = \tau_\theta$.*

Theorem 3.13. ([18]) *A topological space (X, τ) is regular if and only if for each subset $A \subseteq X$, $Cl_\theta(A) = \overline{A}$.*

The following result modify Theorems 3.12 and 3.13 for ω-regular topological spaces:

Theorem 3.14. *For any topological space (X, τ), the following are equivalent:*
a. (X, τ) is ω-regular.
b. $\tau = \tau_{\theta_\omega}$.
c. For each subset $A \subseteq X$, $Cl_{\theta_\omega}(A) = \overline{A}$.

Proof. It follows from Theorems 2.18, 3.11 and 2.23. □

Corollary 3.15. *Every ω-locally indiscrete topological space is ω-regular.*

Proof. Theorems 2.14 and 3.14. □

Corollary 3.16. *Every locally indiscrete topological space is ω-regular.*

Proof. Theorem 2.12 (a) and Corollary 3.15. □

Corollary 3.17. *Every locally countable topological space is ω-regular.*

Proof. Theorem 2.12 (b) and Corollary 3.15. □

Theorem 3.18. ([1]) *Every regular topological space is ω-regular.*

The converse of Theorem 3.18 is not true in general: Consider the topological space in Example 3.9. By Corollary 3.17, $(\mathbb{N}, \tau)$ is ω-regular. On the other hand, it is well known that this topological space is not regular.

Theorem 3.19. *Every anti-locally countable ω-regular topological space is regular.*

Proof. Let (X, τ) be anti-locally countable and ω-regular. We will apply Theorem 3.13. Let $A \subseteq X$. Since (X, τ) is anti-locally countable, then by Theorem 2.17 (a) $Cl_\theta(A) = Cl_{\theta_\omega}(A)$. Also, by Theorem 3.14 we have $Cl_{\theta_\omega}(A) = \overline{A}$. It follows that $Cl_\theta(A) = \overline{A}$. □

The topological space in Example 3.9 is ω-regular but not ω-T_2. Thus, ω-regularity does not imply ω-T_2 in general, however we have the following result:

Theorem 3.20. *Every ω-regular T_1 topological space is ω-T_2.*

Proof. Let (X, τ) be ω-regular and T_1. We apply Theorem 3.2. Let $x \in X$. Since (X, τ) is ω-regular, then by Theorem 3.14, $Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$. Since (X, τ) is T_1, then $\overline{\{x\}} = \{x\}$. Therefore, $Cl_{\theta_\omega}(\{x\}) = \{x\}$. $\square$

To give an example on an ω-T_2 topological space that is not ω-regular, by Theorems 3.8 and 3.19 it is sufficient to give an example of an anti-locally countable T_2 topological space that is not regular. Consider $(\mathbb{R}, \tau_\omega)$ where τ is the usual topology on $\mathbb{R}$. Clearly that $(\mathbb{R}, \tau_\omega)$ is anti-locally countable. On the other hand it is well known that $(\mathbb{R}, \tau_\omega)$ is T_2 but not regular.

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