Filomat 32:3 (2018), 1117–1125
https://doi.org/10.2298/FIL1803117R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weak Solutions for Nonlinear Fractional Differential Equation with Fractional Separated Boundary Conditions in Banach Spaces

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Abstract. This paper deals with nonlinear fractional differential equation with fractional separated boundary conditions. We investigate the existence of weak solutions in Banach spaces. To obtain such result we apply an appropriate fixed point theorem and the technique of measures of weak noncompactness. An example illustrating the theory is given.

1. Introduction

In recent years, several papers have been devoted to the study of the existence of solutions for fractional differential equations, among others we refer to the papers by Agarwal et al. [1, 2], Ahmad et al. [3], Graef et al. [14], Hernández et al. [15]. Moreover, it has be proved that differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, control theory, rheology, chemistry, and so on (see the monograph of Kilbas et al. [16], Podlubny [21], Sabatier et al. [22], and Samko et al. [23]).

In this paper we investigate the existence of weak solutions, for a fractional boundary value problem with fractional separated boundary conditions given by

$$\begin{cases} {}^cD^r x(t) = f(t, x(t)), \quad t \in J = [0, 1],\ 1 < r \leq 2 \\ \alpha_1 x(0) + \beta_1({}^cD^p x(0)) = \gamma_1, \\ \alpha_2 x(1) + \beta_2({}^cD^p x(1)) = \gamma_2, \quad 0 < p < 1, \end{cases} \tag{1}$$

where ${}^cD^r$ denotes the Caputo fractional derivative of order r, f is a given function satisfying some assumptions that will be specified later, and $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ are constants in $\mathbb{R}$, with $\alpha_1 \neq 0$.

This problem was studied recently in [4] in the scalar case using Banach contraction principal, Krasnoselskii's fixed point theorem and the nonlinear alternative of Leray-Schauder type.

Here we extend the results of [4] to cover the abstract case. We establish the existence of solutions of the problem (1) using Mönch's fixed point theorem combined with the technique of measures of weak

2010 *Mathematics Subject Classification*. Primary 26A33; Secondary 34B15, 34G20
Keywords. Caputo fractional derivative, boundary value problem, Pettis integrals, measure of weak noncompactness, fixed point, Banach space
Received: 25 October 2016; Received: 16 December 2016; Accepted: 24 December 2016
Communicated by Ljubiša D.R. Kočinac
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noncompactness, which is an important method for seeking solutions of differential and integral equations. This technique was mainly initiated in the paper of De Blasi [13] and then subsequently developed and used in many papers, for example Allahyari et al. [5], Banaŝ et al. [6–8], Benchohra et al. [9–11], Dhage et al. [12], Lakshmikantham et al. [17], Liao et al. [18] and the references therein. Let us note that the exposition of this method in the framework of problem (1) is new.

2. Preliminaries

In this section, we will state definitions and notations that are used in the remainder of the paper.

Denote by $L^1(J)$ the Banach space of real-valued Lebesgue integrable functions, on the interval J.

E denotes the real Banach space with norm $\|.\|$ and dual E^* also $(E, w) = (E, \sigma(E, E^*))$ denotes the space E with its weak topology.

Let $L^\infty(J)$ be the Banach space of real-valued essentially bounded and measurable functions defined over J equipped with the norm $\|.\|_{L^\infty}$.

$C(J, E)$ is the Banach space of continuous functions $x : J \to E$, with the usual supremum norm.

$$\|x\|_\infty = \sup\{\|x(t)\|, \ t \in J\}.$$

Definition 2.1. A function $h : E \longrightarrow E$ is said to be *weakly sequentially continuous* if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e. for any $(x_n)_n$ in E with $x_n \longrightarrow x$ in (E, w) then $h(x_n) \longrightarrow h(x)$ in (E, w) for each $t \in J$).

Definition 2.2. ([20]) The function $x : J \longrightarrow E$ is said to be *Pettis integrable* on J if and only if there is an element $x_I \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J, E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Proposition 2.3. ([20]) *If $x(.)$ is Pettis integrable and $h(.)$ is a measurable and essentially bounded real-valued function, then $x(.)h(.)$ is Pettis integrable.*

Definition 2.4. ([13]) Let E be a Banach space, Ω_E the bounded subsets of E and B_1 the unit ball of E. The *De Blasi measure of weak noncompactness* is the map $\beta : \Omega_E \longrightarrow [0, \infty)$ defined by

$$\beta(X) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : \ X \subset \varepsilon B_1 + \Omega\}.$$

Properties:

The De Blasi measure of noncompactness satisfies some properties (for more details see [13]).

(a) $A \subset B \Longrightarrow \beta(A) \leq \beta(B)$,

(b) $\beta(A) = 0 \Longrightarrow A$ is relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^w) = \beta(A)$, ($\overline{A}^w$ denotes the weak closure of A),

(e) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) \leq |\lambda|\beta(A)$,

(g) $\beta(\mathrm{conv}(A)) \leq \beta(A)$,

(h) $\beta(\bigcup_{|\lambda|\leq h} \lambda A) = h\beta(A)$.

The following result follows directly from the Hahn-Banach theorem.

Proposition 2.5. *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

Let us now recall the definitions of the Pettis integral and Caputo derivative of fractional order.

Definition 2.6. ([25]) Let $h : J \longrightarrow E$ be a function. The *fractional Pettis integral* of the function h of order $r \in \mathbb{R}_+$ is defined by

$$I^r h(t) = \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)ds,$$

where the sign $\int$ denotes the Pettis integral and Γ is the Gamma function.

Definition 2.7. ([16]) For a function $h : J \longrightarrow E$, the *Caputo fractional-order derivative* of h, is defined by

$$^cD^r h(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{h^{(n)(s)}}{(t-s)^{1-n+r}} ds,$$

where $n = [r] + 1$ and $[r]$ denote the integer part of r.

Theorem 2.8. ([19]) *Let D be a closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(J, E)$ such that $0 \in D$. Assume that $N : D \longrightarrow D$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{\text{conv}}(\{0\} \cup N(V)) \Longrightarrow V \text{ is relatively weakly compact,} \tag{2}$$

holds for every subset $V \subset D$, then N has a fixed point.

3. Main Results

To establish our existence result for the problem (1) we need the following lemma.

Lemma 3.1. *For a given $\sigma \in C(J, E)$, the unique solution of the problem*

$$\begin{cases} ^cD^r x(t) = \sigma(t), \quad t \in J,\ 1 < r \leq 2 \\ \alpha_1 x(0) + \beta_1(^cD^p x(0)) = \gamma_1,\ \alpha_2 x(1) + \beta_2(^cD^p x(1)) = \gamma_2,\ 0 < p < 1, \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} \sigma(s))ds - \frac{t}{v_1}\left(\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} \sigma(s)ds \right. \\ &+ \left. \beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} \sigma(s)ds \right) + \frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}. \end{aligned}$$

where

$$v_1 = \frac{\alpha_2 \Gamma(2-p) + \beta_2}{\Gamma(2-p)}, \quad v_2 = \frac{\gamma_2 \alpha_1 - \alpha_2 \gamma_1}{\alpha_1}.$$

The proof is similar to the one given in [4].
Let us list the following hypothesis:

(H1) For each $t \in J$, the function $f(t, .)$ is weakly sequentially continuous;

(H2) For each $x \in C(J, E)$, the function $f(., x(.))$ is Pettis integrable on J;

(H3) There exist $\rho \in L^{\infty}(J)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ such that

$$\|f(t, x(t))\| \leq \|\rho(t)\|_{L^{\infty}} \psi(\|x\|);$$

(H4) There exists a constant $R > 0$ such that

$$\frac{R}{\|\rho\|_{L^{\infty}} \psi(R)\, \eta + |\frac{v_2}{v_1}| + |\frac{\gamma_1}{\alpha_1}|} > 1,$$

where

$$\eta = \left[\frac{1}{\Gamma(r+1)} + \frac{1}{|v_1|}\left(\frac{|\alpha_2|}{\Gamma(r+1)} + \frac{|\beta_2|}{\Gamma(r-p+1)}\right)\right];$$

(H5) For each bounded set $D \in E$, and each $t \in J$, the following inequality holds

$$\beta(f(s, D)) \leq \rho(t)\beta(D).$$

Theorem 3.2. *Assume that assumptions (H1)-(H5) hold. If*

$$\|\rho\|_{L^{\infty}} < \frac{1}{\eta}, \qquad (3)$$

then the boundary value problem (1) has at least one solution.

Proof. In view of Lemma 3.1, we define an operator $N : C(J, E) \longrightarrow C(J, E)$ by

$$\begin{aligned} Nx(t) &= \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s, x(s))ds - \frac{t}{v_1}\left[\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s, x(s))ds \right. \\ &+ \left. \beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} f(s, x(s))ds\right] + \frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}. \end{aligned}$$

Observe that the fixed points of the operator N are solutions of the problem (1).

First notice that, for $x \in C(J, E)$, we have $f(., x(.)) \in P(J, E)$ (assumption (H2)). Since $s \longmapsto \frac{(t-s)^{r-1}}{\Gamma(r)}$, $s \longmapsto \frac{(1-s)^{r-1}}{\Gamma(r)}$ and $s \longmapsto \frac{(1-s)^{r-1}}{\Gamma(r)}$ are $\in L^{\infty}(J)$ then $\frac{(t-.)^{r-1}}{\Gamma(r)} f(., x(.))$, $\frac{(1-.)^{r-1}}{\Gamma(r)} f(., x(.))$ and $\frac{(1-.)^{r-1}}{\Gamma(r)} f(., x(.))$ for all $t \in J$ are Pettis integrable (Proposition 2.3) and thus, the operator N makes sense.

Let $R > 0$, and consider the set

$$D = \left\{ \begin{array}{ll} x \in C(J,\, E) : \|x\|_{\infty} \leq R & \\ \text{and} & \\ \|x(t_1) - x(t_2)\| \leq \psi(R)\|\rho\|_{L^{\infty}} \left(\frac{t_2^r - t_1^r}{\Gamma(r+1)} + \frac{t_2 - t_1}{|v_1|}\left[\frac{|\alpha_2|}{\Gamma(r+1)} + \frac{|\beta_2|}{\Gamma(r-p+1)}\right]\right) & \\ \qquad + \left|\frac{v_2}{v_1}\right|(t_2 - t_1) & \text{for } t_1,\, t_2 \in J \end{array} \right\}$$

Clearly, the subset D is closed, convex and equicontinuous.

We shall show that N satisfies the assumptions of Theorem 2.8, the proof will be given in three steps.

1. First we show that N maps D into itself.

Take $x \in D$, $t \in J$ and assume that $Nx(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|Nx(t)\| = \varphi(Nx(t))$. Thus

$$\begin{aligned}
\|Nx(t)\| &= \varphi(Nx(t)) = \varphi\Bigg(\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds - \frac{t}{v_1}\bigg[\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds \\
&\quad + \beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} f(s,x(s))ds\bigg] + \frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}\Bigg) \\
&\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} \varphi(f(s,x(s)))ds \\
&\quad + \frac{t}{|v_1|} \int_0^1 \left(\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right) \varphi(f(s,x(s)))ds + \left|\frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}\right| \\
&\leq \|\rho\|_{L^\infty} \psi(R) \left[\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} ds + \frac{t}{|v_1|} \int_0^1 \left(\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right) ds\right] + \left|\frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}\right| \\
&\leq \|\rho\|_{L^\infty} \psi(R)\, \eta + \left|\frac{v_2}{v_1}\right| + \left|\frac{\gamma_1}{\alpha_1}\right| < R.
\end{aligned}$$

Let $t_1, t_2 \in J$, $t_1 < t_2$, $x \in D$, so $Nx(t_2) - Nx(t_1) \neq 0$. Then there exists $\varphi \in E^*$ such that

$$\|Nx(t_2) - Nx(t_1)\| = \varphi(Nx(t_2) - Nx(t_1)).$$

Thus

$$\begin{aligned}
\|Nx(t_2) - Nx(t_1)\| &= \varphi\Bigg(\int_0^{t_2} \frac{(t_2-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds - \frac{t_2}{v_1}\bigg[\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds \\
&\quad + \beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} f(s,x(s))ds\bigg] + \frac{\alpha_1 v_2 t_2 + \gamma_1 v_1}{\alpha_1 v_1} \\
&\quad - \int_0^{t_1} \frac{(t_1-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds + \frac{t_1}{v_1}\bigg[\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s,x(s))ds \\
&\quad + \beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} f(s,x(s))ds\bigg] - \frac{\alpha_1 v_2 t_1 + \gamma_1 v_1}{\alpha_1 v_1}\Bigg),
\end{aligned}$$

which means that

$$
\begin{aligned}
\|Nx(t_2)-Nx(t_1)\| &\leq \varphi\left(\int_0^{t_2}\frac{(t_2-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds-\int_0^{t_1}\frac{(t_2-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds\right)\\
&\quad+\varphi\left(\int_0^{t_1}\frac{(t_2-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds-\int_0^{t_1}\frac{(t_1-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds\right)\\
&\quad+\frac{t_2-t_1}{|v_1|}\varphi\left(\int_0^1\left[\frac{\alpha_2(1-s)^{r-1}}{\Gamma(r)}+\frac{\beta_2(1-s)^{r-p-1}}{\Gamma(r-p)}\right]f(s,x(s))ds)\right)+|\frac{v_2}{v_1}|(t_2-t_1)\\
&\leq \varphi\left(\int_{t_1}^{t_2}\frac{(t_2-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds\right)+\varphi\left(\int_0^{t_1}\frac{(t_2-s)^{r-1}-(t_1-s)^{r-1}}{\Gamma(r)}f(s,x(s))ds\right)\\
&\quad+\frac{t_2-t_1}{|v_1|}\varphi\left(\int_0^1\left[\frac{\alpha_2(1-s)^{r-1}}{\Gamma(r)}+\frac{\beta_2(1-s)^{r-p-1}}{\Gamma(r-p)}\right]f(s,x(s))ds\right)+|\frac{v_2}{v_1}|(t_2-t_1)\\
&\leq \int_{t_1}^{t_2}\frac{(t_2-s)^{r-1}}{\Gamma(r)}\varphi(f(s,x(s)))ds+\int_0^{t_1}\frac{(t_2-s)^{r-1}-(t_1-s)^{r-1}}{\Gamma(r)}\varphi(f(s,x(s)))ds\\
&\quad+\frac{t_2-t_1}{|v_1|}\int_0^1\left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)}+\frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]\varphi(f(s,x(s)))ds+|\frac{v_2}{v_1}|(t_2-t_1)\\
&\leq \int_{t_1}^{t_2}\frac{(t_2-s)^{r-1}}{\Gamma(r)}\|f(s,x(s))\|ds+\int_0^{t_1}\frac{(t_2-s)^{r-1}-(t_1-s)^{r-1}}{\Gamma(r)}\|f(s,x(s))\|ds\\
&\quad+\frac{t_2-t_1}{|v_1|}\int_0^1\left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)}+\frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]\|f(s,x(s))\|ds+|\frac{v_2}{v_1}|(t_2-t_1)\\
&\leq \psi(R)\|\rho\|_{L^\infty}\left(\int_{t_1}^{t_2}\frac{(t_2-s)^{r-1}}{\Gamma(r)}ds+\int_0^{t_1}\frac{(t_2-s)^{r-1}-(t_1-s)^{r-1}}{\Gamma(r)}ds\right.\\
&\quad\left.+\frac{t_2-t_1}{|v_1|}\int_0^1\left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)}+\frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]ds\right)+|\frac{v_2}{v_1}|(t_2-t_1)\\
&\leq \psi(R)\|\rho\|_{L^\infty}\left(\frac{t_2^r-t_1^r}{\Gamma(r+1)}+\frac{t_2-t_1}{|v_1|}\left[\frac{|\alpha_2|}{\Gamma(r+1)}+\frac{|\beta_2|}{\Gamma(r-p+1)}\right]\right)+\left|\frac{v_2}{v_1}\right|(t_2-t_1)
\end{aligned}
$$

Hence $N(D)\subset D$.

2. Then we show that N is weakly sequentially continuous.

Let (x_n) be a sequence in D and let $x_n\longrightarrow x$ in (E,w). Since f satisfies assumption (H1), we have $f(t,x_n(t))$ converges weakly uniformly to $f(t,x(t))$. Hence the Lebesgue Dominated Convergence theorem for Pettis integral implies $Nx_n(t)$ converges weakly uniformly to $Nx(t)$ in (E,w). We do it for each $t\in J$ so $Nx_n\longrightarrow Nx$. Then $N:D\longrightarrow D$ is weakly sequentially continuous.

3. Finally we show that the implication (2) holds.

Let $V\subset D$ such that $V=\overline{\text{conv}}(N(V)\cup\{0\})$. We have $V(t)\subset\overline{\text{conv}}(N(V)\cup\{0\})$ for all $t\in J$. $NV(t)\subset ND(t)$ $t\in J$

is bounded in E. By assumption (H5), and the properties of the measure β we have for each $t \in J$.

$$\begin{aligned}
v(t) &\leq \beta(\overline{\mathrm{conv}}(N(V)(t) \cup \{0\})) \leq \beta(NV(t)) \\
&\leq \beta\Bigg(\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} f(s,V(s))ds - \frac{t}{v_1}\Bigg[\alpha_2 \int_0^1 \frac{(1-s)^{r-1}}{\Gamma(r)} f(s,V(s))ds \\
&\quad +\beta_2 \int_0^1 \frac{(1-s)^{r-p-1}}{\Gamma(r-p)} f(s,V(s))ds) + \frac{\alpha_1 v_2 t + \gamma_1 v_1}{\alpha_1 v_1}\Bigg) \\
&\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} \beta(f(s,V(s)))ds + \frac{t}{|v_1|}\int_0^1 \left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]\beta(f(s,V(s)))ds \\
&\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} \rho(s)\beta(V(s))ds + \frac{t}{|v_1|}\int_0^1 \left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]\rho(s)\beta(V(s))ds \\
&\leq \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} \rho(s)v(s)ds + \frac{t}{|v_1|}\int_0^1 \left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]\rho(s)v(s)ds \\
&\leq \|\rho\|_{L^\infty}\|v\|_\infty \left(\int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} ds + \frac{t}{|v_1|}\int_0^1 \left[\frac{|\alpha_2|(1-s)^{r-1}}{\Gamma(r)} + \frac{|\beta_2|(1-s)^{r-p-1}}{\Gamma(r-p)}\right]ds\right) \\
&\leq \|\rho\|_{L^\infty}\|v\|_\infty \eta.
\end{aligned}$$

This means that

$$\|v\|_\infty (1 - \|\rho\|_{L^\infty}\eta) \leq 0.$$

By (3) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively weakly compact in E. Applying Theorem 2.8 we conclude that N has a fixed point which is a solution of the problem (1). □

4. Example

As an application of our result we consider the following fractional boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}} x_n(t) = \frac{1}{e^{t+3}}(1 + |x_n(t)|), \quad t \in J = [0,1] \\ -x_n(0) +^c D^{\frac{1}{2}} x_n(0) = 1, \\ x_n(1) +^c D^{\frac{1}{2}} x_n(1) = 1, \end{cases} \tag{4}$$

Here, $r = \dfrac{3}{2}$, $p = \dfrac{1}{2}$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = 1$.
Let

$$E = l^1 = \{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\}$$

with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Set $x = (x_1, x_2, ..., x_n, ...)$ and $f = (f_1, f_2, ..., f_n, ...)$,

$$f_n(t, x_n(t)) = \frac{1}{e^{t+3}}(1 + |x_n(t)|), \quad t \in J.$$

Further,

$$v_1 = \frac{\sqrt{\pi} + 2}{\sqrt{\pi}}, \qquad v_2 = 2.$$

For each $x \in \mathbb{R}$ and $t \in J$ we have

$$|f_n(t, x_n(t))| \leq \frac{1}{e^{t+3}}(1 + |x_n(t)|). \tag{5}$$

Hence conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied with

$$\rho(t) = \frac{1}{e^{t+3}}, \qquad t \in J$$

and

$$\psi(u) = 1 + u, \qquad u \in [0, \infty).$$

By (5), for any bounded set $D \subset l^1$, we have

$$\beta(f(t, D)) \leq \frac{1}{e^{t+3}}\beta(D), \quad \text{for each } \ t \in J.$$

Hence $(H5)$ is satisfied.

We have

$$\|\rho\|_{L^\infty}(1 + R)\eta + |\frac{v_2}{v_1}| + |\frac{\gamma_1}{\alpha_1}| < R$$

thus

$$R > \frac{\|\rho\|_{L^\infty}\eta + |\frac{v_2}{v_1}| + |\frac{\gamma_1}{\alpha_1}|}{1 - \|\rho\|_{L^\infty}\eta}.$$

Since $\|\rho\|_{L^\infty} = \frac{1}{e^3}$, $|\frac{v_2}{v_1}| = \frac{2\sqrt{\pi}}{\sqrt{\pi}+2}$, $|\frac{\gamma_1}{\alpha_1}| = 1$, then $\eta \simeq 1.5755$.

Hence (H4) is satisfied for $R > \frac{e^{-3}\eta + \frac{2\sqrt{\pi}}{\sqrt{\pi}+2} + 1}{1 - e^{-3}\eta} \simeq 2.1899$

We have

$$\|\rho\|_{L^\infty} \simeq 0.0498 \leq \frac{1}{\eta} \simeq 0.6347.$$

Hence (3) is satisfied.

Thus, by the conclusion of Theorem 3.2, the boundary value problem (4) has a solution defined on J.

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