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(p, q)-Bivariate-Bernstein-Chlodowsky Operators

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Abstract. In this article, we construct Bivariate-Bernstein -Chlodowsky operators based on (p, q)-integers. We give the basic estimates for these operators. Moreover, we discuss rate of convergence and pointwise approximation in Lipschitz class. In the last, we prove weighted approximation results.

1. Introduction

In 1912, Bernstein [5] gave a sequence of polynomials based on binomial distribution which are known as classical Bernstein operators as follows

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\Big(\frac{k}{n}\Big),\ n \in \mathbb{N}, 0 \leq x \leq 1, \tag{1}$$

where $f \in C[0,1]$ and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. The purpose of these polynomials was to give a simple proof of Weirstrass approximation theorem which plays a central role in the development of operators theory. Using these operators, he proved the poitwise and uniform approximation on $[0,1]$. Bernstein operators defined by (1) and its various generalizations have applications in numerical analysis, computer added geometric design (CAGD) and in solving problem of differential equations. Later on, various linear positive operators have been studied to approximate continuous and Lebesgue measurable functions (see Wafi and Rao ([17], [19], [20]) and Rao and Wafi [18]. Chlodowsky introduced a generalization of operators (1) in 1932 on interval $[0, b_n]$ as follows

$$B_n(f;x) = \sum_{k=0}^{n} f\Big(\frac{kb_n}{n}\Big)\binom{n}{k}\Big(\frac{x}{b_n}\Big)^k\Big(1-\frac{x}{b_n}\Big)^{n-k}, \tag{2}$$

where $0 \leq x \leq b_n$ and b_n is the sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{3}$$

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For the last two decades, q-calculus influenced as a new field of research in operator theory. Lupas [10] introduced q-analogue of classical Bernstein operators and another q-extension of Bernstein operators was given by Philips [14]. The convergence for q-analogue of these polynomials is faster than the classical one.

Recently, the application of (p,q)-calculus emerged in the field of operator theory. First extension using (p,q)-integers for Bernstein operators was given by Mursaleen et al [11] (see also [21], [22] and references therein). (p,q)-analogue in operators increase flexibility in controlling the shapes of curves and surfaces (see [8], [9]) and increase the radius of convergence i.e. radius of convergence is directly proportional to the parameter p ([13]). Now, we recall some basic notion and results from [11] as follows, let $0 < q < p \leq 1$. Then (p,q)-integers for non negative integers n, k are given by

$$[k]_{p,q} = \frac{p^k - q^k}{p-q} \qquad and \qquad [k]_{p,q} = 1 \qquad for \quad k = 0.$$

(p,q)-binomial coefficient

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

and (p,q)-binomial expansion

$$\begin{aligned}(ax+by)^n_{p,q} &= \sum_{k=0}^{n}\binom{n}{k}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k}b^k x^{n-k}y^k,\\ (x+y)^n_{p,q} &= (x+y)(px+py)(p^2x+q^2y)...(p^{n-1}x-q^{n-1}y).\end{aligned}$$

(p,q)-Bernstein-Chlodowsky operators [4] is defined as follows

$$C_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n} c^{p,q}_{n,k}(x) f\left(\frac{[k]_{pq}}{[n]_{pq}p^{k-n}}b_n\right), \tag{4}$$

where

$$c^{p,q}_{n,k}(x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\binom{n}{k}_{p,q}\left(\frac{x}{b_n}\right)^k \prod_{j=0}^{n-k-1}\left(p^j - q^j\frac{x}{b_n}\right)$$

and $\lim\limits_{n\to\infty}\frac{b_n}{[n]_{p,q}} = 0$. In the view of above, we define a Bivariate Bernstein Chlodowsky operators based on (p,q)-integers.

2. Construction of (p,q)-Bivariate Bernstein Chlodowsky Operators

Let $J_1 \times J_2 = [0, c_n] \times [0, c_m]$ and $(x,y) \in J_1 \times J_2$. Then, for a function $f \in C(J_1 \times J_2)$, the (p,q)-Bivariate-Bernstein-Chlodowsky operators $C_{n,m}(f; x, y, p_{12}, q_{12}) = C_{n,m}(f; x, y, p_1, p_2, q_1, q_2)$ are defined as follows

$$\begin{aligned}C_{n,m}(f;x,y,p_{12},q_{12}) &= \frac{1}{p_1^{\frac{n(n-1)}{2}} p_2^{\frac{m(m-1)}{2}}}\sum_{k_1=0}^{n}\sum_{k_2=0}^{m} c^{p_1,q_1}_{n,k_1}(x) c^{p_2,q_2}_{m,k_2}(y)\\ &\times \ f\left(\frac{[k_1]_{p_1q_1}}{[n]_{p_1q_1}p_1^{k_1-n}}b_n, \frac{[k_2]_{p_2q_2}}{[m]_{p_2q_2}p_2^{k_2-m}}b_m\right),\end{aligned} \tag{5}$$

where

$$\begin{aligned} c_{n,k_1}^{p_1,q_1}(x) &= \frac{1}{p_1^{\frac{n(n-1)}{2}}} \binom{n}{k_1}_{p_1,q_1} \left(\frac{x}{b_n}\right)^{k_1} \prod_{j=0}^{n-k_1-1} \left(p_1^j - q_1^j \frac{x}{b_n}\right) \\ c_{m,k_2}^{p_2,q_2}(y) &= \frac{1}{p_2^{\frac{m(m-1)}{2}}} \binom{m}{k_2}_{p_2,q_2} \left(\frac{y}{b_m}\right)^{k_2} \prod_{j=0}^{m-k_2-1} \left(p_2^j - q_2^j \frac{y}{b_m}\right), \end{aligned}$$

and $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{p_1,q_1}} = 0$, $\lim\limits_{m\to\infty} \frac{b_m}{[m]_{p_2,q_2}} = 0$.

Lemma 2.1. *[4] For the operators $C_{n,p,q}$ defined by* (4), *we have*

$$\begin{aligned} C_{n,p,q}(e_0;x) &= 1, \\ C_{n,p,q}(e_1;x) &= x, \\ C_{n,p,q}(e_2;x) &= \frac{p^{n-1}b_n}{[n]_{p,q}}x + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2, \\ C_{n,p,q}(e_3;x) &= \frac{b_n^2 x}{[n]_{p,q}^2}p^{2n-2} + \frac{(2p+q)q[n-1]_{p,q}x^2 b_n}{[n]_{p,q}^2}p^{n-1} + \frac{q^3[n-1]_{p,q}[n-2]_{p,q}x^3}{[n]_{p,q}^2}, \\ C_{n,p,q}(e_4;x) &= \frac{b_n^3 x}{[n]_{p,q}^3}p^{3n-3} + \frac{q(3p^2+3qp+q^3)[n-1]_{p,q}b_n^2 x^2}{[n]_{p,q}^3}p^{2n-4} \\ &+ \frac{q^3(3p^2+2pq+q^2)[n-1]_{p,q}[n-2]_{p,q}b_n x^3}{[n]_{p,q}^3}p^{n-3} + \frac{q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}x^4}{[n]_{p,q}^3}, \end{aligned}$$

where $e_v(t) = t^v$, $v = 0,1,2,3,4$.

Lemma 2.2. *Let $e_{i,j}(x,y) = x^i y^j, 0 \leq i,j \leq 2$ are test functions. Then, we have*

$$\begin{aligned} C_{n,m}(e_{0,0};x,y,p_{12},q_{12}) &= 1, \\ C_{n,m}(e_{1,0};x,y,p_{12},q_{12}) &= x, \\ C_{n,m}(e_{0,1};x,y,p_{12},q_{12}) &= y, \\ C_{n,m}(e_{2,0};x,y,p_{12},q_{12}) &= \frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}x + \frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}x^2, \\ C_{n,m}(e_{0,2};x,y,p_{12},q_{12}) &= \frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}y + \frac{q_2[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}}y^2, \\ C_{n,m}(e_{3,0};x,y,p_{12},q_{12}) &= \frac{b_n^2 x}{[n]_{p_1,q_1}^2}p_1^{2n-2} + \frac{(2p_1+q_1)q_1[n-1]_{p_1,q_1}x^2 b_n}{[n]_{p_1,q_1}^2}p_1^{n-1} \\ &+ \frac{q_1^3[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}x^3}{[n]_{p_1,q_1}^2} \\ C_{n,m}(e_{0,3};x,y,p_{12},q_{12}) &= \frac{b_m^2 y}{[m]_{p_2,q_2}^2}p_2^{2m-2} + \frac{(2p_2+q_2)q_2[m-1]_{p_2,q_2}y^2 b_m}{[m]_{p_2,q_2}^2}p_2^{m-1} \\ &+ \frac{q_2^3[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}y^3}{[m]_{p_2,q_2}^2}, \end{aligned}$$

$$
\begin{aligned}
C_{n,m}(e_{4,0};x,y,p_{12},q_{12}) &= \frac{b_n^3 x}{[n]^3_{p_1,q_1}}p_1^{3n-3} + \frac{q_1(3p_1^2+3q_1p_1+q_1^3)[n-1]_{p_1,q_1}b_n^2x^2}{[n]^3_{p_1,q_1}}p_1^{2n-4}\\
&+ \frac{q_1^3(3p_1^2+2p_1q_1+q_1^2)[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}b_nx^3}{[n]^3_{p_1,q_1}}p_1^{n-3}\\
&+ \frac{q_1^6[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}[n-3]_{p_1,q_1}x^4}{[n]^3_{p_1,q_1}},\\
C_{n,m}(e_{0,4};x,y,p_{12},q_{12}) &= \frac{b_m^3 y}{[m]^3_{p_2,q_2}}p_2^{3m-3} + \frac{q_2(3p_2^2+3q_2p_2+q_2^3)[m-1]_{p_2,q_2}b_m^2y^2}{[m]^3_{p_2,q_2}}\\
&\times p_2^{2m-4} + \frac{q_2^3(3p_2^2+2p_2q_2+q_2^2)[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}b_my^3}{[m]^3_{p_2,q_2}}\\
&\times p_2^{m-3} + \frac{q_2^6[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}[m-3]_{p_2,q_2}y^4}{[m]^3_{p_2,q_2}}.
\end{aligned}
$$

Proof. In view of definition of the operators defined by 5, we have

$$
\begin{aligned}
C_{n,m}(e_{0,0};x,y,p_{12},q_{12}) &= C_n(e_0;x,p_1,q_1)C_n(e_0;y,p_2,q_2),\\
C_{n,m}(e_{1,0};x,y,p_{12},q_{12}) &= C_n(e_1;x,p_1,q_1)C_n(e_0;y,p_2,q_2),\\
C_{n,m}(e_{0,1};x,y,p_{12},q_{12}) &= C_n(e_0;x,p_1,q_1)C_n(e_1;y,p_2,q_2),\\
C_{n,m}(e_{2,0};x,y,p_{12},q_{12}) &= C_n(e_2;x,p_1,q_1)C_n(e_0;y,p_2,q_2),\\
C_{n,m}(e_{0,2};x,y,p_{12},q_{12}) &= C_n(e_0;x,p_1,q_1)C_n(e_2;y,p_2,q_2),\\
C_{n,m}(e_{3,0};x,y,p_{12},q_{12}) &= C_n(e_3;x,p_1,q_1)C_n(e_0;y,p_2,q_2),\\
C_{n,m}(e_{0,3};x,y,p_{12},q_{12}) &= C_n(e_0;x,p_1,q_1)C_n(e_3;y,p_2,q_2),\\
C_{n,m}(e_{4,0};x,y,p_{12},q_{12}) &= C_n(e_4;x,p_1,q_1)C_n(e_0;y,p_2,q_2),\\
C_{n,m}(e_{0,4};x,y,p_{12},q_{12}) &= C_n(e_0;x,p_1,q_1)C_n(e_4;y,p_2,q_2).
\end{aligned}
$$

With the help of these equalities, we can easily prove required result. □

Lemma 2.3. *For the operators given by 5, we have*

$$
\begin{aligned}
C_{n,m}(t-x;x,y,p_{12},q_{12}) &= 0,\\
C_{n,m}(s-y;x,y,p_{12},q_{12}) &= 0,\\
C_{n,m}((t-x)^2;x,y,p_{12},q_{12}) &= \frac{p_1^{n-1}x(b_n-x)}{[n]_{p_1,q_1}},\\
C_{n,m}((s-y)^2;x,y,p_{12},q_{12}) &= \frac{p_2^{m-1}y(b_m-y)}{[n]_{p_1,q_2}},\\
C_{n,m}((t-x)^4;x,y,p_{12},q_{12}) &\le \frac{30p_1^{n-1}b_n}{[n]_{p_1,q_1}}(x^4+x^3+x^2+x),\\
C_{n,m}((s-x)^4;x,y,p_{12},q_{12}) &\le \frac{30p_1^{m-1}b_m}{[m]_{p_2,q_2}}(y^4+y^3+y^2+y).
\end{aligned}
$$

Proof. Using Lemma 2.2, we prove Lemma 2.3. □

3. Main Results

Definition 3.1. *Let $X, Y \subset \mathbb{R}$ be any two given intervals and the set $B(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ is bounded on $X \times Y\}$. For $f \in B(X \times Y)$, let the function $\omega_{total}(f; \cdot, *) : [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ by $\omega_{total}(f; \delta_1, \delta_2) = \sup_{|x-x'|\leq\delta_1, |y-y'|\leq\delta_2} \{|f(x, y) - f(x', y')| : (x, y), (x', y') \in [0, \infty) \times [0, \infty)\}$, is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.*

In order to get the rate of convergence and degree of approximation for the operators $C_{n,m}$, we consider $p_1 = p_n$, $p_2 = p_m$ and $q_1 = q_n$, $q_2 = q_m$ such that $0 < q_n < p_n \leq 1$ and $0 < q_m < p_m \leq 1$ satisfying

$$\lim_{n\to\infty} q_n^n \to a, \lim_{m\to\infty} q_m^m \to b, \lim_{n\to\infty} p_n^n \to c, \lim_{m\to\infty} p_m^m \to d \tag{6}$$

and

$$\lim_{n\to\infty} p_n \to 1, \lim_{m\to\infty} p_m \to 1, \lim_{n\to\infty} q_n \to 1, \lim_{m\to\infty} q_m \to 1, \tag{7}$$

where $0 \leq a, b < c, d < 1$. Here, we recall the following result due to Volkov [16]:

Theorem 3.2. *Let I and J be compact intervals of the real line. Let $L_{n,m} : C(I \times J) \to C(I \times J), (n, m) \in \mathbb{N} \times \mathbb{N}$ be linear positive operators. If*

$$\lim_{n,m\to\infty} L_{n,m}(e_{ij}) = e_{x,y}, (i, j) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{n,m\to\infty} L_{n,m}(e_{20} + e_{02}) = e_{20} + e_{02},$$

uniformly on $I \times J$, then the sequence $(L_{n,m}f)$ converges to f uniformly on $I \times J$ for any $f \in C(I \times J)$.

Theorem 3.3. *Let $e_{ij}(x, y) = x^i y^j (0 \leq i+j \leq 2, i, j \in \mathbb{N})$ be the test functions defined on $J_1 \times J_2$ and $(p_n), (q_n), (p_m), (q_m)$ be the sequences defined by (6) and (7). If*

$$\lim_{n,m\to\infty} (C_{n,m}e_{ij})(x, y) = e_{ij}(x, y), (i, j) \in \{(0, 0), (1, 0), (0, 1)\}$$

and

$$\lim_{n,m\to\infty} (C_{n,m}(e_{20} + e_{02})(x, y) = e_{20}(x, y) + e_{02}(x, y),$$

uniformly on $J_1 \times J_2$, then

$$\lim_{n,m\to\infty} (C_{n,m}f)(x, y) = f(x, y),$$

uniformly for any $f \in C(J_1 \times J_2)$.

Proof. Using Lemma 2.2, it is obvious for $i = j = 0$

$$\lim_{n,m\to\infty} (C_{n,m}e_{00})(x, y) = e_{00}(x, y).$$

For $i = 1$ and $j = 0$, we have

$$\begin{aligned} \lim_{n,m\to\infty} (C_{n,m}e_{10})(x, y) &= x. \\ \lim_{n,m\to\infty} (C_{n,m}e_{10})(x, y) &= e_{00}(x, y). \end{aligned}$$

For $i = 0$ and $j = 1$, we have

$$\begin{aligned}\lim_{n,m\to\infty}(C_{n,m}e_{01})(x,y) &= y.\\ \lim_{n,m\to\infty}(C_{n,m}e_{01})(x,y) &= e_{01}(x,y)\end{aligned}$$

and

$$\begin{aligned}\lim_{n,m\to\infty}(C_{n,m}(e_{20}+e_{02})(x,y) &= \lim_{n,m\to\infty}\left\{\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}x+\frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}x^2+\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}y+\frac{q_2[m-1]_{p_2,q_2}}{[n]_{p_2,q_2}}y^2\right\}\\ \lim_{n,m\to\infty}(C_{n,m}(e_{20}+e_{02})(x,y) &= e_{20}(x,y)+e_{02}(x,y).\end{aligned}$$

From Theorem 3.2, we completes the proof of Theorem 3.3.

Theorem 3.4. *[15] Let $L : C([0,\infty)\times[0,\infty)) \to B([0,\infty)\times[0,\infty))$ be a linear positive operator. For any $f \in C(X\times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$\begin{aligned}|(Lf)(x,y)-f(x,y)| &\le |Le_{0,0}(x,y)-1||f(x,y)|+\Big[Le_{0,0}(x,y)+\delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot-x))^2(x,y)}\\ &+ \delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(*-y))^2(x,y)}\\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{0,0})^2(x,y)(L(\cdot-x))^2(x,y)(L(*-y))^2(x,y)}\Big]\omega_{total}(f;\delta_1,\delta_2),\end{aligned}$$

holds.

Theorem 3.5. *Let $f \in C(J_1 \times J_2)$ and $(x,y) \in J_1 \times J_2$. Then, for $(n,m) \in \mathbb{N}$ and for any $\delta_1, \delta_2 > 0$, we have*

$$|(C_{n,m}f)(x,y)-f(x,y)| \le 4\omega_{total}(f;\delta_1,\delta_2),$$

where $\delta_1 = \sqrt{C_{n,m}((t-x)^2,x,y,p_{12},q_{12})}$ and $\delta_2 = \sqrt{C_{n,m}((s-y)^2,x,y,p_{12},q_{12})}$.

Proof. From Theorem 3.4, we have

$$\begin{aligned}|(C_{n,m}f)(x,y) &- f(x,y)|\\ &\le \Big[1++\delta_1^{-1}\sqrt{C_{n,m}((t-x)^2)(x,y))}+\delta_2^{-1}\sqrt{C_{n,m}((s-y)^2)(x,y)}\\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{C_{n,m}((t-x)^2)(x,y))C_{n,m}((s-y)^2)(x,y)}\Big]\omega_{total}(f;\delta_1,\delta_2).\end{aligned}$$

On choosing $\delta = \sqrt{C_{n,m}((t-x)^2)(x,y)}$ and $\delta_2 = \sqrt{C_{n,m}((s-y)^2)(x,y)}$, we get the required result. □

Now, we shall prove degree of approximation for the operators $C_{n,m}$ in Lipschitz class. We consider the Lipschitz class $Lip_M(\gamma_1,\gamma_2)$ in terms of two variables as follows:

$$|f(t,s)-f(x,y)| \le M|t-x|^{\gamma_1}|s-y|^{\gamma_2},$$

where $M > 0$, $0 < \gamma_1, \gamma_2 \le 1$ and for any $(t,s), (x,y) \in J_1 \times J_2$.

Theorem 3.6. *For $f \in Lip_M(\gamma_1,\gamma_2)$, we have*

$$|C_{n,m}(f;q_n,q_m,p_n,p_m;x,y)-f(x,y)| \le M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y)$$

where $\delta_n(x) = C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m;x,y)$ and
$\delta_m(y) = C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m;x,y)$.

Proof. Since $f \in Lip_M(\gamma_1, \gamma_2)$, we can write

$$\begin{aligned}
|C_{n,m}(f; q_n, q_m, p_n, p_m; x, y) &- f(x,y)| \\
&\leq C_{n,m}(|f(t,s) - f(x,y)|; q_n, q_m, p_n, p_m; x, y) \\
&\leq MC_{n,m}(|t-x|^{\gamma_1}|s-y|^{\gamma_2}; q_n, q_m, p_n, p_m; x, y) \\
&= MC_{n,m}(|t-x|^{\gamma_1}; q_n, q_m, p_n, p_m; x, y) \\
&\times C_{n,m}(|s-y|^{\gamma_2}; q_n, q_m, p_n, p_m; x, y).
\end{aligned}$$

Using Hölder inequality with $\alpha_1 = \frac{2}{\gamma_1}, \beta_1 = \frac{2}{2-\gamma_1}$ and $\alpha_2 = \frac{2}{\gamma_2}, \beta_2 = \frac{2}{2-\gamma_2}$, respectively, we get

$$\begin{aligned}
|C_{n,m}(f; q_n, q_m, p_n, p_m; x, y) - f(x,y)| &\leq \left\{C_{n,m}((t-x)^2; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{\gamma_1}{2}} \\
&\times \left\{C_{n,m}(1; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{2}{2-\gamma_1}} \\
&\times \left\{C_{n,m}((s-x_2)^2; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{\gamma_2}{2}} \\
&\times \left\{C_{n,m}(1; q_n, q_n, p_n, p_m; x, y)\right\}^{\frac{2}{2-\gamma_2}} \\
&= M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y).
\end{aligned}$$

Which completes the proof of Theorem 3.6. □

Here, we discuss degree of approximation in weighted space for the operator defined by (5). We recall some basic notions from [7] as
$B_\rho([0,\infty) \times [0,\infty))$ is the space of all functions defined on $\mathbb{R}_+^2 = [0,\infty) \times [0,\infty)$ with the condition $|f(x,y)| \leq M_f\rho(x,y)$, where M_f is a positive constant depending on f and $\rho(x,y) = 1 + x^2 + y^2$ is a weight function. $C_\rho([0,\infty) \times [0,\infty)) = \{f : f$ is a continuous function belongs to $B_\rho([0,\infty) \times [0,\infty))\}$ equipped with the norm $\|f\|_\rho = \sup\limits_{(x,y)\in\mathbb{R}_+^2} \frac{|f(x,y)|}{\rho(x,y)}$ and $C_\rho^k([0,\infty) \times [0,\infty)) = \left\{f : f \in C_\rho \text{ and } \lim\limits_{x,y\to\infty} \frac{|f(x,y)|}{\rho(x,y)} < k\right\}$. For all $f \in C_\rho^k$, the weighted modulus of continuity is defined as

$$\omega_\rho(f; \delta_1, \delta_2) = \sup_{(x,y)\in\mathbb{R}_+^2} \sup_{|h_1|\leq\delta_1, |h_2|\leq\delta_2} \frac{|f(x+h_1, y+h_2) - f(x,y)|}{\rho(x,y)\rho(h_1,h_2)}$$

and

$$|f(t,s) - f(x,y)| \leq 8(1 + x^2 + y^2)\omega_\rho(f; \delta_n, \delta_m)\left(1 + \frac{|t-x|}{\delta_n}\right)\left(1 + \frac{|s-y|}{\delta_m}\right)(1 + (t-x)^2)(1 + (s-y)^2). \quad (8)$$

Theorem 3.7. *If the operators $C_{n,m}$ defined by* (5) *satisfying the conditions*

$$\lim_{n,m\to\infty} \|C_{n,m}(1; q_n, q_m, p_n, p_m, x, y) - 1\|| = 0,$$
$$\lim_{n,m\to\infty} \|C_{n,m}(t; q_n, q_m, p_n, p_m, x, y) - x\|| = 0,$$
$$\lim_{n,m\to\infty} \|C_{n,m}(s; q_n, q_m, p_n, p_m, x, y) - y\|| = 0,$$

and

$$\lim_{n,m\to\infty} \|C_{n,m}(t^2 + s^2; q_n, q_m, p_n, p_m, x, y) - (x^2 + y^2)\|| = 0.$$

Then

$$\lim_{n,m\to\infty} \|C_{n,m}(f; q_n, q_m, p_n, p_m, x, y) - f\|| = 0,$$

for each $f \in C_\rho^k([0,\infty) \times [0,\infty))$.

Proof. In view of Lemma 2.2, we completes the proof of Theorem 3.7. □

Theorem 3.8. *Let $f \in C_\rho^k([0,\infty)\times[0,\infty))$. Then, we have*

$$\sup_{(x,y)\in\mathbb{R}_+^2} \frac{|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y)-f(x,y)|}{(1+x^2+y^2)^3} \leq K\omega_\rho(f;\delta_n,\delta_m)$$

holds for the large value of n,m where $\delta_n = o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)$ and $\delta_m = o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)$.

Proof. From (8) and the operators (5), we have

$$\begin{aligned}
|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y) &- f(x,y)| \\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m) \\
&\times \left(1+\frac{C_{n,m}(|t-x|;q_n,q_m,p_n,p_m,x,y)}{\delta_n}\right) \\
&\times \left(1+\frac{C_{n,m}(|s-y|;q_n,q_m,p_n,p_m,x,y)}{\delta_n}\right) \\
&\times (1+C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m,x,y)) \\
&\times (1+C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m,x,y)).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y)-f(x,y)| \\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Bigg[1+C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m,x,y) \\
&+\frac{\sqrt{C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m,x,y)}}{\delta_n} \\
&\times\frac{\sqrt{C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m,x,y)C_{n,m}((t-x)^4;q_n,q_m,p_n,p_m,x,y)}}{\delta_n}\Bigg] \\
&\times\Bigg[1+C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m,x,y)+\frac{\sqrt{C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m,x,y)}}{\delta_m} \\
&\times\frac{\sqrt{C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m,x,y)C_{n,m}((s-y)^4;q_n,q_m,p_n,p_m,x,y)}}{\delta_n}\Bigg].
\end{aligned} \tag{9}$$

From Lemma 2.3, we have

$$\begin{aligned}
C_{n,m}((t-x)^2;q_n,q_m,p_n,p_m,x,y) &\leq o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)x, \\
C_{n,m}((s-y)^2;q_n,q_m,p_n,p_m,x,y) &\leq o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)y, \\
C_{n,m}((t-x)^4;q_n,q_m,p_n,p_m,x,y) &\leq o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)(x^4+x^3+x^2+x), \\
C_{n,m}((s-x)^4;q_n,q_m,p_n,p_m,x,y) &\leq o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)(y^4+y^3+y^2+y).
\end{aligned} \tag{10}$$

Combining (9) and all identities in (10), we have

$$
\begin{aligned}
&|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)x+\frac{\sqrt{o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)x}}{\delta_n}\right.\\
&\left.\times\frac{\sqrt{o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)xo\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)(x^4+x^3+x^2+x)}}{\delta_n}\right]\\
&\times\left[1+o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)y+\frac{\sqrt{o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)y}}{\delta_m}\frac{\sqrt{o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)yo\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)(y^4+y^3+y^2+y)}}{\delta_n}\right].
\end{aligned}
$$

□

Choosing $\delta_n=o\left(\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}\right)$ and $\delta_m=o\left(\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}\right)$, we find

$$
\begin{aligned}
&|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+\delta_n x+\sqrt{x}\frac{\sqrt{x(x^4+x^3+x^2+x)}}{\delta_n}\right]\left[1+\delta_m y+\sqrt{y}\frac{\sqrt{y(y^4+y^3+y^2+y)}}{\delta_n}\right].
\end{aligned}
$$

For sufficiently large large value of n and m, we have

$$
\sup_{(x,y)\in\mathbb{R}_+^2}\frac{|C_{n,m}(f;q_n,q_m,p_n,p_m,x,y)-f(x,y)|}{(1+x^2+y^2)^3}\leq K\omega_\rho(f;\delta_n,\delta_m),
$$

where K is a positive constant independent of n, m and $\delta_n<1$, $\delta_m<1$.

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