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Boundary Schwarz Lemma for Holomorphic Functions

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Abstract. In this paper, a boundary version of Schwarz lemma is investigated. We take into consideration a function $f(z)$ holomorphic in the unit disc and $f(0) = 0$ such that $\left|\Re f\right| < 1$ for $|z| < 1$, we estimate a modulus of angular derivative of $f(z)$ function at the boundary point b with $f(b) = 1$, by taking into account their first nonzero two Maclaurin coefficients. Also, we shall give an estimate below $\left|f'(b)\right|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_0 \neq 0$. Moreover, two examples for our results are considered.

1. Introduction

The Schwarz lemma is one of the most important results in complex analysis. This lemma, named after Hermann Amandus Schwarz, is as results in complex analysis about holomorphic functions defined on the unit disc. The classical Schwarz lemma can be stated as follows [7]:

Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, with $f(0) = 0$ and $\left|f(z)\right| < 1$. Then $\left|f(z)\right| \leq |z|$, with strict inequality for all $z \neq 0$ in D unless f has the form $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Also $\left|f'(0)\right| \leq 1$, with equality only for $f(z) = \alpha z$ with $|\alpha| = 1$.

Let $f(z)$ be the function, which is holomorphic in the unit disc and satisfies the relations $f(0) = 0$ and $\left|\Re f(z)\right| < 1$ for $|z| < 1$. Therefore, we have

$$\Re\left(e^{\frac{i\pi}{2}f(z)}\right) > 0, \quad |z| < 1.$$

Hence the function

$$\varphi(z) = \frac{e^{\frac{i\pi}{2}f(z)} - 1}{e^{\frac{i\pi}{2}f(z)} + 1}$$

is holomorphic in D that satisfies the inequality $\left|\varphi(z)\right| < 1$ for $|z| < 1$ and $\varphi(0) = 0$. Applying the Schwarz lemma for the function $\varphi(z)$, we obtain

$$\left|f'(0)\right| \leq \frac{4}{\pi}. \tag{1.1}$$

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The equality (1.1) holds if and only if

$$f(z) = \frac{2}{i\pi} \ln\left(\frac{1 + ze^{i\theta}}{1 - ze^{i\theta}}\right),$$

where θ is a real number.

Thus, Schwarz lemma gives a better bound under stronger hypothesis.

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $\left|f(b)\right| = 1$ and $f'(b)$ exists, then $\left|f'(b)\right| \geq 1$, which is known as the Schwarz lemma on the boundary.

In [20], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Lemma 1.1 (The boundary Schwarz lemma). *Let $f : D \to D$ be holomorphic function with $f(0) = 0$. Assume that there is a $b \in \partial D$ so that f extends continuously to b, $\left|f(b)\right| = 1$, and $f'(b)$ exists. Then*

$$\left|f'(b)\right| \geq \frac{2}{1 + \left|f'(0)\right|} \tag{1.2}$$

and

$$\left|f'(b)\right| \geq 1 \tag{1.3}$$

Inequality (1.2) is sharp, with equality possible for each value of $\left|f'(0)\right|$. Also, $\left|f'(b)\right| > 1$ unless $f(z) = ze^{i\theta}$, θ real.

Inequality (1.3) and its generalizations have important applications in geometric theory of functions (see, e.g., [8], [22]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [3], [5], [6], [9], [10], [13], [14], [20], [21], [23] and references therein).

For our main results we need the following lemma known as Julia-Wolff lemma [22].

Lemma 1.2 (Julia-Wolff lemma). *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $\left|f(b)\right| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq \left|f'(b)\right| \leq \infty$.*

V. N. Dubinin [5] strengthened the inequality $\left|f'(b)\right| \geq 1$ by involving zeros of the function $f(z)$. S. G. Krantz and D. M. Burns [11] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. According to M. Mateljević's studies, some other types of results which are related to the subject can be found in (see, e.g., [15], [16], [17] and [19]). In addition, [18] was posed on ResearchGate where is discussed concerning results in more general aspects. Also, M. Jeong [10] got some inequalities at a boundary point for a different form of holomorphic functions and showed the sharpness of these inequalities. In addition, M. Jeong found a necessary and sufficient condition for a holomorphic map to have fixed points only on the boundary of the unit disc and compared its derivatives at fixed points to get some relations among them [9]. X. Tang, T. Liu and J. Lu [14] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

As seen in [12], an aplication of the Schwarz lemma is obtained. A boundary Schwarz lemma is formed for pluriharmonic mapping between unit balls and any dimension by Y. Liu, S. Dai and Y. Pan.

Taishun Liu, Jianfei Wang, Xiaomin Tang [24] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$. They then applied their new Schwarz lemma to study problems from the geometric function theory in several complex variables.

Furthermore, for historical background about the Schwarz lemma and its application on the boundary of the unit disc we refer to (see [3], [23], [25]).

2. Main Results

In this section, the different versions of the boundary Schwarz Lemma have been discussed. Assuming the existence of angular limit on a boundary point, it has been obtained some inequalities about estimating below the modulus of the derivative of the holomorphic function. Also, it has been shown that these inequalities are sharp.

Theorem 2.1. *Let $f(z)$ be a holomorphic function in the unit disc D, $f(0)=0$ and $\left|\Re f(z)\right|<1$ for $|z|<1$. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $f(b)=1$. Then, we have the inequality*

$$\left|f'(b)\right| \geq \frac{2}{\pi}. \tag{2.1}$$

The inequality (2.1) *is sharp, with equality for the function*

$$f(z)=\frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Proof. Let

$$\varphi(z)=\frac{e^{\frac{i\pi}{2}f(z)}-1}{e^{\frac{i\pi}{2}f(z)}+1}.$$

$\varphi(z)$ is a holomorphic function in D, $\varphi(0)=0$ and $\left|\varphi(z)\right|<1$ for $|z|<1$. For $b\in\partial D$ and $f(b)=1$, we take

$$\left|\varphi(b)\right|=\left|\frac{e^{\frac{i\pi}{2}f(b)}-1}{e^{\frac{i\pi}{2}f(b)}+1}\right|=\left|\frac{e^{\frac{i\pi}{2}}-1}{e^{\frac{i\pi}{2}}+1}\right|=\left|\frac{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}-1}{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}+1}\right|=\left|\frac{i-1}{i+1}\right|=1.$$

From (1.3), we obtain

$$1\leq\left|\varphi'(b)\right|=\left|\frac{i\pi f'(b)e^{\frac{i\pi}{2}f(b)}}{\left(e^{\frac{i\pi}{2}f(b)}+1\right)^2}\right|=\frac{\pi\left|f'(b)\right|\left|e^{\frac{i\pi}{2}}\right|}{\left|e^{\frac{i\pi}{2}}+1\right|^2}=\frac{\pi}{2}\left|f'(b)\right|$$

and

$$\left|f'(b)\right|\geq\frac{2}{\pi}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=\frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Then

$$f'(z)=\frac{2}{i\pi}\frac{2}{1-z^2},$$

$$f'(i)=\frac{2}{i\pi}\frac{2}{1-i^2}=\frac{2}{i\pi}$$

and

$$\left|f'(i)\right|=\frac{2}{\pi}.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f'(b)\right| \geq \frac{1}{\pi}\frac{16}{4+\pi\left|f'(0)\right|}. \tag{2.2}$$

The equality in (2.2) *occurs for the function*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 2.1. Therefore, on account of (1.2), we obtain

$$\frac{2}{1+\left|\varphi'(0)\right|} \leq \left|\varphi'(b)\right| = \left|\frac{i\pi f'(b)e^{\frac{i\pi}{2}f(b)}}{\left(e^{\frac{i\pi}{2}f(b)}+1\right)^2}\right| = \frac{\pi\left|f'(b)\right|\left|e^{\frac{i\pi}{2}}\right|}{\left|e^{\frac{i\pi}{2}}+1\right|^2} = \frac{\pi}{2}\left|f'(b)\right|.$$

Also, since

$$\left|\varphi'(0)\right| = \frac{\pi}{4}\left|f'(0)\right|,$$

we take

$$\frac{2}{1+\frac{\pi}{4}\left|f'(0)\right|} \leq \frac{\pi}{2}\left|f'(b)\right|.$$

Thus, we have

$$\left|f'(b)\right| \geq \frac{1}{\pi}\frac{16}{4+\pi\left|f'(0)\right|}.$$

To show that the inequality (2.2) is sharp, take the holomorphic function

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Then

$$\left|f'(i)\right| = \frac{2}{\pi}.$$

Since $\left|f'(0)\right| = \frac{4}{\pi}$, (2.2) is satisfied with equality. That is;

$$\frac{1}{\pi}\frac{16}{4+\pi\left|f'(0)\right|} = \frac{1}{\pi}\frac{16}{4+\pi\frac{4}{\pi}} = \frac{2}{\pi}.$$

□

If we know that the second coefficient in the expension of the function $f(z) = c_1z + c_2z^2 + c_3z^3 + ...$, then we obtain new inequalities of Schwarz lemma at the boundary by taking into account c_2.

Theorem 2.3. *Let $f(z) = c_1z + c_2z^2 + ...$ be a holomorphic function in the unit disc D and $\left|\Re f(z)\right| < 1$ for $|z| < 1$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = 1$. Then, we have*

$$\left|f'(b)\right| \geq \frac{2}{\pi}\left(1+\frac{2\left(4-\pi\left|c_1\right|\right)^2}{16-\pi^2\left|c_1\right|^2+4\pi\left|c_2\right|}\right). \tag{2.3}$$

The inequality (2.3) *is sharp with extremal function*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Proof. Let $h(z) = z$, $z \in D$ and $\varphi(z)$ be as in the proof of Theorem 2.1. $h(z)$ is holomorphic in D and $|h(z)| < 1$ for $|z| < 1$. Lindelöf principle implies that for each $z \in D$, we have $\left|\varphi(z)\right| \leq |h(z)|$. Thus

$$\phi(z) = \frac{\varphi(z)}{h(z)}$$

is a holomorphic function in D and $\left|\phi(z)\right| < 1$ for $|z| < 1$. In particular, we take

$$\left|\phi(0)\right| = \frac{\pi}{4}\left|c_1\right| \tag{2.4}$$

and

$$\left|\phi'(0)\right| = \frac{\pi}{4}\left|c_2\right|.$$

In addition, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = \left|\varphi'(b)\right| \geq |h'(b)| = \frac{bh'(b)}{h(b)}.$$

The composite function

$$G(z) = \frac{\phi(z) - \phi(0)}{1 - \overline{\phi(0)}\phi(z)}$$

satisfies the hypothesis of the Schwarz lemma on the boundary, from where we obtain estimate

$$\frac{2}{1 + |G'(0)|} \leq |G'(b)| = \frac{1 - \left|\phi(0)\right|^2}{\left|1 - \overline{\phi(0)}\phi(b)\right|^2}\left|\phi'(b)\right| \leq \frac{1 + \left|\phi(0)\right|}{1 - \left|\phi(0)\right|}\left\{\left|\varphi'(b)\right| - 1\right\}.$$

Since

$$|G'(0)| = \frac{4\pi\left|c_2\right|}{16 - \pi^2\left|c_1\right|^2},$$

we have that

$$\frac{2}{1 + \frac{4\pi|c_2|}{16 - \pi^2|c_1|^2}} \leq \frac{4 + \pi\left|c_1\right|}{4 - \pi\left|c_1\right|}\left\{\frac{\pi}{2}\left|f'(b)\right| - 1\right\}.$$

Therefore, we get the inequality (2.3).

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Then

$$\left|f'(i)\right| = \frac{2}{\pi}$$

and since $\left|f'(0)\right| = \frac{4}{\pi} = \left|c_1\right|$, (2.3) is satisfied with equality. □

If $f(z)$ has no zeros different from $z = 0$ in Theorem 2.3, the inequality (2.3) can be further strengthened as follow:

Theorem 2.4. *Let* $f(z) = c_1 z + c_2 z^2 + ...$, $\Re c_1 = 0$, $-1 < \Im c_1 < 0$ *be a holomorphic function in the unit disc D and* $\left|\Re f(z)\right| < 1$ *for* $|z| < 1$ *and* $f(z)$ *has no zeros in D except* $z = 0$. *Suppose that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = 1$. *Then, we have*

$$\left|f'(b)\right| \geq \frac{2}{\pi}\left(1 - \frac{2\left|c_1\right|\left(\ln\left|\frac{\pi c_1}{4}\right|\right)^2}{2\left|c_1\right|\left(\ln\left|\frac{\pi c_1}{4}\right|\right) - \left|c_2\right|}\right) \tag{2.5}$$

and

$$\left|c_2\right| \leq 2\left|c_1\left(\ln\left|\frac{\pi c_1}{4}\right|\right)\right|. \tag{2.6}$$

The equality in (2.5) *occurs for the function*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right)$$

and the equality in (2.6) *occurs for the function*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1 + ze^{\frac{1+z}{1-z}\left(\ln\frac{i\pi c_1}{4}\right)}}{1 - ze^{\frac{1+z}{1-z}\left(\ln\frac{i\pi c_1}{4}\right)}}\right),$$

where $\Re c_1 = 0$, $-1 < \Im c_1 < 0$ *and* $\left(\ln\frac{i\pi c_1}{4}\right) < 0$.

Proof. Having in mind inequality (2.4), we denote by $\ln\phi(z)$ the holomorphic branch of the logarithm normed by the condition and since $\Re c_1 = 0$, $-1 < \Im c_1 < 0$, we have

$$\ln\phi(0) = \ln\left(\frac{i\pi c_1}{4}\right) = \ln\left|\frac{i\pi c_1}{4}\right| + i\arg\left(\frac{i\pi c_1}{4}\right) = \ln\left|\frac{\pi c_1}{4}\right| < 0.$$

Consider the function

$$p(z) = \frac{\ln\phi(z) - \ln\phi(0)}{\ln\phi(z) + \ln\phi(0)}.$$

$p(z)$ is holomorphic function in D, $p(0) = 0$, $\left|p(z)\right| < 1$ for $|z| < 1$ and $\left|p(b)\right| = 1$ for $b \in \partial D$. Therefore, from (1.2), we obtain

$$\begin{aligned}\frac{2}{1 + \left|p'(0)\right|} &\leq \left|p'(b)\right| = \frac{\left|2\ln\phi(0)\right|}{\left|\left(\ln\phi(b) + \ln\phi(0)\right)^2\phi(b)\right|}\left|\frac{b\varphi'(b)}{\varphi(b)} - \frac{bh'(b)}{h(b)}\right| \\ &= \frac{-2\ln\phi(0)}{\arg^2\phi(b) + \ln^2\phi(0)}\left\{\left|\varphi'(b)\right| - \left|h'(b)\right|\right\}.\end{aligned}$$

Replacing $\arg^2\phi(b)$ by zero, we take

$$\frac{2}{1 + \left|p'(0)\right|} \leq \frac{-2}{\ln\phi(0)}\left\{\frac{\pi}{2}\left|f'(b)\right| - 1\right\}.$$

Since

$$\left|p'(0)\right| = \frac{-1}{2\ln\left(\left|\frac{\pi c_1}{4}\right|\right)}\frac{\left|c_2\right|}{\left|c_1\right|},$$

we obtain

$$\frac{2}{1-\frac{1}{2\ln\left(\left|\frac{\pi c_1}{4}\right|\right)}\frac{|c_2|}{|c_1|}} \leq \frac{-2}{\ln\left(\left|\frac{\pi c_1}{4}\right|\right)}\left\{\frac{\pi}{2}\left|f'(b)\right|-1\right\}.$$

Therefore, we take the inequality (2.5) with an obvious equality case.

Similary, $p(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq \left|p'(0)\right| = \frac{-1}{2\ln\left(\left|\frac{\pi c_1}{4}\right|\right)}\frac{|c_2|}{|c_1|}$$

and

$$|c_2| \leq 2\left|c_1 \ln\left(\left|\frac{\pi c_1}{4}\right|\right)\right|.$$

The equality in (2.6) is obtained for the function

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+ze^{\frac{1+z}{1-z}\left(\ln\frac{i\pi c_1}{4}\right)}}{1-ze^{\frac{1+z}{1-z}\left(\ln\frac{i\pi c_1}{4}\right)}}\right)$$

as show simple calculations. □

In Theorem 2.3 and 2.4, there are both c_1 and c_2 in the right side of the inequalities. But, if we use (1.3) instead of (1.2), we obtain weaker but more simpler inequality (not including c_2).

Theorem 2.5. *Under the same assumptions as in Theorem 2.4, we have*

$$\left|f'(b)\right| \geq \frac{2}{\pi}\left(1-\frac{1}{2}\ln\left(\left|\frac{\pi c_1}{4}\right|\right)\right). \tag{2.7}$$

In addition, the result is sharp and the extremal function is

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

Proof. From proof of the Theorem 2.4, using the inequality (1.3) for the function $p(z)$, we obtain

$$1 \leq \left|p'(b)\right| = \frac{-2\ln\phi(0)}{\arg^2\phi(b)+\ln^2\phi(0)}\left\{\left|\varphi'(b)\right|-|h'(b)|\right\}.$$

Replacing $\arg^2\phi(b)$ by zero, we take

$$1 \leq \frac{-2}{\ln\phi(0)}\left\{\frac{\pi}{2}\left|f'(b)\right|-1\right\}.$$

Thus, we obtain the inequality (2.7) with an obvious equality case. □

In the following Theorem, we shall give an estimate below $\left|f'(b)\right|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_0 \neq 0$.

Theorem 2.6. *Let* $f(z) = c_1z + c_2z^2 + ...$ *be a holomorphic function in the unit disc D,* $f(z_0) = 0$ *for* $0 < |z_0| < 1$ *and* $\left|\Re f(z)\right| < 1$ *for* $|z| < 1$*. Suppose that, for some* $b \in \partial D$*, f has an angular limit* $f(b)$ *at b,* $f(b) = 1$*. Then, we have*

$$\left|f'(b)\right| \geq \frac{2}{\pi}\left(1+\frac{1-|z_0|^2}{|b-z_0|^2}+\frac{4|z_0|-\pi\left|f'(0)\right|}{4|z_0|+\pi\left|f'(0)\right|}\right. \tag{2.8}$$
$$\left.\times\left[1+\frac{16|z_0|^2+\pi^2\left|f'(z_0)\right|\left|f'(0)\right|\left(1-|z_0|^2\right)-4\pi\left|f'(z_0)\right|\left(1-|z_0|^2\right)-4\pi\left|f'(0)\right|}{16|z_0|^2+\pi^2\left|f'(z_0)\right|\left|f'(0)\right|\left(1-|z_0|^2\right)+4\pi\left|f'(z_0)\right|\left(1-|z_0|^2\right)+4\pi\left|f'(0)\right|}\frac{1-|z_0|^2}{|b-z_0|^2}\right]\right).$$

The inequality (2.8) *is sharp, with equality for each possible values* $\left|f'(0)\right| = \frac{4}{\pi}c$ *and* $\left|f'(z_0)\right| = \frac{4}{\pi}d\left(0 \leq c \leq \frac{4}{\pi}|z_0|, 0 \leq d \leq \frac{4}{\pi}\frac{|z_0|}{1-|z_0|^2}\right)$.

Proof. Let

$$k(z) = \frac{z-z_0}{1-\overline{z_0}z}$$

and $g : D \to D$ is holomorphic function and a point $z_0 \in D$. So, we have

$$\left|g(z)\right| \leq \frac{\left|g(z_0)\right| + |k(z)|}{1+\left|g(z_0)\right||k(z)|}. \tag{2.9}$$

If $m : D \to D$ is holomorphic function and $0 < |z_0| < 1$, letting

$$g(z) = \frac{m(z)-m(0)}{z\left(1-\overline{m(0)}m(z)\right)}$$

in (2.9), we obtain

$$\left|\frac{m(z)-m(0)}{\left(1-\overline{m(0)}m(z)\right)}\right| \leq |z|\frac{\left|\frac{m(z_0)-m(0)}{z_0\left(1-\overline{m(0)}m(z_0)\right)}\right| + |k(z)|}{1+\left|\frac{m(z_0)-m(0)}{z_0\left(1-\overline{m(0)}m(z_0)\right)}\right||k(z)|}$$

and

$$|m(z)| \leq \frac{|m(0)| + |z|\frac{|C|+|k(z)|}{1+|C||k(z)|}}{1+|m(0)||z|\frac{|C|+|k(z)|}{1+|C||k(z)|}}, \tag{2.10}$$

where

$$C = \frac{m(z_0)-m(0)}{z_0\left(1-\overline{m(0)}m(z_0)\right)}.$$

Without loss of generality, we will assume that $b = 1$. If we take

$$m(z) = \frac{\varphi(z)}{z\frac{z-z_0}{1-\overline{z_0}z}},$$

then we obtain

$$m(0) = \frac{\varphi'(0)}{-z_0}, \quad m(z_0) = \frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}$$

and

$$C = \frac{\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0} + \frac{\varphi'(0)}{z_0}}{z_0\left(1-\frac{\overline{\varphi'(0)}}{z_0}\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right)},$$

where $|C| \leq 1$. Let $|m(0)| = \gamma$ and

$$D = \frac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right| + \left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1 + \left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)}.$$

By (2.10), we take

$$\left|\varphi(z)\right| \leq |z|\,|k(z)|\,\frac{\gamma + |z|\,\frac{|D|+|k(z)|}{1+|D||k(z)|}}{1 + \gamma\,|z|\,\frac{|D|+|k(z)|}{1+|D||k(z)|}}$$

and

$$\frac{1 - \left|\varphi(z)\right|}{1 - |z|} \geq \frac{1 + \gamma\,|z|\,\frac{|D|+|k(z)|}{1+|D||k(z)|} - \gamma\,|z|\,|k(z)| - |z|^2\,|k(z)|\,\frac{|D|+|k(z)|}{1+|D||k(z)|}}{(1 - |z|)\left(\gamma\,|z|\,\frac{|D|+|k(z)|}{1+|D||k(z)|}\right)} = \eta.$$

Let $\lambda(z) = 1 + \gamma\,|z|\,\frac{|D|+|k(z)|}{1+|D||k(z)|}$ and $s(z) = 1 + D\,|k(z)|$. Then

$$\eta = \frac{1 - |z|^2\,|k(z)|}{(1 - |z|)\,\lambda(z)s(z)} + D\,|k(z)|\,\frac{1 - |z|^2}{(1 - |z|)\,\lambda(z)s(z)} + D\gamma\,|z|\,\frac{1 - |k(z)|^2}{(1 - |z|)\,\lambda(z)s(z)}. \tag{2.11}$$

Since

$$\lim_{z\to 1}\lambda(z) = 1 + \gamma, \quad \lim_{z\to 1}s(z) = 1 + D$$

and

$$1 - |k(z)|^2 = 1 - \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|^2 = \frac{\left(1 - |z|^2\right)\left(1 - |z_0|^2\right)}{\left|1 - \overline{z_0}z\right|^2},$$

passing to the angular limit in (2.11) gives

$$\begin{aligned}\left|\varphi'(1)\right| &\geq \frac{2}{(1 + \gamma)\,(1 + D)}\left(1 + \frac{1 - |z_0|^2}{|1 - z_0|^2} + D + \gamma D\frac{1 - |z_0|^2}{|1 - z_0|^2}\right)\\ &= 1 + \frac{1 - |z_0|^2}{|1 - z_0|^2} + \frac{1 - \gamma}{1 + \gamma}\left(1 + \frac{1 - D}{1 + D}\frac{1 - |z_0|^2}{|1 - z_0|^2}\right).\end{aligned}$$

In addition, since

$$\frac{1 - \gamma}{1 + \gamma} = \frac{1 - |m(0)|}{1 + |m(0)|} = \frac{1 - \left|\frac{\varphi'(0)}{z_0}\right|}{1 + \left|\frac{\varphi'(0)}{z_0}\right|} = \frac{|z_0| - \left|\varphi'(0)\right|}{|z_0| - \left|\varphi'(0)\right|} = \frac{4\,|z_0| - \pi\,\left|f'(0)\right|}{4\,|z_0| + \pi\,\left|f'(0)\right|}$$

and

$$
\begin{aligned}
\frac{1-D}{1+D} &= \frac{1-\dfrac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1-\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)}}{1+\dfrac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1-\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)}} \\
&= \frac{|z_0|\left(1-\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)-\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|-\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1-\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)+\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|} \\
&= \frac{|z_0|\left(1-\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|\left|\frac{\left(1-|z_0|^2\right)}{z_0}\right|\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|\right)-\left|\frac{\left(1-|z_0|^2\right)}{z_0}\right|\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|-\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|}{|z_0|\left(1-\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|\left|\frac{\left(1-|z_0|^2\right)}{z_0}\right|\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|\right)+\left|\frac{\left(1-|z_0|^2\right)}{z_0}\right|\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|+\frac{\pi}{4}\left|\frac{f'(0)}{z_0}\right|},
\end{aligned}
$$

we obtain

$$
\left|\varphi'(1)\right| \geq \left(1+\frac{1-|z_0|^2}{|b-z_0|^2}+\frac{4|z_0|-\pi\left|f'(0)\right|}{4|z_0|+\pi\left|f'(0)\right|}\left[1+\frac{16|z_0|^2+\pi^2\left|f'(z_0)\right|\left|f'(0)\right|\left(1-|z_0|^2\right)-4\pi\left|f'(z_0)\right|\left(1-|z_0|^2\right)-4\pi\left|f'(0)\right|}{16|z_0|^2+\pi^2\left|f'(z_0)\right|\left|f'(0)\right|\left(1-|z_0|^2\right)+4\pi\left|f'(z_0)\right|\left(1-|z_0|^2\right)+4\pi\left|f'(0)\right|}\frac{1-|z_0|^2}{|b-z_0|^2}\right]\right).
$$

From definition of $\varphi(z)$, we have

$$
\varphi'(z) = \frac{i\pi f'(z)e^{\frac{i\pi}{2}f(z)}}{\left(1+e^{\frac{i\pi}{2}f(z)}\right)^2}
$$

and

$$
\left|\varphi'(1)\right| = \frac{\pi}{2}\left|f'(b)\right|.
$$

Therefore, we obtain the inequality (2.8).

Now, we shall show that the inequality (2.8) is sharp.

Since $m(z) = \frac{\varphi(z)}{z\frac{z-z_0}{1-\overline{z}_0 z}}$ is holomorphic function in the unit disc D and $|m(z)| \leq 1$ for $|z| < 1$, we obtain

$$
\left|\varphi'(0)\right| \leq |z_0|
$$

and

$$
\left|\varphi'(z_0)\right| \leq \frac{|z_0|}{1-|z_0|^2}.
$$

Now, we take $z_0 \in (-1, 0)$ and arbitrary two numbers c and d, such that $0 \leq c \leq \frac{4}{\pi}|z_0|$, $0 \leq d \leq \frac{4}{\pi}\frac{|z_0|}{1-|z_0|^2}$. Let

$$
F = \frac{d\frac{1-|z_0|^2}{z_0}+\frac{c}{z_0}}{z_0\left(1+cd\frac{1-|z_0|^2}{z_0^2}\right)} = \frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}.
$$

The function

$$
\varsigma(z) = iz\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{F+\frac{z-z_0}{1-\overline{z}_0 z}}{1+F\frac{z-z_0}{1-\overline{z}_0 z}}}{1-\frac{c}{z_0}z\frac{F+\frac{z-z_0}{1-\overline{z}_0 z}}{1+F\frac{z-z_0}{1-\overline{z}_0 z}}}
$$

is holomorphic in D and $|\varsigma(z)| < 1$ for $|z| < 1$.

Let

$$\frac{e^{\frac{i\pi}{2}f(z)}-1}{e^{\frac{i\pi}{2}f(z)}+1} = iz\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{F+\frac{z-z_0}{1-\overline{z_0}z}}{1+F\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{F+\frac{z-z_0}{1-\overline{z_0}z}}{1+F\frac{z-z_0}{1-\overline{z_0}z}}}.$$

Therefore, we take $\left|f'(0)\right| = \frac{4}{\pi}c$ and

$$\frac{i\pi f'(z_0)e^{\frac{i\pi}{2}f(z_0)}}{\left(e^{\frac{i\pi}{2}f(z_0)}+1\right)^2} = i\frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+Fz_0}{1-\frac{c}{z_0}z_0F} = i\frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}z_0}{1-\frac{c}{z_0}z_0\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}}$$

$$\left|f'(z_0)\right| = \frac{4}{\pi}d.$$

With the simple calculations, we obtain

$$\begin{aligned}\varsigma'(1) &= i\left(1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-F^2}{(1+F)^2}\right)\left(1-\frac{c}{z_0}\right)+\frac{c}{z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-F^2}{(1+F)^2}\right)\left(-\frac{c}{z_0}+1\right)}{\left(-\frac{c}{z_0}+1\right)^2}\right)\\
&= i\left(1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{1+\frac{c}{z_0}}{1-\frac{c}{z_0}}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-F}{1+F}\right)\right)\\
&= i\left(1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right)\right),\end{aligned}$$

$$\frac{i\pi f'(1)e^{\frac{i\pi}{2}f(1)}}{\left(e^{\frac{i\pi}{2}f(1)}+1\right)^2} = i\left[1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right)\right],$$

and

$$\frac{\pi f'(1)e^{\frac{i\pi}{2}f(1)}}{\left(e^{\frac{i\pi}{2}f(1)}+1\right)^2} = \left[1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right)\right].$$

Since $z_0 \in (-1, 0)$, the last equality show that (2.8) is sharp. □

3. Examples

Example 3.1. *Let us consider a function $f(z)$ given by*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+z}{1-z}\right).$$

From here, we have that

$$f(z) = -\frac{2i}{\pi}\ln\left(\frac{1+z}{1-z}\right) = -\frac{2i}{\pi}\left(\ln\left|\frac{1+z}{1-z}\right| + i\arg\left(\frac{1+z}{1-z}\right)\right)$$

and

$$f(z) = -\frac{2i}{\pi}\ln\left|\frac{1+z}{1-z}\right| + \frac{2}{\pi}\arg\left(\frac{1+z}{1-z}\right).$$

Therefore, we take

$$\Re f(z) = \frac{2}{\pi}\arg\left(\frac{1+z}{1-z}\right),$$

$$\left|\Re f(z)\right| = \frac{2}{\pi}\left|\arg\left(\frac{1+z}{1-z}\right)\right| < \frac{2}{\pi}\frac{\pi}{2} = 1$$

and

$$\left|\Re f(z)\right| < 1.$$

Thus, f(z) satisfies the conditions of theorems which is given above except Theorem 2.6. In addition, for some $b \in \partial D$, we have

$$f(i) = \frac{2}{i\pi}\ln\left(\frac{1+i}{1-i}\right) = \frac{2}{i\pi}\ln(i) = \frac{2}{i\pi}(\ln|i| + i\arg i) = \frac{2}{i\pi}\frac{\pi i}{2}$$

and

$$f(i) = 1.$$

Example 3.2. *Let us define f(z) by*

$$f(z) = \frac{2}{i\pi}\ln\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right),$$

where

$$\varsigma(z) = iz\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0} + z\frac{F+\frac{z-z_0}{1-\overline{z_0}z}}{1+F\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{F+\frac{z-z_0}{1-\overline{z_0}z}}{1+F\frac{z-z_0}{1-\overline{z_0}z}}}.$$

Hence, we get

$$f(z) = -\frac{2i}{\pi}\ln\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right) = -\frac{2i}{\pi}\left(\ln\left|\frac{1+\varsigma(z)}{1-\varsigma(z)}\right| + i\arg\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right)\right)$$

and

$$f(z) = -\frac{2i}{\pi}\ln\left|\frac{1+\varsigma(z)}{1-\varsigma(z)}\right| + \frac{2}{\pi}\arg\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right).$$

Thus, we obtain

$$\Re f(z) = \frac{2}{\pi}\arg\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right),$$

$$\left|\Re f(z)\right| = \frac{2}{\pi}\left|\arg\left(\frac{1+\varsigma(z)}{1-\varsigma(z)}\right)\right| < \frac{2}{\pi}\frac{\pi}{2} = 1$$

and

$$\left|\Re f(z)\right| < 1.$$

So, f(z) satisfies the conditions of Theorem 2.6. In addition, for some $b \in \partial D$, we have

$$f(1) = \frac{2}{i\pi}\ln\left(\frac{1+\varsigma(1)}{1-\varsigma(1)}\right) = \frac{2}{i\pi}\ln\left(\frac{1+i}{1-i}\right) = 1.$$

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