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On the Mazur-Ulam Theorem in Non-Archimedean Fuzzy Anti-2-Normed Spaces

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Abstract. We study the notion of a non-Archimedean fuzzy anti-2-normed space over a non-Archimedean field and prove that Mazur-Ulam theorem holds under some conditions in non-Archimedean fuzzy anti-2-normed spaces.

1. Introduction

A mapping $f : X \longrightarrow Y$ is called an *isometry* if f satisfies

$$d_Y(f(x),\ f(y)) = d_X(x,\ y)$$

for all $x,\ y \in X$, where $d_X(\cdot,\ \cdot)$ and $d_Y(\cdot,\ \cdot)$ denote the metrics in the spaces X and Y, respectively.

The theory of isometric mappings had been originated in the classical paper ([13]) by Mazur and Ulam in 1932.

Mazur-Ulam Theorem. Every isometry f of a normed real linear space X onto a normed real linear space is a linear mapping up to translation, that is, $x \mapsto f(x) - f(0)$ is linear, which amounts to the definition that f is affine.

The Mazur-Ulam theorem is not true for a normed complex vector space. In addition, the onto assumption is also essential. Without this assumption, Baker([2]) proved that an isometry from a normed real linear space into a strictly convex normed real linear space is affine.

In 1984, Katsaras([10]) and Wu and Fang([17]) introduced the notion of a fuzzy norm on linear space and also Wu and Fang gave the generalization of Kolmogoroff normalized theorem for fuzzy topological linear space. Gähler and Gähler([8]) defined fuzzy norm of a fuzzy real number as a difference of its positive and negative parts. Also, Gähler([6, 7]) introduced a new approach for the theory of 2-norm and n-norm on a linear space. Chu([5]) studied the Mazur-Ulam theorem in linear 2-normed spaces. Recently, Moslehian and Sadeghi([15]) introduced the Mazur-Ulam theorem in the non-Archimedean strictly convex normed spaces. Choy et al.([4]) investigated the Mazur-Ulam theorem by using the interior preserving mappings in linear 2-normed spaces and also proved the theorem on non-Archimedean 2-normed spaces over a linear ordered non-Archimedean field without the strict convexity assumption. Moreover, Mirmostafaee and Moslehian([14]) introduced a non-Archimedean fuzzy norm on a linear space over a non-Archimedean field.

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In particular, Amyari and Sadeghi([1]) proved Mazur-Ulam theorem under condition of strict convexity in non-Archimedean 2-normed spaces. Jebril and Samanta([9]) introduced a fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta([3]) and investigated their various properties. Many mathematicians considered the fuzzy normed spaces in different branches of pure and applied mathematics.

Recently, Park and Alaca introduced the new result of the Mazur-Ulam theorem for 2-isometry in the framework of 2-fuzzy 2-normed linear spaces; see([16]). They proved the theorem without the condition of preserving collinearity of 2-isometry.

In this paper, we investigate the notion of a non-Archimedean fuzzy 2-normed space over a linear ordered non-Archimedean field. We prove that Mazur-Ulam theorem holds on a non-Archimedean fuzzy 2-normed space without preserving collinearity of 2-isometry and also with interior preserving fuzzy 2-isometry. Note that our proof of the theorem extended the cases where it is a non-Archimedean fuzzy 2-normed space.

2. Non-Archimedean Fuzzy Anti-2-Normed Space

In this section, we introduce a non-Archimedean fuzzy anti-2-normed space.

Definition 2.1. A *non-Archimedean field* is a field $\mathcal{K}$ equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$ satisfying the following properties:

1. $|a| \geq 0$ and equality holds if and only if $a = 0$,
2. $|ab| = |a|\,|b|$,
3. $|a + b| \leq \max\{|a|, |b|\}$

for all $a, b \in \mathcal{K}$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$; see([12]). We call it a *non-Archimedean trivial valuation*. Also, the most important examples of non-Archimedean spaces are p-adic numbers; see([14]).

Definition 2.2. Let X be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \times X \longrightarrow [0, \infty)$ is said to be a *non-Archimedean 2-norm* if it satisfies the following properties:

1. $\|x, y\| = 0$ if and only if x, y are linearly dependent
2. $\|x, y\| = \|y, x\|$
3. $\|cx, y\| = |c|\,\|x, y\|$
4. $\|x, y + z\| \leq \max\{\|x, y\|, \|x, z\|\}$,

for all $x, y, z \in X$ and $c \in \mathcal{K}$. Then $(X, \|\cdot\|)$ is called a *non-Archimedean 2-normed space*.

Definition 2.3. Let X be a linear space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $N : X^2 \times \mathbb{R} \longrightarrow [0, 1]$ is said to be a *non-Archimedean fuzzy anti-2-norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(aN1) $N(x, y, t) = 1$ for $t \leq 0$,
(aN2) for $t > 0$, $N(x, y, t) = 0$ if and only if x and y are linearly dependent,
(aN3) $N(x, y, t) = N(y, x, t)$,
(aN4) $N(cx, y, t) = N(x, y, \frac{t}{|c|})$ for $c \neq 0$,
(aN5) $N(x, y + z, \max\{s, t\}) \leq \max\{N(x, y, s), N(x, z, t)\}$,
(aN6) $N(x, y, *)$ is a non-increasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, y, t) = 0$.

The pair (X, N) is called a *non-Archimedean fuzzy anti-2-normed space.*

The property (aN4) implies that $N(-x, y, t) = N(x, y, t)$ for all $x, y \in X$ and $t > 0$. It is easy to show that (aN5) is equivalent the following condition:

$$N(x, y + z, t) \leq \max\{N(x, y, t), N(x, z, t)\}, \text{ for all } x, y, z \in X \text{ and } t \in \mathbb{R}.$$

Example 2.4. Let $(X, \|\cdot\,,\cdot\|)$ be a non-Archimedean 2-normed space. Define

$$N(x,y,t) = \begin{cases} \frac{\|x,\,y\|}{t+\|x,\,y\|} & \text{when } t > 0\,, \\ 1 & \text{when } t \le 0\,, \end{cases}$$

where $x, y \in X$ and $t \in \mathbb{R}$. Then (X, N) is a non-Archimedean fuzzy anti-2-normed space. Indeed,

(aN1) The definition of N implies that $N(x,y,t) = 1$ for $t \le 0$.

(aN2) Let $t > 0$. $N(x,y,t) = 0 \Leftrightarrow \|x,y\| = 0 \Leftrightarrow x$ and y are linearly dependent.

(aN3) $N(x,y,t) = \frac{\|x,\,y\|}{t+\|x,\,y\|} = \frac{\|y,\,x\|}{t+\|y,\,x\|} = N(y,x,t)$.

(aN4)

$$N(cx,y,t) = \frac{\|cx\,,y\|}{t+\|cx\,,y\|} = \frac{\|x\,,y\|}{\frac{t}{|c|}+\|x\,,y\|} = N(x,\ y,\ \frac{t}{|c|})$$

for all $c \neq 0$.

(aN5) Let $x,\ y\,, z \in X$ and $s\,, t \in \mathbb{R}$. If $s\,, t \le 0$, then $N(x, y+z, \max\{s,t\}) = 1 = \max\{N(x,y,s)\,, N(x,z,t)\}$. If $s < 0 < t$, then $N(x, y+z, \max\{s,t\}) \le 1 = \max\{N(x,y,s)\,, N(x,z,t)\}$. Now, let $s\,, t > 0$. We may assume that $\max\{s,t\} = s \ge t$. Since $\|x\,, y+z\| \le \max\{\|x\,,y\|\,,\|x\,,z\|\}$, we may consider two cases where (a) $\max\{\|x\,,y\|\,,\|x\,,z\|\} = \|x\,,y\|$ and (b) $\max\{\|x\,,y\|\,,\|x\,,z\|\} = \|x\,,z\|$. The case (a) :

$$\begin{aligned} N(x,\ y+z\,, \max\{s,t\}) &= \frac{\|x,\ y+z\|}{s+\|x,\ y+z\|} \\ &\le \frac{\|x,\ y\|}{s+\|x,\ y\|} \\ &\le \max\{N(x,y,s)\,, N(x,z,t)\}\,. \end{aligned}$$

The case (b) :

$$\begin{aligned} N(x,\ y+z\,, \max\{s,t\}) &= \frac{\|x,\ y+z\|}{s+\|x,\ y+z\|} \\ &\le \frac{\|x,\ z\|}{s+\|x,\ z\|} \\ &\le \frac{\|x,\ z\|}{t+\|x,\ z\|} \\ &\le \max\{N(x,y,s)\,, N(x,z,t)\}\,. \end{aligned}$$

These cases imply that

$$N(x, y+z, \max\{s,t\}) \le \max\{N(x,y,s)\,, N(x,z,t)\}\,.$$

(aN6) Let $s < t \le 0$ and let $s \le 0 < t$. These cases imply that $N(x,y,s) = 1 \ge N(x,y,t)$. If $0 < s < t$, then

$$N(x,y,s) - N(x,y,t) = \frac{(s-t)\|x,y\|}{(s+\|x,y\|)(t+\|x,y\|)} \le 0\,.$$

Hence $N(x,y,*)$ is a non-increasing function of $\mathbb{R}$. Also,

$$\lim_{t\to\infty} N(x,y,t) = \lim_{t\to\infty} \frac{\|x,\ y\|}{t+\|x,\ y\|} = 0\,,$$

for all $x\,, y \in X$.

Definition 2.5. A non-Archimedean fuzzy anti-2-normed space is said to be *strictly convex* if $N(x, y+z, \max\{s,t\}) = \max\{N(x,y,s)\,, N(x,z,t)\}$ and $N(x,y,s) = N(x,z,t)$ imply $y = z$ and $s = t$.

Definition 2.6. Let (X, N) and (Y, N) be two non-Archimedean fuzzy anti-2-normed spaces. We call $f : (X, N) \longrightarrow (Y, N)$ a *fuzzy 2-isometry* if $N(a - c, b - c, t) = N(f(a) - f(c), f(b) - f(c), t)$, for all $a, b, c \in X$ and $t > 0$.

For given points x, y and $z \in X$, $\triangle xyz$ denotes the triangle determined by x, y and z. A point $(x + y + z)/3$ is called a *barycenter* of $\triangle xyz$. If p is a point of a set $\{t_1 x + t_2 y + t_3 z \,|\, t_1 + t_2 + t_3 = 1, t_i \in \mathcal{K}, t_i > 0, i = 1, 2, 3\}$, then p is called an *interior point* of $\triangle xyz$.

Definition 2.7. Let (X, N) and (Y, N) be two non-Archimedean fuzzy anti-2-normed spaces. We call $f : (X, N) \longrightarrow (Y, N)$ *an interior preserving mapping of the triangle* if $f(p)$ is an interior point of $\triangle f(x) f(y) f(z)$, where p is an interior point of $\triangle xyz$.

3. Mazur-Ulam Theorem

We first denote the set of all elements of $\mathcal{K}$ whose norms are 1 by C, that is,

$$C = \{r \in \mathcal{K} \,|\, |r| = 1\}.$$

We recall the definition of ordered non-Archimedean field $\mathcal{K}$.

Definition 3.1. A field $\mathbb{K}$ is *orderable* if there exists a non-empty $\mathbb{K}_+ \subset \mathbb{K}$ such that
(1) $0 \notin \mathbb{K}_+$
(2) for all $x, y \in \mathbb{K}_+, x + y \in \mathbb{K}_+$ and $xy \in \mathbb{K}_+$
(3) for all $x (\neq 0) \in \mathbb{K}, x \in \mathbb{K}_+$ or $-x \in \mathbb{K}_+$.

Provided that $\mathbb{K}$ is orderable, we can fix a set $\mathbb{K}_+$ that satisfies a strict order relation on $\mathbb{K}$ by $x <_{\mathbb{K}_+} y$ if and only if $y - x \in \mathbb{K}_+$. Then we call $(\mathbb{K}, <_{\mathbb{K}_+})$ an ordered field.

Definition 3.2. Let $\mathcal{K}$ be an ordered field and a non-Archimedean field equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$. Then we call $\mathcal{K}$ an *ordered non-Archimedean field.*

In this section, we will prove Mazur-Ulam theorem on non-Archimedean fuzzy anti-2-normed space X over a linear ordered non-Archimedean field $\mathcal{K}$.

Lemma 3.3. *Let (X, N) be a non-Archimedean fuzzy anti-2-normed space over a linear ordered non-Archimedean field $\mathcal{K}$. Then*

$$N(x, y, t) = N(x, y + rx, t), \text{ for all } r \in \mathcal{K}.$$

Proof. Let $x, y \in X$ and let $r \in \mathcal{K}$. Without loss generality, we may assume $t > 0$. Then

$$N(x, y + rx, t) \leq \max\{N(x, y, t), N(x, rx, t)\} = N(x, y, t).$$

Conversely,

$$\begin{aligned} N(x, y, t) = N(x, y + rx - rx, t) &\leq \max\{N(x, y + rx, t), N(x, rx, t)\} \\ &= N(x, y + rx, t). \end{aligned}$$

Thus $N(x, y, t) = N(x, y + rx, t)$ for all $r \in \mathcal{K}$. □

Lemma 3.4. *Let (X, N) be a non-Archimedean fuzzy anti-2-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $C = \{2^n \,|\, n \in \mathbb{Z}\}$ and let $a, b, c \in X$ and $t > 0$. Suppose X is strictly convex. Then $\alpha = \frac{a+b}{2}$ is the unique element of X such that*

$$N(a - c, a - \alpha, t) = N(b - \alpha, b - c, t) = N(a - c, b - c, t)$$

where $N(a - c, b - c, t) \neq 0$ and $\alpha \in \{sa + (1 - s)b \,|\, s \in \mathcal{K}\}$.

Proof. Let $\alpha = \frac{a+b}{2} \in X$ and $t > 0$. By Lemma 3.3, we have

$$\begin{aligned} N(a-c, a-\alpha, t) &= N(a-c, a-\frac{a+b}{2}, t) = N(a-c, \frac{a-b}{2}, t) \\ &= N(a-c, a-b, |2|t) = N(a-c, a-b, t) \\ &= N(a-c, b-c, t)\,. \end{aligned}$$

Similarly,

$$\begin{aligned} N(b-\alpha, b-c, t) &= N(b-\frac{a+b}{2}, b-c, t) = N(b-a, b-c, t) \\ &= N(a-c, b-c, t)\,. \end{aligned}$$

Hence we have $N(a-c, a-\alpha, t) = N(a-c, b-c, t) = N(b-\alpha, b-c, t)$, that is, the existence part holds. To show the uniqueness part, assume that β is such an element of X such that

$$N(a-c, a-\beta, t) = N(b-\beta, b-c, t) = N(a-c, b-c, t)$$

where $N(a-c, b-c, t) \neq 0$ and $\alpha \in \{sa + (1-s)b \,|\, s \in \mathcal{K}\}$. Hence we may let

$$\beta = sa + (1-s)b$$

where $s \in \mathcal{K}$. We may assume $s \neq 0$ and $s \neq 1$.

$$\begin{aligned} N(a-c, b-c, t) &= N(a-c, a-\beta, t) = N(a-c, a-(sa+(1-s)b), t) \\ &= N(a-c, a-b, \frac{t}{|1-s|}) = N(a-c, b-c, \frac{t}{|1-s|})\,. \end{aligned}$$

Similarly, we have

$$N(a-c, b-c, t) = N(b-\beta, b-c, t) = N(a-c, b-c, \frac{t}{|s|})\,,$$

that is,

$$N(a-c, b-c, t) = N(a-c, b-c, \frac{t}{|1-s|}) = N(a-c, b-c, \frac{t}{|s|})\,.$$

We note that

$$\begin{aligned} & N(a-c+a-c, b-c, \max\{\frac{t}{|s|}, \frac{t}{|1-s|}\}) \\ & \leq \max\{N(a-c, b-c, \frac{t}{|s|}), N(a-c, b-c, \frac{t}{|1-s|})\} \\ & = N(a-c, b-c, \frac{t}{|s|}) = N(a-c, b-c, \frac{t}{|1-s|})\,, \end{aligned}$$

and

$$\begin{aligned} & N(a-c+a-c, b-c, \max\{\frac{t}{|s|}, \frac{t}{|1-s|}\}) \\ & = N(2(a-c), b-c, \max\{\frac{t}{|s|}, \frac{t}{|1-s|}\}) \\ & = N(a-c, b-c, \max\{\frac{t}{|s|}, \frac{t}{|1-s|}\})\,. \end{aligned}$$

The previous note implies that

$$N(a-c,b-c,t)=N(a-c,b-c,\frac{t}{|s|})=N(a-c,b-c,\frac{t}{|1-s|})\,.$$

The strict convexity of X implies that $|s|=|1-s|=1$. Then there exist elements t_1 and t_2 in $\mathbb{Z}$ such that $1-s=2^{t_1}$ and $s=2^{t_2}$. Since $2^{t_1}+2^{t_2}=1$, we know that $t_1\,,t_2\ <0$. Without loss of generality, we let $1-s=2^{-n_1}$ and $s=2^{-n_2}$ with $n_1\geq n_2$. If $n_1\gneq n_2$ then

$$1=2^{-n_1}+2^{-n_2}=2^{-n_1}(1+2^{n_1-n_2})\,.$$

Hence $2^{n_1}=1+2^{n_1-n_2}$. This is a contradiction. Thus $n_1=n_2$, that is, $s=\frac{1}{2}$. This implies that $\beta=\frac{a+b}{2}=\alpha$. Therefore the proof is completed. □

Theorem 3.5. *Let X and Y be non-Archimedean fuzzy anti-2-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $C=\{2^n\,|\,n\in\mathbb{Z}\}$. Let X and Y be strict convexities. If $f:X\longrightarrow Y$ is a fuzzy 2-isometry, then $f(x)-f(0)$ is additive.*

Proof. Let $g(x)=f(x)-f(0)$. Since f is a fuzzy 2-isometry, so is g. Since $g:X\longrightarrow Y$ is a fuzzy 2-isometry, we have

$$\begin{aligned}&N(g(a)-g(c),\,g(a)-g(\frac{a+b}{2}),\,t)=N(a-c,\,a-\frac{a+b}{2},\,t)\\&=N(a-c,\,a-b,\,t)=N(a-c,\,b-c,\,t)\\&=N(g(a)-g(c),\,g(b)-g(c),\,t)\,.\end{aligned}$$

Similarly, we get $N(g(b)-g(\frac{a+b}{2}),\,g(b)-g(c),\,t)=N(g(a)-g(c),\,g(b)-g(c),\,t)$. Hence

$$\begin{aligned}N(g(a)-g(c),\,g(a)-g(\frac{a+b}{2}),\,t)&=N(g(b)-g(\frac{a+b}{2}),\,g(b)-g(c),\,t)\\&=N(g(a)-g(c),\,g(b)-g(c),\,t)\,.\end{aligned}$$

We note that

$$\begin{aligned}N(g(\frac{a+b}{2})-g(b),\,g(a)-g(b),\,t)&=N(\frac{a+b}{2}-b,\,a-b,\,t)\\&=N(a-b,\,a-b,\,t)=0\,.\end{aligned}$$

By Definition 2.3, we have $g(\frac{a+b}{2})-g(b)=s(g(a)-g(b))$, that is,

$$g(\frac{a+b}{2})=sg(a)+(1-s)g(b)$$

for some $s\in\mathcal{K}$. The uniqueness of Lemma 3.4 implies that $g(\frac{a+b}{2})=\frac{g(a)+g(b)}{2}$ for all $a,\,b\in X$. Thus $f(x)-f(0)$ is additive, as desired. □

In the following, we will investigate that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. By using this result, we will prove a Mazur-Ulam theorem on non-Archimedean fuzzy anti-2-normed space X over a linear ordered non-Archimedean field $\mathcal{K}$ with $C=\{3^n\,|\,n\in\mathbb{Z}\}$.

Lemma 3.6. *Let $(X\,,N)$ be a non-Archimedean fuzzy anti-2-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $C=\{3^n\,|\,n\in\mathbb{Z}\}$ and let $a,b,c\in X$ and $t>0$. Suppose X is strictly convex. Then $\alpha=\frac{a+b+c}{3}$ is the unique element of X such that*

$$N(a-\alpha,b-\alpha,t)=N(b-\alpha,c-\alpha,t)=N(a-\alpha,c-\alpha,t)=N(a-b,a-c,t)$$

where $\alpha\in\{t_1a+t_2b+t_3c\,|\,t_1+t_2+t_3=1\,,t_i\in\mathcal{K}\,,t_i>0\,,i=1,2,3\}$.

Proof. Let $\alpha = \frac{a+b+c}{3} \in X$ and $t > 0$. By Lemma 3.3, we have

$$\begin{aligned}
N(a-\alpha, b-\alpha, t) &= N(a-\alpha, b-a, t) = N(2a-b-c, b-a, |3|t) \\
&= N(a-c, b-a, t) = N(a-c, b-c, t) \\
&= N(a-b, a-c, t).
\end{aligned}$$

Similarly, we get

$$N(b-\alpha, c-\alpha, t) = N(a-b, a-c, t) \text{ and } N(a-\alpha, c-\alpha, t) = N(a-b, a-c, t).$$

Hence we have $N(a-\alpha, b-\alpha, t) = N(b-\alpha, c-\alpha, t) = N(a-\alpha, c-\alpha, t) = N(a-b, a-c, t)$, that is, the existence part holds. To show the uniqueness part, assume that β is such an element of X such that

$$N(a-\beta, b-\beta, t) = N(b-\beta, c-\beta, t) = N(a-\beta, c-\beta, t) = N(a-b, a-c, t)$$

where $\beta \in \{t_1 a + t_2 b + t_3 c \,|\, t_1 + t_2 + t_3 = 1, t_i \in \mathcal{K}, t_i > 0, i = 1, 2, 3\}$. Hence we may let $\beta = s_1 a + s_2 b + s_3 c$ where $s_1 + s_2 + s_3 = 1$. Then we have

$$\begin{aligned}
N(a-b, a-c, t) &= N(a-\beta, b-\beta, t) = N((1-s_1)a - s_2 b - (1-s_1-s_2)c,\ b-a, t) \\
&= N((1-s_1-s_2)a - (1-s_1-s_2)c,\ b-a,\ t) \\
&= N(a-c,\ b-a,\ \frac{t}{|1-s_1-s_2|}) \\
&= N(a-b,\ a-c,\ \frac{t}{|1-s_1-s_2|}).
\end{aligned}$$

Also, we have

$$\begin{aligned}
N(a-b, a-c, t) &= N(a-\beta, c-\beta, t) = N((1-s_1)a - s_2 b - (1-s_1-s_2)c,\ c-a,\ t) \\
&= N(s_2 a - s_2 b,\ c-a,\ t) \\
&= N(a-b,\ a-c,\ \frac{t}{|s_2|}).
\end{aligned}$$

Similarly, we get

$$N(a-b, a-c, t) = N(b-\beta,\ c-\beta,\ t) = N(a-b,\ a-c,\ \frac{t}{|s_1|}),$$

that is,

$$N(a-b, a-c, \frac{t}{|1-s_1-s_2|}) = N(a-b, a-c, \frac{t}{|s_2|}) = N(a-b, a-c, \frac{t}{|s_1|}).$$

We note that

$$\begin{aligned}
&N(a-b+a-b+a-b, a-c, \max\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\}) \\
&\leq \max\{N(a-b, a-c, \frac{t}{|s_1|}), N(a-b, a-c, \frac{t}{|s_2|}), N(a-b, a-c, \frac{t}{|1-s_1-s_2|})\} \\
&= N(a-b, a-c, \frac{t}{|s_1|}) = N(a-b, a-c, \frac{t}{|s_2|}) = N(a-b, a-c, \frac{t}{|1-s_1-s_2|}),
\end{aligned}$$

and

$$\begin{aligned}
&N(a-b+a-b+a-b, a-c, \max\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\}) \\
&= N(3(a-b), a-c, \max\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\}) \\
&= N(a-b, a-c, \max\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\}).
\end{aligned}$$

The strict convexity of X implies that $|s_1| = |s_2| = |1 - s_1 - s_2| = 1$. Then there exist elements k_1, k_2 and k_3 in $\mathbb{Z}$ such that $s_1 = 3^{k_1}, s_2 = 3^{k_2}$ and $1 - s_1 - s_2 = 3^{k_3}$. Since $3^{k_1} + 3^{k_2} + 3^{k_3} = 1$, we know that $k_1, k_2, k_3 < 0$. Without loss of generality, we let $s_1 = 3^{-n_1}, s_2 = 3^{-n_2}$ and $1 - s_1 - s_2 = 3^{-n_3}$ with $n_1 \geq n_2 \geq n_3$. Then

$$1 = 3^{-n_1} + 3^{-n_2} + 3^{-n_3} = 3^{-n_1}(1 + 3^{n_1 - n_2} + 3^{n_1 - n_3}).$$

Hence $3^{n_1} = 1 + 3^{n_1-n_2} + 3^{n_1-n_3}$. This is a contradiction. Thus $s_1 = s_2 = s_3 = \frac{1}{3}$. This implies that $\beta = \frac{a+b+c}{3} = \alpha$. Therefore the proof is completed. □

Theorem 3.7. *Let X and Y be non-Archimedean fuzzy anti-2-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $C = \{3^n \mid n \in \mathbb{Z}\}$. Let X and Y be strict convexities. If $f : X \longrightarrow Y$ is an interior preserving fuzzy 2-isometry, then $f(x) - f(0)$ is additive.*

Proof. Let $g(x) = f(x) - f(0)$. Since f is a fuzzy 2-isometry, so is g. For a, b and $c \in X$, let $\triangle abc$ be a triangle determined by the points a, b and c, and let x be an interior point of $\triangle abc$. Since f is an interior preserving mapping, we may write

$$f(x) = s_1 f(a) + s_2 f(b) + s_3 f(c),$$

where $s_i \in \mathcal{K}, s_i > 0\ (i = 1, 2, 3)$ with $s_1 + s_2 + s_3 = 1$. Then we have

$$\begin{aligned} g(x) &= s_1 f(a) + s_2 f(b) + s_3 f(c) - f(0) \\ &= s_1(f(a) - f(0)) + s_2(f(b) - f(0)) + s_3(f(c) - f(0)) \\ &= s_1 g(a) + s_2 g(b) + s_3 g(c). \end{aligned}$$

Hence $g(x)$ is an interior point of $\triangle g(a)g(b)g(c)$, that is, g is also an interior preserving mapping.

Since $g : X \longrightarrow Y$ is a fuzzy 2-isometry, we have

$$\begin{aligned} &N(g(a) - g(\frac{a+b+c}{3}), g(b) - g(\frac{a+b+c}{3}), t) \\ &= N(a - \frac{a+b+c}{3}, b - \frac{a+b+c}{3}, t) \\ &= N(a - b, b - \frac{a+b+c}{3}, t) = N(a - b, 2b - a - c, |3|t) \\ &= N(a - b, a - c, t) = N(g(a) - g(b), g(a) - g(c), t). \end{aligned}$$

Similarly, we get

$$\begin{aligned} &N(g(b) - g(\frac{a+b+c}{3}), g(c) - g(\frac{a+b+c}{3}), t) \\ &= N(g(a) - g(b), g(a) - g(c), t) \\ &= N(g(a) - g(\frac{a+b+c}{3}), g(c) - g(\frac{a+b+c}{3}), t). \end{aligned}$$

Since $\frac{a+b+c}{3}$ is an interior point of the triangle $\triangle abc$ and g is an interior preserving mapping, $g(\frac{a+b+c}{3})$ is an interior point of the triangle $\triangle g(a)g(b)g(c)$. By the uniqueness of Lemma 3.6, we have

$$g(\frac{a+b+c}{3}) = \frac{g(a) + g(b) + g(c)}{3}$$

for all $a, b, c \in X$. Thus $f(x) - f(0)$ is additive, as desired. □

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