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Local Properties of Absolute Matrix Summability of Factored Fourier Series

Hikmet Seyhan Özarslan[a], Şebnem Yıldız[b]

[a]*Erciyes University, Department of Mathematics, 38039 Kayseri, Turkey*
[b]*Ahi Evran University, Department of Mathematics, 40100 Kırşehir, Turkey*

Abstract. In this paper, we introduce two new general theorems on $\varphi - \left|A, p_n\right|_k$ summability factors of infinite and Fourier series. By using these theorems, we obtain some new results regarding other important summability methods and investigate conversions between them.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [17]).
The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{3}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\mid \bar{N}, p_n \mid_k$ summability is the same as $|C, 1|_k$ (resp. $\mid \bar{N}, p_n \mid$) summability.

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Email addresses: seyhan@erciyes.edu.tr; hseyhan38@gmail.com (Hikmet Seyhan Özarslan), sebnemyildiz@ahievran.edu.tr; sebnem.yildiz82@gmail.com (Şebnem Yıldız)

2. Known Results

In [23], Özarslan has proved following theorem dealing with Riesz summability of infinite series.

Theorem 2.1. *Let $k \geq 1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n) satisfy the following conditions*

$$\sum_{n=1}^{m} p_n|\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} P_n|\Delta\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{5}$$

$$p_{n+1} = O(p_n), \tag{6}$$

then the series $\sum a_n\lambda_n P_n$ is summable $|\bar{N}, p_n|_k$.

Definition 2.2. *Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where*

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v \quad n = 0, 1, ... \tag{7}$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \left|A, p_n\right|_k$, $k \geq 1$, if (see [26])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|\bar{\Delta}A_n(s)\right|^k < \infty, \tag{8}$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s). \tag{9}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \left|A, p_n\right|_k$ summability reduces to $\left|A, p_n\right|_k$ summability (see [29]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $\left|\bar{N}, p_n\right|_k$ summability. Furthermore, if we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\varphi - \left|A, p_n\right|_k$ reduces to $|C, 1|_k$ summability (see [16]). Finally, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $\left|R, p_n\right|_k$ summability (see [6]).

3. On the Summability Factors of Fourier Series

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$.
The Fourier series of $f(t)$ is

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=0}^{\infty} C_n(t), \tag{10}$$

where (a_n) and (b_n) denote the Fourier coefficients. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x, it is not affected by the values it takes outside the interval), also it is known that the convergence of the Fourier series can be ensured by local hypothesis, that is to say, the behavior of the convergence of Fourier series for a particular value of x depends on the behavior of the function in the immediate neighborhood

of this point only. Hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f .
On the other hand it is known that absolute convergence of a Fourier series is not a local property. Also Bosanquet and Kestelman [15] showed that even summability $|C, 1|$ of a Fourier series a given point is not a local property of the generating function.
Mohanty [22] demonstrated that the $|R, logn, 1|$ summability of the factored Fourier series

$$\sum \frac{C_n(t)}{log(n+1)} \tag{11}$$

at $t = x$, is a local property of the generating function of $\sum C_n(t)$. Later on Matsumoto [20] improved this result by replacing the series (11) by

$$\sum \frac{C_n(t)}{\{loglog(n+1)\}^{1+\epsilon}}, \quad \epsilon > 0. \tag{12}$$

Generalizing the above result Bhatt [2] proved the following theorem.

Theorem 3.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, logn, 1|$ of the series $\sum C_n(t)\lambda_n logn$ at a point can be ensured by a local property.*

Many works have been done dealing with Fourier series (see [1], [4]-[5], [7]-[14], [18]-[25], [27]-[28], [30]-[31]). Among them, the following theorem has been given in [23] as the result of Theorem 2.1.

Theorem 3.2. *Let $k \geq 1$. The summability $| \bar{N}, p_n |_k$ of the series $\sum C_n(t)\lambda_n P_n$ at a point is a local property of the generating function if the conditions* (4) *and* (5) *are satisfied.*

4. Main Results

The aim of this paper is to generalize Theorem 2.1 and Theorem 3.2 for $\varphi - |A, p_n|_k$ summability methods under different conditions by using general summability factors by applicating to Fourier series.
Before stating the main theorem, we must first introduce some further notations.
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{13}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{14}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{15}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{16}$$

Now, we shall prove the following theorems.

Theorem 4.1. *If $A = (a_{nv})$ is a positive normal matrix such that*

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{17}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v+1, \tag{18}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{19}$$

and $\left(\frac{\varphi_n p_n}{P_n}\right)$ be a non-increasing sequence. If all the conditions of Theorem 2.1 are satisfied and (φ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} p_v |\lambda_v| \quad = \quad O(1) \quad as \quad m \to \infty, \tag{20}$$

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} P_v |\Delta\lambda_v| \quad = \quad O(1) \quad as \quad m \to \infty, \tag{21}$$

then the series $\sum a_n \lambda_n P_n$ is summable $\varphi - \left|A, p_n\right|_k$, $k \geq 1$.

It should be noted that if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.1.

Theorem 4.2. *Let $k \geq 1$. The summability $\varphi - \left|A, p_n\right|_k$ of the series $\sum C_n(t)\lambda_n P_n$ at a point is a local property of the generating function if all conditions of Theorem 4.1 are satisfied.*

We need the following lemma for the proof of Theorem 4.1.

Lemma 4.3. *([23]) If the sequences (λ_n) and (p_n) satisfy the conditions (4) and (5) of Theorem 2.1, then $P_m|\lambda_m| = O(1) \quad as \quad m \to \infty$.*

Proof of Theorem 4.1
Without any loss of generality we may assume that $a_0 = s_0 = 0$.
Let (I_n) denote the A-transform of the series $\sum_{n=1}^{\infty} a_n P_n \lambda_n$. Then, by (15) and (16), we have

$$\bar{\Delta} I_n \quad = \quad \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v = \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v P_v) \sum_{r=1}^{v} a_r + \hat{a}_{nn}\lambda_n P_n \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v P_v) s_v + a_{nn}\lambda_n P_n s_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v P_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v P_v s_v - \sum_{v=1}^{n-1} \hat{a}_{n,v+1} p_{v+1}\lambda_{v+1} s_v + a_{nn}\lambda_n P_n s_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 4.1, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{22}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid I_{n,1} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \Delta_v(\hat{a}_{nv}) \mid |\lambda_v| P_v \mid s_v \mid \right\}^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \Delta_v(\hat{a}_{nv}) \mid |\lambda_v|^k P_v^k \mid s_v \mid^k \right\} \times \left\{ \sum_{v=1}^{n-1} \mid \Delta_v(\hat{a}_{nv}) \mid \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} \mid \Delta_v(\hat{a}_{nv}) \mid |\lambda_v|^k P_v^k \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^k P_v^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \mid \Delta_v(\hat{a}_{nv}) \mid \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\lambda_v|^k P_v^k a_{vv} = O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\lambda_v|^k P_v^k \frac{p_v}{P_v} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} (|\lambda_v| P_v)^{k-1} p_v |\lambda_v| = O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} p_v |\lambda_v| \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1 and Lemma 4.3.
Now, again using Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid I_{n,2} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid |\Delta\lambda_v| P_v \mid s_v \mid \right\}^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid^k |\Delta\lambda_v| P_v \mid s_v \mid^k \right\} \times \left\{ \sum_{v=1}^{n-1} |\Delta\lambda_v| P_v \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid\mid \hat{a}_{n,v+1} \mid^{k-1} |\Delta\lambda_v| P_v \right\} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid |\Delta\lambda_v| P_v \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v| P_v \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \mid \hat{a}_{n,v+1} \mid \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\Delta\lambda_v| P_v \sum_{n=v+1}^{m+1} \mid \hat{a}_{n,v+1} \mid = O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} |\Delta\lambda_v| P_v \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1.

Again, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \mid I_{n,3} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid p_{v+1} |\lambda_{v+1}| |s_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid^k p_v |\lambda_v| \right\} \times \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid^{k-1} \mid \hat{a}_{n,v+1} \mid p_v |\lambda_v| \right\} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} a_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} \mid \hat{a}_{n,v+1} \mid p_v |\lambda_v| \right\} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \mid \hat{a}_{n,v+1} \mid \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} p_v |\lambda_v| \sum_{n=v+1}^{m+1} \mid \hat{a}_{n,v+1} \mid \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v} \right)^{k-1} p_v |\lambda_v| = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1.
Finally, since $P_n|\lambda_n| = O(1)$ as $n \to \infty$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} \mid I_{n,4} \mid^k &= \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^k |\lambda_n|^k P_n^k \mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} a_{nn}^{k-1} |\lambda_n|^{k-1} |\lambda_n| P_n^k \frac{p_n}{P_n} \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} p_n |\lambda_n| = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1. This completes the proof of Theorem 4.1.

Proof of Theorem 4.2
Since the behavior of the Fourier series for a particular value of x, as far as convergence is concerned, depends on the behavior of the function in the immediate neighbourhood of this point only, Theorem 4.2 is an immediate consequence of Theorem 4.1.

5. Conclusions

If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability. If we set $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we obtain a new result dealing with $\left|R, p_n\right|_k$ summability method. Additionally, if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for dealing with $\varphi - |C, 1|_k$ summability. Furthermore, if we take $\varphi_n = n, a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

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