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On $\mathcal{I}A$-Density of Points and Some of its Consequences

Pratulananda Das[a], Kumardipta Bose[a], Sayan Sengupta[a]

[a] *Department of Mathematics, Jadavpur University, Kolkata - 700032, West Bengal, India*

Abstract. In this note, continuing in the line of [2] we further consider a more general approach and for $y \in \mathbb{R}$ and a sequence $x = (x_n) \in \ell^\infty$ we define the more general notion of $\mathcal{I}A$-density of indices of those x_n's which are close to y, denoted by $\mathcal{I}\delta_A(y)$ where A is a non-negative regular matrix. Connections are drawn between $\mathcal{I}\delta_A(y)$ and particular limit points of $((Ax)_n)$. Our main result states that if $x = (x_n)$ is a bounded sequence, $\mathcal{I}\delta_A(y)$ exists for every $y \in \mathbb{R}$ and $\sum\limits_{y \in D} \mathcal{I}\delta_A(y) = 1$ then $\mathcal{I} - \lim_{n\to\infty}(Ax)_n = \sum_{y\in D} \mathcal{I}\delta_A(y) \cdot y$ provided both finitely exists. This is an improvement of the alternative version of famous Osikiewicz Theorem given in [2].

1. Introduction

Before we assert what we have done in this paper it is necessary to understand the history behind this investigation. For $n, m \in \mathbb{N}$ with $n < m$, let $[n, m]$ denote the set $\{n, n+1, n+2, \ldots, m\}$. Let $A \subset \mathbb{N}$. Define

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n} \quad \text{and} \quad \underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

The numbers $\overline{d}(A)$ and $\underline{d}(A)$ are called the upper natural density and the lower natural density of A, respectively. If $\overline{d}(A) = \underline{d}(A)$, then this common value is called the natural density of A and we denote it by $d(A)$. Let $\mathcal{I}_d$ be the family of all subsets of $\mathbb{N}$ which have natural density 0. Then $\mathcal{I}_d$ is a proper nontrivial admissible ideal of subsets of $\mathbb{N}$. The notion of natural density was used by Fast [7] and Scoenberg [23] to define the notion of statistical convergence.

Osikiewicz had developed the ideas of finite and infinite splices in [20]. Let $E_1, E_2, E_3, \ldots, E_k, \ldots$ be a partition of $\mathbb{N}$ into countable number of sequences. Let $y_1, y_2, y_3, \ldots, y_k, \ldots$ be distinct real numbers. Let (x_n) be such that

$$\lim_{n\to\infty, n\in E_i} x_n = y_i.$$

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Email addresses: pratulananda@yahoo.co.in (Pratulananda Das), gabuaktafaltu6ele@hotmail.com (Kumardipta Bose), mathematics.sayan1729@gmail.com (Sayan Sengupta)

Then (x_n) is called an infinite-splice (In the same way Osikiewicz defined an finite splice taking finite number of sequences and finite number of distinct real numbers). Osikiewicz then considered a regular matrix summability method A and the notion of A-density the details of which are presented in the next section. He proved the following result.

Theorem 1.1 (Osikiewicz[20]). *Assume that A is non-negative regular summability matrix. Assume that $(x_n) \in \ell^\infty$ is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty, n\in E_i} x_n$. Assume that $\delta_A(E_i)$ exists for each i and*

$$\sum_i \delta_A(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_i y_i \delta_A(E_i).$$

Very recently in [2] a new approach was made to study the Osikiewicz Theorem by defining the notion of the A-density of a point and an alternative version of the same result was established. In fact it was shown that the assumptions of Osikiewicz Theorem imply those of the following Theorem

Theorem 1.2. *[2] Suppose that $x = (x_n)$ is a bounded sequence, $\delta_A(y)$ exists for every $y \in \mathbb{R}$ and $\sum_{y\in D} \delta_A(y) = 1$. Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in D} \delta_A(y) \cdot y.$$

On the other hand recently the notion of A density was further generalized to the notion of $\mathcal{I}A$ density in [21, 22] using a nontrivial proper admissible ideal $\mathcal{I}$ of $\mathbb{N}$. Continuing the investigation from [2], in this note we define for $y \in \mathbb{R}$ and a sequence $x = (x_n) \in \ell^\infty$ the more general notion of $\mathcal{I}A$-density of indices of those x_n's which are close to y, denoted by $\mathcal{I}\delta_A(y)$ where A is a non-negative regular matrix and establish a more general version of Theorem 2.

2. Basic Definitions and Results

We first present the necessary definitions and notations which will form the background of this article. We will also establish some important results which will be used later to prove the main results of the paper.

If $x = (x_n)$ is a sequence and $A = (a_{n,k})$ is a summability matrix, then by Ax we denote the sequence $((Ax)_1, (Ax)_2, (Ax)_3, \dots)$ where $(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k$. The matrix A is called regular if $\lim_{n\to\infty} x_n = L$ implies $\lim_{n\to\infty} (Ax)_n = L$. The well-known Silverman-Töeplitz theorem characterizes regular matrices in the following way. A matrix A is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$,

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$,

(iii) $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{n,k}| < \infty$.

For a non-negative regular matrix A and $E \subset \mathbb{N}$, following Freedman and Sember [11], the A-density of E, denoted by $\delta_A(E)$, is defined as follows

$$\overline{\delta_A}(E) = \limsup_{n\to\infty} \sum_{k\in E} a_{n,k} = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \mathbb{1}_E(k) = \limsup_{n\to\infty} (A\mathbb{1}_E)_n,$$

$$\underline{\delta_A}(E) = \liminf_{n\to\infty} \sum_{k\in E} a_{n,k} = \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\mathbb{1}_E(k) = \liminf_{n\to\infty} (A\mathbb{1}_E)_n$$

where $\mathbb{1}_E$ is a 0-1 sequence such that $\mathbb{1}_E(k) = 1 \iff k \in E$. If $\overline{\delta_A}(E) = \underline{\delta_A}(E)$ then we say that the A-density of E exists and it is denoted by $\delta_A(E)$. Clearly, if A is the Cesàro matrix i.e.

$$a_{n,k} = \begin{cases} \frac{1}{n} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

then δ_A coincides with the natural density.

Throughout by ℓ^∞ we denote the set of all bounded sequences of reals.

We have already stated The original Osikiewicz Theorem in the introduction, namely Theorem 1.

In [2] another version was proved which has also been stated, namely Theorem 2, which was based on a new approach where the authors had defined for a sequence (x_n) a density $\delta_A(y)$ of indices of those x_n which are close to y which was not dealt with till then in the literature. This was a more general approach than that of Osikiewicz.

Fix $(x_n) \in \ell^\infty$. For $y \in \mathbb{R}$ let

$$\overline{\delta_A}(y) = \lim_{\varepsilon\to 0^+} \overline{\delta_A}(\{n : |x_n - y| \leq \varepsilon\})$$

and

$$\underline{\delta_A}(y) = \lim_{\varepsilon\to 0^+} \underline{\delta_A}(\{n : |x_n - y| \leq \varepsilon\}).$$

If $\overline{\delta_A}(y) = \underline{\delta_A}(y)$, then the common value is denoted by $\delta_A(y)$.

Now recall that a non-empty family $\mathcal{I}$ of subsets of $\mathbb{N}$ is an ideal in $\mathbb{N}$ if for $A, B \subset \mathbb{N}$, (i)$A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$; (ii)$A \in \mathcal{I}, B \subset A \Rightarrow B \in \mathcal{I}$. Further if $\bigcup_{A\in\mathcal{I}} A = \mathbb{N}$ i.e. $\{n\} \in \mathcal{I}\ \forall\ n \in \mathbb{N}$, then $\mathcal{I}$ is called admissible or free. A non-empty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is a filter if (i) $\phi \notin \mathcal{F}$; (ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$; (iii)$A \in \mathcal{F}, B \supset A \Rightarrow B \in \mathcal{F}$. We can construct the filter $\mathcal{F}(\mathcal{I})$ associated with an ideal $\mathcal{I}$ as follows: $\mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus B : B \in \mathcal{I}\}$. Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

An ideal $\mathcal{I}$ is said to be a P-ideal if for any sequence of sets (D_n) from $\mathcal{I}$, there is another sequence of sets (C_n) in $\mathcal{I}$ such that $D_n \triangle C_n$ is finite for all n and $\bigcup_n C_n \in \mathcal{I}$. Equivalently $\mathcal{I}$ is a P-ideal if and only if for each sequence (A_n) of sets from $\mathcal{I}$ there exists $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$.

We now recall the following definitions.

(i) (x_n) is $\mathcal{I}$-convergent to y if for any $\varepsilon > 0$, $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$ [16].
(ii) A point y is called an $\mathcal{I}$-cluster point of (x_n) if $\{n : |x_n - y| \leq \varepsilon\} \notin \mathcal{I}$ for any $\varepsilon > 0$ [16].
(iii) y is called an $\mathcal{I}$-limit point of (x_n) if there is a set $B \subset \mathbb{N}$, $B \notin \mathcal{I}$, such that $\lim_{n\in B} x_n = y$ [16].

Recall that an $\mathcal{I}$-limit point is an $\mathcal{I}$-cluster point of a sequence which is again a general limit point but the converses are not generally true.
(iv) We define the $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of a sequence (x_n) as follows (see [5, 17]):

Let $B_x = \{\beta \in \mathbb{R} : \{k \in \mathbb{N} : x_k > \beta\} \notin \mathcal{I}\}$ and $C_x = \{\alpha \in \mathbb{R} : \{k \in \mathbb{N} : x_k < \alpha\} \notin \mathcal{I}\}$. Then

$$\mathcal{I} - \limsup x_n = \begin{cases} \sup B_x & \text{if } B_x \neq \phi \\ -\infty & \text{if } B_x = \phi \end{cases}$$

Similarly

$$\mathcal{I} - \liminf x_n = \begin{cases} \inf C_x & \text{if } C_x \neq \phi \\ \infty & \text{if } C_x = \phi \end{cases}$$

For a set $E \subset \mathbb{N}$ we define the $\mathcal{I}A$ upper density of E by

$$\mathcal{I}\overline{\delta_A}(E) = \mathcal{I} - \limsup_{n\to\infty} \sum_{k\in E} a_{n,k}.$$

Similarly the $\mathcal{I}$-A lower density is defined (see [21, 22]). Then it is easy to show that the family

$$\mathcal{J}_{\mathcal{I},A} = \{E \subset \mathbb{N} : \mathcal{I}\overline{\delta_A}(E) = 0\}$$

forms a proper admissible ideal of $\mathbb{N}$.

Definition 2.1. *[22] A sequence (x_n) of real numbers is said to converge $\mathcal{I}A-$statistically to x if for any given $\epsilon > 0$, $\mathcal{I}\overline{\delta_A}(E_\epsilon) = 0$ where*

$$E_\epsilon = \{n \in \mathbb{N} : |x_n - x| \geq \epsilon\}.$$

The first thing we do in this note is to introduce the following two notions in line of (iv) above (which can be called $\mathcal{I}A$-limit superior and limit inferior for convenience):

Let $B_x = \{\beta \in \mathbb{R} : \{k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{k,j}x_j > \beta\} \notin \mathcal{I}\}$ and $C_x = \{\alpha \in \mathbb{R} : \{k \in \mathbb{N} : \sum_{j=1}^{\infty} a_{k,j}x_j < \alpha\} \notin \mathcal{I}\}$. Then

$$\mathcal{I}A - \limsup x_n = \begin{cases} \sup B_x & \text{if } B_x \neq \phi \\ -\infty & \text{if } B_x = \phi \end{cases}$$

Similarly

$$\mathcal{I}A - \liminf x_n = \begin{cases} \inf C_x & \text{if } C_x \neq \phi \\ \infty & \text{if } C_x = \phi \end{cases}$$

Remark 2.2. *There is no connection between the notion of $\mathcal{I}A-$statistical convergence considered in [21, 22] and the notions of $\mathcal{I}A-$ limit superior and inferior because of which we do away with the term "statistical" in the above definitions.*

Below we obtain a characterisation for $\mathcal{I}A$-limit superior and limit inferior.

Lemma 2.3. *For a regular non-negative matrix $A = (a_{k,j})$ and a sequence (x_n) of real numbers, $\mathcal{I}A - \limsup(x)_n = \beta$(finite) if and only if for arbitrary $\varepsilon > 0$*

$$\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > \beta - \varepsilon\} \notin \mathcal{I} \text{ and } \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > \beta + \varepsilon\} \in \mathcal{I}.$$

Similarly $\mathcal{I}A - \liminf(x)_n = \alpha$(finite) if and only if for arbitrary $\varepsilon > 0$

$$\{k : \sum_{j=1}^{\infty} a_{k,j}x_j < \alpha + \varepsilon\} \notin \mathcal{I} \text{ and } \{k : \sum_{j=1}^{\infty} a_{k,j}x_j < \alpha - \varepsilon\} \in \mathcal{I}.$$

Proof. Let $\mathcal{I} - \limsup(Ax)_n = \beta$. Then $\beta = \sup\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$. So $\beta + \varepsilon \notin \{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$. So clearly $\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > \beta + \varepsilon\} \in \mathcal{I}$. Again, as $\beta = \sup\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$, so for $\varepsilon > 0$ $\exists$ $y_0 \in \{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$ such that $\beta - \varepsilon < y_0 \leq \beta$. That means that there is a $y_0 \in \mathbb{R}$ with $\beta - \varepsilon < y_0 \leq \beta$ such that $\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y_0\} \notin \mathcal{I}$. Subsequently it follows that $\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > \beta - \varepsilon\} \supseteq \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y_0\} \notin \mathcal{I}$.

Conversely suppose that the stated conditions hold. Choose $y_0 \in \{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$. Now if $y_0 > \beta$, then from the given condition it follows that $\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y_0\} \in \mathcal{I}\}$ which is not true. So $y_0 \leq \beta$ which consequently implies that β is an upper bound to the set $\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$. Again let y_1 be any upper bound of the set $\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$. If $\beta > y_1$ then we can choose a $\eta > 0$ such that $\beta > y_1 + \eta > y_1$. Now the given condition implies that $\{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y_1 + \eta\} \notin \mathcal{I}$. Consequently $y_1 + \eta \in \{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$, which shows that y_1 can not be an upper bound of $\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$ which is a contradiction. Therefore $\beta = \sup\{y \in \mathbb{R} : \{k : \sum_{j=1}^{\infty} a_{k,j}x_j > y\} \notin \mathcal{I}\}$.

The proof for $\mathcal{I}A - \liminf$ is similar and so is omitted. $\square$

3. Main Results

In this section we introduce the main notion of this paper and establish some of its interesting consequences including the general version of Osikiewicz Theorem. It is important to note that all these results can be proved without any additional assumption on the ideal.

We first define the main concepts of $\mathcal{I}A$-densities at a point where the upper $\mathcal{I}A$-density is defined by

$$\mathcal{I}\overline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \mathcal{I}\overline{\delta_A}\{n : |x_n - y| \leq \varepsilon\}$$

and the lower $\mathcal{I}A$-density is defined by

$$\mathcal{I}\underline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \mathcal{I}\underline{\delta_A}\{n : |x_n - y| \leq \varepsilon\}.$$

If $\mathcal{I}\overline{\delta_A}(y) = \mathcal{I}\underline{\delta_A}(y)$, then the common value is denoted by $\mathcal{I}\delta_A(y)$.

We start with the following observation.

Lemma 3.1. *Suppose that $\mathcal{I}\delta_A(y)$ exists for any $y \in \mathbb{R}$. Then the set $D = \{y \in \mathbb{R} : \mathcal{I}\delta_A(y) > 0\}$ is countable and* $\sum_{y \in D} \mathcal{I}\delta_A(y) \leq 1$.

Proof. Let (r_n) be a strictly monotonically decreasing sequence converging to 1. For fixed $m \in \mathbb{N}$ let $D_m = \{y \in \mathbb{R} : \mathcal{I}\delta_A(y) \geq \frac{1}{m}\}$. Let $y_1, \ldots, y_l \in D_m$ be distinct. Then for $\varepsilon = \min_{i \neq j} \dfrac{|y_i - y_j|}{3} > 0$ the sets $E_i = \{n : |x_n - y_i| \leq \varepsilon\}$ are pairwise disjoint and $\mathcal{I}\underline{\delta_A}(E_i) \geq \frac{1}{m}$. Since A is also regular so we can choose a n_0 such that

$$\sum_{k=1}^{\infty} a_{n,k} \leq r_p$$

for $n \geq n_0$ and for all $i = 1, \ldots, l$ where p is fixed. Again for a fixed $\tau > 0$ (such that $m\tau < 1$)

$$\{n : \sum_{k \in E_i} a_{n,k} < \frac{1}{m} - \tau\} \in \mathcal{I}.$$

So

$$\bigcup_{j=1}^{l} \{n : \sum_{k \in E_j} a_{n,k} < \frac{1}{m} - \tau\} \in \mathcal{I}.$$

As E_i's are pairwise disjoint we get

$$\{n : \sum_{k \in E_1 \cup \ldots \cup E_l} a_{n,k} < \frac{l}{m} - l\tau\} = \{n : \sum_{j=1}^{l} \sum_{k \in E_j} a_{n,k} < \frac{l}{m} - l\tau\} \subset \bigcup_{j=1}^{l} \{n : \sum_{k \in E_j} a_{n,k} < \frac{1}{m} - \tau\} \in \mathcal{I}.$$

Note that as $\mathcal{I}$ is proper and free hence $\{n : \sum\limits_{k \in E_1 \cup \ldots \cup E_l} a_{n,k} \geq \frac{l}{m} - l\tau\} \in \mathcal{F}(\mathcal{I})$ and so $\{n : \sum\limits_{k \in E_1 \cup \ldots \cup E_l} a_{n,k} \geq \frac{l}{m} - l\tau\} \cap \{n_0 + 1, n_0 + 2, \ldots\} \in \mathcal{F}(\mathcal{I})$. Consequently we can find a $n_1 > n_0$ such that

$$\sum_{k \in \bigcup\limits_{j=1}^{l} E_j} a_{n_1,k} \geq \frac{l}{m} - l\tau$$

and simultaneously

$$\sum_{k=1}^{\infty} a_{n_1,k} \leq r_p.$$

Hence we must have $l \leq \frac{mr_p}{1-m\tau}$ which shows that D_m must be finite. Clearly then $D = \bigcup\limits_{m} D_m$ is countable. Again for arbitrary $\varepsilon_0 > 0$ we get $B \in \mathcal{F}(\mathcal{I})$ and for $n \in B$

$$\sum_{y \in D_m} \mathcal{I}\delta_A(y) = \sum_{j=1}^{l} \mathcal{I}\delta_A(y_j) = \sum_{j=1}^{l} [\lim_{\varepsilon \to 0^+} \mathcal{I}\underline{\delta_A}(E_j)]$$

$$\leq \sum_{j=1}^{l} \mathcal{I}\underline{\delta_A}(E_j) = \sum_{j=1}^{l} [\mathcal{I} - \liminf_{n} \sum_{k \in E_j} a_{n,k}] \leq \sum_{j=1}^{l} [\sum_{k \in E_j} a_{n,k} + \frac{\varepsilon_0}{l}] = \sum_{k \in \bigcup\limits_{j=1}^{l} E_j} a_{n,k} + \varepsilon_0 \leq r_p + \varepsilon_0$$

Letting $\varepsilon_0 \to 0$ we get $\sum\limits_{y \in D_m} \mathcal{I}\delta_A(y) \leq r_p$. So

$$\sum_{y \in D} \mathcal{I}\delta_A(y) = \lim_{m \to \infty} \sum_{y \in D_m} \mathcal{I}\delta_A(y) \leq r_p.$$

Finally letting $p \to \infty$ we have $\sum\limits_{y \in D} \mathcal{I}\delta_A(y) \leq 1$. $\square$

Note that in general, one cannot prove that $D = \{y \in \mathbb{R} : \mathcal{I}\delta_A(y) > 0\}$ is nonempty. Also the above lemma would not remain true if one would change $\mathcal{I}\delta_A(y)$ to $\mathcal{I}\overline{\delta_A}(y)$, that is $\overline{D} := \{y \in \mathbb{R} : \mathcal{I}\overline{\delta_A}(y) > 0\}$ need not be countable. An example in this respect is given in [2] for $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$.

The next result extends Theorem 6 [2] which simultaneously presents a more general version of a slight improvement of The Osikiewicz Theorem. The method which we use in our proof is similar to that of Osikiewicz, but not analogous as we use essentially new arguments in line of [2] with necessary nontrivial modifications which arise due to presence of ideals.

Theorem 3.2. *Suppose that $x = (x_n)$ is a bounded sequence, $\mathcal{I}\delta_A(y)$ exists for every $y \in \mathbb{R}$ and $\sum_{y\in D} \mathcal{I}\delta_A(y) = 1$. Then*

$$\mathcal{I} - \lim_{n\to\infty}(Ax)_n = \sum_{y\in D} \mathcal{I}\delta_A(y) \cdot y$$

provided both finitely exist.

Proof. Since (x_n) is bounded, there is $M > 0$ such that $|x_n| \le M$ for every $n \in \mathbb{N}$. Let $D = \{y_i\}_i$ where y_i's are distinct. Let $\varepsilon > 0$ be given and let $r \in \mathbb{N}$ be such that $\sum_{i=1}^{r} \mathcal{I}\delta_A(y_i) > 1 - \varepsilon$ and $|\sum_{i=r+1}^{\infty} \mathcal{I}\delta_A(y_i) \cdot y_i| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $1/N < \min\{\min_{1\le i\ne j\le r} |y_i - y_j|, \varepsilon/r\}$ and such that the set $E_i := \{j : |x_j - y_i| < 1/N\}$ have the following property

$$\mathcal{I}\delta_A(y_i) - \frac{\varepsilon}{r(M+1)} \le \mathcal{I}\underline{\delta_A}(E_i) \le \mathcal{I}\overline{\delta_A}(E_i) \le \mathcal{I}\delta_A(y_i) + \frac{\varepsilon}{r(M+1)}$$

for $i = 1, \ldots, r$. Observe that $E_1, \ldots, E_r$ are pairwise disjoint. Now let $B_i \in \mathcal{F}(\mathcal{I})$ be such that

$$\mathcal{I}\underline{\delta_A}(E_i) - \frac{1}{N} < \sum_{k\in E_i} a_{n,k} < \mathcal{I}\overline{\delta_A}(E_i) + \frac{1}{N}$$

for every $n \in B_i$ for all $i = 1, \ldots, r$. Let $B = \bigcap_{i=1}^{r} B_i \in \mathcal{F}(\mathcal{I})$. Therefore for all $n \in B$ and $i = 1, \ldots, r$

$$\mathcal{I}\delta_A(y_i) - \frac{1}{N} - \frac{\varepsilon}{r(M+1)} < \sum_{k\in E_i} a_{n,k} < \mathcal{I}\delta_A(y_i) + \frac{1}{N} + \frac{\varepsilon}{r(M+1)}$$

and consequently

$$|\sum_{k\in E_i} a_{n,k} - \mathcal{I}\delta_A(y_i)| < \frac{1}{N} + \frac{\varepsilon}{r(M+1)}. \tag{1}$$

Then for these n we have

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k}x_k \le \sum_{k\in E_1} a_{n,k} \cdot \left(y_1 + \frac{1}{N}\right) + \cdots + \sum_{k\in E_r} a_{n,k} \cdot \left(y_r + \frac{1}{N}\right) + \sum_{k\in(E_1\cup\cdots\cup E_r)^c} a_{n,k} \cdot M.$$

Since A is regular, we can choose a $m_1 \in \mathbb{N}$ such that for all $n \ge m_1$

$$\sum_{k=1}^{\infty} a_{n,k} < 1 + \varepsilon.$$

Now observe that

$$1 + \varepsilon > \sum_{k=1}^{\infty} a_{n,k} = \sum_{k\in E_1\cup\cdots\cup E_r} a_{n,k} + \sum_{k\in(E_1\cup\cdots\cup E_r)^c} a_{n,k}$$

where from above we have

$$\sum_{k\in(E_1\cup\cdots\cup E_r)} a_{n,k} = \sum_{j=1}^{r}\sum_{k\in E_j} a_{n,k} > \sum_{j=1}^{r} \mathcal{I}\delta_A(y_j) - \frac{r}{N} - \frac{\varepsilon}{M+1} > 1 - \frac{r}{N} - (1 + \frac{1}{M+1}) \cdot \varepsilon.$$

Now let $B_1 = B \cap \{m_1, m_1 + 1, m_1 + 2, ...\}$. Then $B_1 \in \mathcal{F}(\mathcal{I})$. Therefore for $n \in B_1$ we have

$$\sum_{k\in(E_1\cup\cdots\cup E_r)^c} a_{n,k} \le (1 + \varepsilon) - (1 - \frac{r}{N} - (1 + \frac{1}{M+1})\varepsilon) = \frac{r}{N} + (2 + \frac{1}{M+1})\varepsilon.$$

Consequently we get for $n \in B_1$,

$$(Ax)_n \leq \sum_{k\in E_1} a_{n,k} \cdot \Big(y_1 + \frac{1}{N}\Big) + \cdots + \sum_{k\in E_r} a_{n,k} \cdot \Big(y_r + \frac{1}{N}\Big) + \frac{Mr}{N} + (2 + \frac{1}{M+1})M\varepsilon$$

and analogously

$$(Ax)_n \geq \sum_{k\in E_1} a_{n,k} \cdot \Big(y_1 - \frac{1}{N}\Big) + \cdots + \sum_{k\in E_r} a_{n,k} \cdot \Big(y_r - \frac{1}{N}\Big) - \frac{Mr}{N} - (2 + \frac{1}{M+1})M\varepsilon.$$

Thus

$$(Ax)_n - \sum_{i=1}^{r} \sum_{k\in E_i} a_{n,k} \cdot \Big(y_i + \frac{1}{N}\Big) \leq \frac{Mr}{N} + (2 + \frac{1}{M+1})M\varepsilon \tag{2}$$

$$(Ax)_n - \sum_{i=1}^{r} \sum_{k\in E_i} a_{n,k} \cdot \Big(y_i - \frac{1}{N}\Big) \geq -\frac{Mr}{N} - (2 + \frac{1}{M+1})M\varepsilon \tag{3}$$

Hence using (1) and (2), for $n \in B_1$ we get

$$\begin{aligned}
(Ax)_n - \sum_{i} \mathcal{I}\delta_A(y_i) \cdot y_i &= (Ax)_n - \sum_{i}^{r} \mathcal{I}\delta_A(y_i) \cdot y_i - \sum_{r+1}^{\infty} \mathcal{I}\delta_A(y_i) \cdot y_i \\
&\leq (Ax)_n - \sum_{i}^{r} \mathcal{I}\delta_A(y_i) \cdot y_i + \left|\sum_{r+1}^{\infty} \mathcal{I}\delta_A(y_i) \cdot y_i\right| \leq (Ax)_n - \sum_{i=1}^{r} \mathcal{I}\delta_A(y_i) \cdot y_i + \varepsilon \\
&\leq \Big((Ax)_n - \sum_{k\in E_i} a_{n,k} \sum_{i=1}^{r} (y_i + \frac{1}{N})\Big) + \sum_{i=1}^{r} \Big(\sum_{k\in E_i} a_{n,k} \cdot (y_i + \frac{1}{N}) - \mathcal{I}\delta_A(y_i) \cdot y_i\Big) + \varepsilon \\
&\leq \sum_{i=1}^{r} \left(\Big(\sum_{k\in E_i} a_{n,k} - \mathcal{I}\delta_A(y_i)\Big) \cdot \Big(y_i + \frac{1}{N}\Big)\right) + \sum_{i=1}^{r} \mathcal{I}\delta_A(y_i) + \frac{Mr}{N} + (2M + \frac{M}{M+1} + 1)\varepsilon \\
&\leq \sum_{i=1}^{r} \left(\Big(\sum_{k\in E_i} a_{n,k} - \mathcal{I}\delta_A(y_i)\Big) \cdot \Big(|y_i| + \frac{1}{N}\Big)\right) + \frac{r}{N} + \frac{Mr}{N} + (2M + \frac{M}{M+1} + 1)\varepsilon \\
&\leq r \cdot \Big(\frac{1}{N} + \frac{\varepsilon}{r(M+1)}\Big) \cdot (M + 1 + \frac{1}{N}) + \frac{r}{N} + \frac{Mr}{N} + (2M + \frac{M}{M+1} + 1)\varepsilon \\
&\leq \varepsilon.(M + 2 + M + 1 + 2M + \frac{M}{M+1} + 1) + \varepsilon^2.(\frac{M+2}{M+1}) \\
&\leq (4M + 6) \cdot \varepsilon \quad \text{[without any loss of generality taking } \varepsilon < 1]
\end{aligned}$$

Analogously from (1) and (3) for $n \in B_1$ we get

$$\begin{aligned}
(Ax)_n - \sum_{i} \mathcal{I}\delta_A(y_i) \cdot y_i &\geq -r \cdot \Big(\frac{1}{N} + \frac{\varepsilon}{r(M+1)}\Big) \cdot (M + 1 + \frac{1}{N}) - \frac{r}{N} - \frac{Mr}{N} - (2M + \frac{M}{M+1} + 1)\varepsilon \\
&\leq (4M + 6) \cdot \varepsilon
\end{aligned}$$

So we obtain for any $\varepsilon > 0 \ \exists \, B_1 \in \mathcal{F}(\mathcal{I})$ such that

$$\left|(Ax)_n - \sum_{i} \mathcal{I}\delta_A(y_i) \cdot y_i\right| \leq (4M + 6) \cdot \varepsilon$$

for all $n \in B_1$. Therefore

$$\Big\{n : |(Ax)_n - \sum_i \mathcal{I}\delta_A(y_i) \cdot y_i| > (4M+6) \cdot \varepsilon\Big\} \in \mathcal{I}.$$

Hence $\mathcal{I} - \lim_n (Ax)_n = \sum_i \mathcal{I}\delta_A(y_i) \cdot y_i$. $\square$

In Proposition 8 [2] it was observed that for a bounded sequence (x_n) and for $y \in \mathbb{R}$, $\overline{\delta_A}(y) = 1$ implies that y is a limit point of the sequence $((Ax)_n)$. Now a natural question arises what can we conclude if for $y \in \mathbb{R}$, $\mathcal{I}\overline{\delta_A}(y) = 1$. The following example shows that the condition $\mathcal{I}\overline{\delta_A}(y) = 1$ is not sufficient for y to be an $\mathcal{I}$-limit point of $((Ax)_n)$.

Example 3.3. Let $\{P_k\}$ be a partition of $\mathbb{N}$ into infinite sets. Let $\mathcal{I}$ be an ideal defined by

$$B \in \mathcal{I} \Leftrightarrow B \cap P_k \quad \text{is finite for all but finitely many} \quad k.$$

We define a bounded sequence by

$$x_n = \frac{1}{k} \Leftrightarrow n \in P_k.$$

Note that for every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$B_\varepsilon = \{n \in \mathbb{N} : |x_n - 0| \leq \varepsilon\} = \bigcup_{k > k_0} P_k.$$

Thus, $B_\varepsilon \in \mathcal{F}(\mathcal{I})$ and consequently $\mathcal{I} - \limsup_n \chi_{B_\varepsilon}(n) = 1$. Again for A be the identity matrix

$$\mathcal{I}\overline{\delta_A}(0) = \lim_{\varepsilon \to 0} \mathcal{I}\overline{\delta_A}(B_\varepsilon) = \lim_{\varepsilon \to 0}(\mathcal{I} - \limsup_n \sum_{k \in B_\varepsilon} a_{n,k}) = \lim_{\varepsilon \to 0}(\mathcal{I} - \limsup_n \chi_{B_\varepsilon}(n)) = \lim_{\varepsilon \to 0} 1 = 1$$

Now we show that 0 is not an $\mathcal{I}$-limit point of $(Ax)_n$. Suppose to the contrary that 0 is an $\mathcal{I}$-limit point of $(Ax)_n$. Then there is $B \notin \mathcal{I}$ such that $\lim_{n \in B} (Ax)_n = 0$. Since $B \notin \mathcal{I}$, there is infinitely many k such that $B \cap P_k$ is infinite. Take any of them, say k_0 is such that $B \cap P_{k_0}$ is infinite. Then for every $n \in B \cap P_{k_0}$, $(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k = a_{n,n} x_n = 1.\frac{1}{k_0} = \frac{1}{k_0}$. Thus the sequence $(Ax)_{n \in B}$ contains infinitely many values $\frac{1}{k_0}$, hence it cannot be convergent to 0, a contradiction.

However we can derive the following conclusion which is interesting.

Proposition 3.4. *Assume that $x = (x_n)$ is bounded. If $\mathcal{I}\overline{\delta_A}(y) = 1$, then y is an $\mathcal{I}$-cluster point of the sequence $\big((Ax)_n\big)$.*

Proof. Since (x_n) is bounded, there is $M > 0$ such that $|x_n| \leq M$ for every $n \in \mathbb{N}$. Let $y \in \mathbb{R}$ be such that $\mathcal{I}\overline{\delta_A}(y) = 1$. Let $N \in \mathbb{N}$. Let $E_N = \{j \in \mathbb{N} : |x_j - y| < 1/N\}$. Note that $\mathcal{I}\overline{\delta_A}(E_N) = 1$. Then $B_N = \{k : \sum_{j \in E_N} a_{k,j} > \mathcal{I}\overline{\delta}_A(E_N) - \frac{1}{N}\} \notin \mathcal{I}$. Again from regularity of A we can find a $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^{\infty} a_{k,j} \leq 1 + \frac{1}{N}$ holds for $k \geq k_0$. Let $A_N = B_N \cap \{k_0+1, k_0+2, ...\}$. Then $A_N \notin \mathcal{I}$ and clearly for $k \in A_N$

$$\sum_{j=1}^{\infty} a_{k,j} < 1 + \frac{1}{N}.$$

alongwith

$$\sum_{j\in E_N} a_{k,j} > 1 - \frac{1}{N}.$$

Let $\varepsilon > 0$ be arbitrarily chosen. Choose $N_0 \in \mathbb{N}$ such that $N_0 > 1$ and $\frac{|y|+2+2M}{N_0} < \varepsilon$. Also for $N \in \mathbb{N}$, we note that $y - \frac{1}{N} < x_k < y + \frac{1}{N}$ for $k \in E_N$ and $-M \leq x_k \leq M$ for $k \notin E_N$. So for $n \in A_{N_0}$, we have

$$\sum_{k\in E_{N_0}} a_{n,k} \cdot \Big(y - \frac{1}{N_0}\Big) - \sum_{k\notin E_{N_0}} a_{n,k} \cdot M$$

$$\leq \sum_{k=1}^{\infty} a_{n,k} \cdot x_k = \sum_{k\in E_{N_0}} a_{n,k} \cdot x_k + \sum_{k\notin E_{N_0}} a_{n,k} \cdot x_k$$

$$\leq \sum_{k\in E_{N_0}} a_{n,k} \cdot \Big(y + \frac{1}{N_0}\Big) + \sum_{k\notin E_{N_0}} a_{n,k} \cdot M.$$

Observe that for $n \in A_{N_0}$

$$\sum_{k\notin E_{N_0}} a_{n,k} = \sum_{k=1}^{\infty} a_{n,k} - \sum_{k\in E_{N_0}} a_{n,k} < 1 + \frac{1}{N_0} - (1 - \frac{1}{N_0}) = \frac{2}{N_0}.$$

Therefore we get

$$-\frac{2M+2+|y|}{N_0} \leq \Big(1 - \frac{1}{N_0}\Big) \cdot \Big(y - \frac{1}{N_0}\Big) - \frac{2}{N_0} \cdot M - y$$

$$\leq \sum_{k=1}^{\infty} a_{n,k} \cdot x_k - y$$

$$\leq \Big(1 + \frac{1}{N_0}\Big) \cdot \Big(y + \frac{1}{N_0}\Big) + \frac{2}{N_0} \cdot M - y \leq \frac{2M+2+|y|}{N_0}.$$

Hence for $n \in A_{N_0}$

$$|\sum_{k=1}^{\infty} a_{n,k} \cdot x_k - y| \leq \frac{2M+2+|y|}{N_0} < \varepsilon.$$

This shows that $\{n : |(Ax)_n - y| < \varepsilon\} \supset A_{N_0} \notin \mathcal{I}$. Since this is true for any $\varepsilon > 0$, it follows that y is an $\mathcal{I}$-cluster point of $\big((Ax)_n\big)$. $\square$

References

[1] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (1) (2007), 715–729.

[2] A. Bartoszewicz, P. Das, S. Glab, On matrix summability of spliced sequences and A-density of points, Linear Algebra Appl., 487 (2015), 22 - 42.

[3] A. Bartoszewicz, S. Głąb, A. Wachowicz, Remarks on ideal boundedness, convergence and variation of sequences. J. Math. Anal. Appl. 375 (2011), no. 2, 431–435.

[4] P. Das, Some further results on ideal convergence in topological spaces, Topology Appl., 159 (2012), 2621–2625.

[5] K. Demirci, $\mathcal{I}$-limit superior and limit inferior, Math. Commun., 6 (2001), 165 - 172.

[6] A. Faisant, G. Grekos, V. Toma, On the statistical variation of sequences, J. Math. Anal. Appl., 306 (2) (2005) 432–439.

[7] H. Fast, Sur la convergence statistique. Colloq. Math. 2 (1951), 241–244 (1952).

[8] R. Filipów, N. Mrożek, I. Recław, P. Szuca, Ideal convergence of bounded sequences, J. Symbolic Logic, 72 (2) (2007), 501?-512.

[9] R. Filipów, P. Szuca, On some questions of Drewnowski and Łuczak concerning submeasures on $\mathbb{N}$, J. Math. Anal. Appl. 371 (2) (2010), 655–660.

[10] R. Filipów, P. Szuca, Density versions of Schur's theorem for ideals generated by submeasures, J. Combin. Theory Ser. A, 117 (7) (2010), 943–956.
[11] A.R. Freedman and J.J. Sember, Densities and summability, Pacific J. Math. 95 (1981), 293–305.
[12] J. A. Fridy, On statistical convergence. Analysis 5 (1985), no. 4, 301–313.
[13] J. A. Fridy, Statistical limit points. Proc. Amer. Math. Soc. 118 (1993), no. 4, 1187–1192.
[14] R. Henstock, The efficiency of matrices for bounded sequences, J. London Math. Soc., 25 (1950), 27–33.
[15] J. Jasiński, I. Recław, On spaces with the ideal convergence property, Colloq. Math., 111 (1) (2008), 43–50
[16] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-convergence, Real Anal. Exchange, 26 (2) (2000/2001), 669-685.
[17] B.K. Lahiri, P. Das, Further results on I-limit superior and limit inferior, Math. Commun., 8 (2)(2003), 151 - 156.
[18] B. K. Lahiri and P. Das, $\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces, Math. Bohemica, 130 (2005), 153-160.
[19] N. Mrożek, Ideal version of Egorov's theorem for analytic P-ideals, J. Math. Anal. Appl., 349 (2) (2009), 452–458.
[20] J. A. Osikiewicz, Summability of spliced sequences. Rocky Mountain J. Math. 35 (2005), no. 3, 977–996.
[21] E. Savas, P. Das, S. Dutta, A note on strong matrix summability via ideals, Appl. Math. Lett. 25 (4) (2012), 733 - 738.
[22] E. Savas, P. Das, S. Dutta, A note on some generalized summability methods, Acta Math. Univ. Comenianae, Vol. LXXXII (2) (2013), 297 - 304.
[23] I. J. Schoenberg, The integrability of certain functions and related summability methods. Amer. Math. Monthly 66 (1959) 361–375.
[24] S. Solecki, Analytic ideals and their applications. Ann. Pure Appl. Logic **99** (1?3) (1999) 51–72.