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Integral Inequalities for Differentiable p-Harmonic Convex Functions

Muhammad Aslam Noor[a], Khalida Inayat Noor[a], Sabah Iftikhar[a]

[a]*Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*

Abstract. In this paper, we consider a new class of harmonic convex functions, which is called p-harmonic convex function. Several new Hermite-Hadamard, midpoint, Trapezoidal and Simpson type inequalities for functions whose derivatives in absolute value are p-harmonic convex are obtained. Some special cases are discussed. The ideas and techniques of this paper may stimulate further research.

1. Introduction

It is well known fact that convexity theory has played an important and significant role in the development of several branches of pure and applied sciences. Recall that the minimum of a differentiable convex function can be characterized by variational inequalities, which unable to study a wide class of unrelated problems in a unified and general framework. For the formulation, applications, numerical methods and other aspects of variational inequalities, see the references. It has been shown [17] that a function f is a convex function, if and only if, the function f satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a,b],$$

which is called the Hermite-Hadamard inequality, see [9, 10]. For the applications of the Hermite-Hadamard inequalities, see [1, 5–7, 14, 15, 17–19, 22, 24, 26] and the references therein.

Convex functions and convex sets are generalized in various directions using quite different techniques and ideas. An important class of convex functions is called the harmonic convex functions, which was introduced and studied by Anderson et al. [2] and Iscan [12]. Let a real function f be defined on some nonempty interval $I = [a,b] \subseteq \mathbb{R} \setminus \{0\}$. The function f is said to be harmonic convex on $I = [a,b] \subseteq \mathbb{R} \setminus \{0\}$ if it satisfies the inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a,b]. \tag{1}$$

This double inequality is known as Hermite-Hadamard integral inequality for harmonic convex functions [12]. Both inequalities hold in the reversed direction if f is harmonic concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of harmonic convexity and it follows

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Email addresses: noormaslam@hotmail.com (Muhammad Aslam Noor), khalidanoor@hotmail.com (Khalida Inayat Noor), sabah.iftikhar22@gmail.com (Sabah Iftikhar)

easily from Jensens inequality. For some results which generalize, improve and extend the inequalities, we refer the reader to the recent paper [8] and references therein.

Motivated and inspired by the on going research in this direction, Noor et al. [24] introduced and studied an other class of harmonic functions, which is called p-harmonic convex function. Several new integral inequalities for p-harmonic convex functions have been obtained. For suitable and appropriate choice of the parameter p, one can obtain several new classes of convex functions and their variant forms.

In this paper, we derive several Hermite-Hadamard, midpoint, Trapezoidal and Simpson type inequalities for differentiable p-harmonic convex functions. From our results, one can obtain a wide class of known and new inequalities for harmonic convex functions and their variant forms. Some of the results obtained in this paper can be viewed as significant and important refinement of the previously known results.

We now recall some known results and basic concepts.

Definition 1.1. [26]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx + (1-t)y} \in I, \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 1.2. [12]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a harmonic convex function, if and only if,*

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Definition 1.3. [24]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be p-harmonic convex set, if*

$$\Big[\frac{x^p y^p}{tx^p + (1-t)y^p}\Big]^{\frac{1}{p}} \in I, \quad \forall x, y \in I, t \in [0, 1], \qquad p \neq 0.$$

Definition 1.4. [24]. *Let $[a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be p-harmonic convex function, if*

$$f\Big(\Big[\frac{x^p y^p}{tx^p + (1-t)y^p}\Big]^{\frac{1}{p}}\Big) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0, 1].$$

Note that for $t = \frac{1}{2}$, we have

$$f\Big(\Big[\frac{2x^p y^p}{x^p + y^p}\Big]^{\frac{1}{p}}\Big) \leq \frac{f(x) + f(y)}{2}, \qquad \forall x, y \in I,$$

which is called Jensen type p-harmonic convex function.

Remark 1.5. *Let $[a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. If $[a, b] \subset (0, \infty)$ and $p \in \mathbb{R} \setminus \{0\}$, then Definition 1.4 reduces to definition 2.1 in [3].*

Theorem 1.6. [24]. *Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a p-harmonic convex function on the interval [a,b]. Then*

$$\begin{aligned} f\Big(\Big[\frac{2a^p b^p}{a^p + b^p}\Big]^{\frac{1}{p}}\Big) &\leq \frac{1}{2}\Big[f\Big(\Big[\frac{4a^p b^p}{a^p + 3b^p}\Big]^{\frac{1}{p}}\Big) + f\Big(\Big[\frac{4a^p b^p}{3a^p + b^p}\Big]^{\frac{1}{p}}\Big)\Big] \\ &\leq \frac{a^p b^p}{p(b^p - a^p)} \int_a^b \frac{f(x)}{x^{1+\frac{1}{p}}} \mathrm{d}x \leq \frac{1}{2}\Big[f\Big(\Big[\frac{2a^p b^p}{a^p + b^p}\Big]^{\frac{1}{p}}\Big) + \frac{f(a) + f(b)}{2}\Big] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{2}$$

Remark 1.7. *If $p = 1$, then p-harmonic convex function reduces to harmonic convex function and if $p = -1$, then p-harmonic convex functions become convex functions. For other appropriate and suitable choice of the parameter, one can obtain different classes of convex functions. This shows that the p-harmonic convex functions are more general and include harmonic convex functions and convex functions as special cases.*

2. Main results

We need the following result in order to prove our main results.

Lemma 2.1. *Let* $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I^o *of* I. *If* $f' \in L[a,b]$ *and* $\lambda \in [0,1]$, *then*

$$\begin{aligned}
&(1-\lambda)f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
=\ &\frac{(b^p-a^p)}{2p(a^pb^p)}\Bigg[\int_0^{\frac{1}{2}}(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
+\ &\int_{\frac{1}{2}}^1(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\Bigg].
\end{aligned}$$

Proof. Let

$$\begin{aligned}
I\ =\ &\frac{b^p-a^p}{2p(a^pb^p)}\int_0^{\frac{1}{2}}(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
+\ &\int_{\frac{1}{2}}^1(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\Bigg]\\
=\ &\frac{b^p-a^p}{2p(a^pb^p)}\int_0^{\frac{1}{2}}(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
+\ &\frac{b^p-a^p}{2p(a^pb^p)}\int_{\frac{1}{2}}^1(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
=\ &I_1+I_2
\end{aligned}$$

Now

$$\begin{aligned}
I_1\ =\ &\frac{b^p-a^p}{2p(a^pb^p)}\int_0^{\frac{1}{2}}(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
=\ &\frac{1-\lambda}{2}f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}+\frac{\lambda}{2}f\big(a\big)-\int_0^{\frac{1}{2}}f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
I_2\ =\ &\frac{b^p-a^p}{2p(a^pb^p)}\int_{\frac{1}{2}}^1(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
=\ &\frac{\lambda}{2}f\big(b\big)+\frac{1-\lambda}{2}f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}-\int_{\frac{1}{2}}^1f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t
\end{aligned}$$

Thus

$$I=I_1+I_2=(1-\lambda)f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x$$

which is the required result. □

Theorem 2.2. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set and let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\rightarrow\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $q\geq 1$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
\leq\ &\frac{(b^p-a^p)}{2p(a^pb^p)}\Big[(\kappa_1(p,a,b))^{1-\frac{1}{q}}[\kappa_3(p,a,b)|f'(a)|^q+\kappa_5(p,a,b)|f'(b)|^q]^{\frac{1}{q}}\\
+\ &(\kappa_2(p,b,a))^{1-\frac{1}{q}}[\kappa_6(p,b,a)|f'(a)|^q+\kappa_4(p,b,a)|f'(b)|^q]^{\frac{1}{q}}\Big],
\end{aligned}\tag{3}$$

where

$$\kappa_1(p,a,b)=\int_0^{\frac{1}{2}}\big|2t-\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t,$$

$$\kappa_2(p,b,a)=\int_{\frac{1}{2}}^1\big|2t-2+\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t,$$

$$\kappa_3(p,a,b)=\int_0^{\frac{1}{2}}(1-t)\big|2t-\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t,\tag{4}$$

$$\kappa_4(p,b,a)=\int_{\frac{1}{2}}^1 t\big|2t-2+\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t,\tag{5}$$

$$\kappa_5(p,a,b)=\int_0^{\frac{1}{2}} t\big|2t-\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t,\tag{6}$$

$$\kappa_6(p,b,a)=\int_{\frac{1}{2}}^1(1-t)\big|2t-2+\lambda\big|\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{1+\frac{1}{p}}\mathrm{d}t.\tag{7}$$

Proof. Using Lemma 2.1 and the power mean inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
\leq\ &\frac{b^p-a^p}{2p(a^pb^p)}\Big[\int_0^{\frac{1}{2}}\Big|(2t-\lambda)\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big|\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|\mathrm{d}t\\
+\ &\int_{\frac{1}{2}}^1\Big|(2t-2+\lambda)\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big|\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|\mathrm{d}t\Big]\\
\leq\ &\frac{b^p-a^p}{2p(a^pb^p)}\Big[\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\\
&\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\
+\ &\Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\\
&\Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b^p-a^p}{2p(a^pb^p)}\Big[\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\\
&\quad\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big[(1-t)|f'(a)|^q+t|f'(b)|^q\Big]\mathrm{d}t\Big)^{\frac{1}{q}}\\
&+\Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\\
&\quad\Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}\Big[(1-t)|f'(a)|^q+t|f'(b)|^q\Big]\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\
&= \frac{b^p-a^p}{2p(a^pb^p)}\Big[(\kappa_1(p,a,b))^{1-\frac{1}{q}}[\kappa_3(p,a,b)|f'(a)|^q+\kappa_5(p,a,b)|f'(b)|^q]^{\frac{1}{q}}\\
&+ (\kappa_2(p,b,a))^{1-\frac{1}{q}}[\kappa_6(p,b,a)|f'(a)|^q+\kappa_4(p,b,a)|f'(b)|^q]^{\frac{1}{q}}\Big],
\end{aligned}
$$

which is the required result. □

If $q = 1$, then Theorem 2.2 reduces to the following result.

Corollary 2.3. . *Let $I = [a, b] \subset \mathbb{R}\setminus\{0\}$ be p-harmonic convex set and let $f : I = [a, b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|$ is p-harmonic convex function on I and $\lambda \in [0, 1]$, then*

$$
\begin{aligned}
&\Big|(1-\lambda)f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\Big|\\
&\leq \frac{(b^p-a^p)}{2p}\Big[(\kappa_3(p,a,b)+\kappa_6(p,b,a))|f'(a)|+(\kappa_4(p,b,a)+\kappa_5(p,a,b))|f'(b)|\Big],
\end{aligned}
$$

where $\kappa_3, \kappa_4, \kappa_5, \kappa_6$ are given by (4)-(7).

Theorem 2.4. *Let $I = [a, b] \subset \mathbb{R}\setminus\{0\}$ be p-harmonic convex set. Let $f : I = [a, b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-harmonic convex function on I, $r, q > 1$, $\frac{1}{r}+\frac{1}{q} = 1$ and $\lambda \in [0, 1]$, then*

$$
\begin{aligned}
&\Big|(1-\lambda)f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\Big|\\
&\leq \frac{(b^p-a^p)}{2a^pb^p}\Big[(\kappa_7(r,p;a,b))^{\frac{1}{r}}\Big(\frac{|f'(a)|^q+\big|f'\big(\big[\frac{2a^pb^p}{a^p+b^p}\big]^{\frac{1}{p}}\big)\big|^q}{4}\Big)^{\frac{1}{q}}\\
&+ (\kappa_8(r,p;b,a))^{\frac{1}{r}}\Big(\frac{\big|f'\big(\big[\frac{2a^pb^p}{a^p+b^p}\big]^{\frac{1}{p}}\big)\big|^q+|f'(b)|^q}{4}\Big)^{\frac{1}{q}}\Big], \qquad (8)
\end{aligned}
$$

where

$$
\kappa_7(r,p;a,b) = \int_0^{\frac{1}{2}}|2t-\lambda|^r\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{r+\frac{r}{p}}\mathrm{d}t, \qquad (9)
$$

$$
\kappa_8(r,p;b,a) = \int_{\frac{1}{2}}^1|2t-2+\lambda|^r\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{r+\frac{r}{p}}\mathrm{d}t. \qquad (10)
$$

Proof. Using Lemma 2.1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+\frac{1}{p}}}\mathrm{d}x\right| \\
\leq\;& \frac{b^p-a^p}{2p(a^pb^p)}\left[\int_0^{\frac{1}{2}}\left|(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right.\\
+\;& \left.\int_{\frac{1}{2}}^1\left|(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right]\\
\leq\;& \frac{b^p-a^p}{2p(a^pb^p)}\left[\left(\int_0^{\frac{1}{2}}\left|(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
+\;& \left.\left(\int_{\frac{1}{2}}^1\left|(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
=\;& \frac{b^p-a^p}{2p(a^pb^p)}\left[\left(\int_0^{\frac{1}{2}}\left|(2t-\lambda)\right|^r\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{r+\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{a^pb^p}{p(b^p-a^p)}\int_a^{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+\frac{1}{p}}}\mathrm{d}x\right)^{\frac{1}{q}}\right.\\
+\;& \left.\left(\int_{\frac{1}{2}}^1\left|(2t-2+\lambda)\right|^r\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{r+\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{a^pb^p}{p(b^p-a^p)}\int_{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}^b\frac{|f'(x)|^q}{x^{1+\frac{1}{p}}}\mathrm{d}t\right)^{\frac{1}{q}}\right] \qquad (11)
\end{aligned}$$

Using the p-harmonic convexity of $|f'|^q$, we obtain the following inequalities from inequality (1.6)

$$\frac{2a^pb^p}{p(b^p-a^p)}\int_a^{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+\frac{1}{p}}}\mathrm{d}x\leq\frac{1}{2}\left[|f'(a)|^q+\left|f'\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\right|^q\right], \qquad (12)$$

and

$$\frac{2a^pb^p}{p(b^p-a^p)}\int_{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}^b\frac{|f'(x)|^q}{x^{1+\frac{1}{p}}}\mathrm{d}x\leq\frac{1}{2}\left[\left|f'\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\right|^q+|f'(b)|^q\right]. \qquad (13)$$

A combination of (11)-(13) gives the required inequality (8). □

Theorem 2.5. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set and let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $r,q>1$, $\frac{1}{r}+\frac{1}{q}=1$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
\leq\;& \frac{(b^p-a^p)}{2p(a^pb^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{\frac{1}{r}}\left[\left(\kappa_9(q,p;a,b)|f'(a)|^q+\kappa_{11}(q,p;a,b)|f'(b)|^q\right)^{\frac{1}{q}}\right.\\
+\;& \left.\left(\kappa_{12}(q,p;b,a)|f'(a)|^q+\kappa_{10}(q,p;b,a)|f'(b)|^q\right)^{\frac{1}{q}}\right], \qquad (14)
\end{aligned}$$

where

$$\kappa_9(q,p;a,b) = \int_0^{\frac{1}{2}}(1-t)\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{q+\frac{q}{p}}\mathrm{d}t, \tag{15}$$

$$\kappa_{10}(q,p;b,a) = \int_{\frac{1}{2}}^1 t\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{q+\frac{q}{p}}\mathrm{d}t, \tag{16}$$

$$\kappa_{11}(q,p;a,b) = \int_0^{\frac{1}{2}} t\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{q+\frac{q}{p}}\mathrm{d}t, \tag{17}$$

$$\kappa_{12}(q,p;b,a) = \int_{\frac{1}{2}}^1(1-t)\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{q+\frac{q}{p}}\mathrm{d}t. \tag{18}$$

Proof. Using Lemma 2.1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+\frac{1}{p}}}\mathrm{d}x\right|\\
\leq\;& \frac{b^p-a^p}{2p(a^pb^p)}\Big[\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|\Big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|\mathrm{d}t\\
+\;& \int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|\Big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|\mathrm{d}t\Big]\\
\leq\;& \frac{b^p-a^p}{2p(a^pb^p)}\Big[\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\
+\;& \Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}}f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\
=\;& \frac{b^p-a^p}{2p(a^pb^p)}\Big[\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{q+\frac{q}{p}}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\
+\;& \Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{q+\frac{q}{p}}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\
\leq\;& \frac{b^p-a^p}{2p(a^pb^p)}\Big[\Big(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{q+\frac{q}{p}}\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]\Big)^{\frac{1}{q}}\\
+\;& \Big(\int_{\frac{1}{2}}^1\big|(2t-2+\lambda)\big|^r\mathrm{d}t\Big)^{\frac{1}{r}}\Big(\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{q+\frac{q}{p}}\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]\Big)^{\frac{1}{q}}\Big]\\
=\;& \frac{b^p-a^p}{2p(a^pb^p)}\times\Big(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\Big)^{\frac{1}{r}}\Big[\Big(\kappa_9(q,p;a,b)|f'(a)|^q+\kappa_{11}(q,p;a,b)|f'(b)|^q\Big)^{\frac{1}{q}}\\
+\;& \Big(\kappa_{12}(q,p;b,a)|f'(a)|^q+\kappa_{10}(q,p;b,a)|f'(b)|^q\Big)^{\frac{1}{q}}\Big],
\end{aligned}$$

which is the required result. □

3. Applications

In this section, we obtain some important applications of our main results.

I. For $\lambda = 0$, Theorem 2.4 reduces to the following result.

Corollary 3.1. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set. Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $r,q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq\frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\kappa_7(r,p;a,b))^{\frac{1}{r}}\Bigg(\frac{|f'(a)|^q+\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q}{4}\Bigg)^{\frac{1}{q}}+(\kappa_8(r,p;b,a))^{\frac{1}{r}}\Bigg(\frac{\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q+|f'(b)|^q}{4}\Bigg)^{\frac{1}{q}}\Bigg],$$

where $\kappa_7(r,p;b,a)$ and $\kappa_8(r,p;b,a)$ are given by (2.7) *and* (2.8) *respectively.*

II. For $\lambda=1$, Theorem 2.4 reduces to the following result.

Corollary 3.2. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set. Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $r,q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq\frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\kappa_7(r,p;a,b))^{\frac{1}{r}}\Bigg(\frac{|f'(a)|^q+\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q}{4}\Bigg)^{\frac{1}{q}}+(\kappa_8(r,p;b,a))^{\frac{1}{r}}\Bigg(\frac{\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q+|f'(b)|^q}{4}\Bigg)^{\frac{1}{q}}\Bigg],$$

where $\kappa_7(r,p;b,a)$ and $\kappa_8(r,p;b,a)$ are given by (2.7) *and* (2.8) *respectively.*

III. For $\lambda=\frac{1}{3}$, Theorem 2.4 reduces to the following result.

Corollary 3.3. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set. Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $r,q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}+f(b)\Big]-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq\frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\kappa_7(r,p;a,b))^{\frac{1}{r}}\Bigg(\frac{|f'(a)|^q+\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q}{4}\Bigg)^{\frac{1}{q}}+(\kappa_8(r,p;b,a))^{\frac{1}{r}}\Bigg(\frac{\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q+|f'(b)|^q}{4}\Bigg)^{\frac{1}{q}}\Bigg],$$

where $\kappa_7(r,p;b,a)$ and $\kappa_8(r,p;b,a)$ are given by (2.7) *and* (2.8) *respectively.*

IV. For $\lambda=\frac{1}{2}$, Theorem 2.4 reduces to the following result.

Corollary 3.4. *Let $I=[a,b]\subset\mathbb{R}\setminus\{0\}$ be p-harmonic convex set. Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $r,q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{1}{4}\Big[f(a)+2f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac{1}{p}}+f(b)\Big]-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq\frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\kappa_7(r,p;a,b))^{\frac{1}{r}}\Bigg(\frac{|f'(a)|^q+\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q}{4}\Bigg)^{\frac{1}{q}}+(\kappa_8(r,p;b,a))^{\frac{1}{r}}\Bigg(\frac{\Big|f'\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big)\Big|^q+|f'(b)|^q}{4}\Bigg)^{\frac{1}{q}}\Bigg],$$

where $\kappa_7(r,p;b,a)$ and $\kappa_8(r,p;b,a)$ are given by (2.7) *and* (2.8) *respectively.*

V. For $\lambda=0$, Theorem 2.5 reduces to the following result.

Corollary 3.5. *Let $I = [a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-harmonic convex function on I, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| f\left(\frac{2a^p b^p}{a^p + b^p}\right)^{\frac{1}{p}} - \frac{p(a^p b^p)}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1+p}} \mathrm{d}x \right| \\
\leq\ & \frac{(b^p - a^p)}{2p(a^p b^p)} \times \left(\frac{1}{2(r+1)}\right)^{\frac{1}{r}} \Big[\Big(\kappa_9(q, p; a, b)|f'(a)|^q + \kappa_{11}(q, p; a, b)|f'(b)|^q \Big)^{\frac{1}{q}} \\
+\ & \Big(\kappa_{12}(q, p; b, a)|f'(a)|^q + \kappa_{10}(q, p; b, a)|f'(b)|^q \Big)^{\frac{1}{q}} \Big],
\end{aligned}$$

where $\kappa_9(q, p; a, b)$, $\kappa_{10}(q, p; b, a)$, $\kappa_{11}(q, p; a, b)$ and $\kappa_{12}(q, p; b, a)$ are given by (2.12)-(2.15).

VI. For $\lambda = 1$, Theorem 2.5 reduces to the following result.

Corollary 3.6. *Let $I = [a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-harmonic convex function on I, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{p(a^p b^p)}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1+p}} \mathrm{d}x \right| \\
\leq\ & \frac{(b^p - a^p)}{2p(a^p b^p)} \times \left(\frac{1}{2(r+1)}\right)^{\frac{1}{r}} \Big[\Big(\kappa_9(q, p; a, b)|f'(a)|^q + \kappa_{11}(q, p; a, b)|f'(b)|^q \Big)^{\frac{1}{q}} \\
+\ & \Big(\kappa_{12}(q, p; b, a)|f'(a)|^q + \kappa_{10}(q, p; b, a)|f'(b)|^q \Big)^{\frac{1}{q}} \Big],
\end{aligned}$$

where $\kappa_9(q, p; a, b)$, $\kappa_{10}(q, p; b, a)$, $\kappa_{11}(q, p; a, b)$ and $\kappa_{12}(q, p; b, a)$ are given by (2.12)-(2.15).

VII. For $\lambda = \frac{1}{3}$, Theorem 2.5 reduces to the following result.

Corollary 3.7. *Let $I = [a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-harmonic convex function on I, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2a^p b^p}{a^p + b^p}\right)^{\frac{1}{p}} + f(b) \right] - \frac{p(a^p b^p)}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1+p}} \mathrm{d}x \right| \\
\leq\ & \frac{(b^p - a^p)}{p(a^p b^p)} \times \left(\frac{1 + 2^{p+1}}{6^{p+1}(r+1)}\right)^{\frac{1}{r}} \Big[\Big(\kappa_9(q, p; a, b)|f'(a)|^q + \kappa_{11}(q, p; a, b)|f'(b)|^q \Big)^{\frac{1}{q}} \\
+\ & \Big(\kappa_{12}(q, p; b, a)|f'(a)|^q + \kappa_{10}(q, p; b, a)|f'(b)|^q \Big)^{\frac{1}{q}} \Big],
\end{aligned}$$

where $\kappa_9(q, p; a, b)$, $\kappa_{10}(q, p; b, a)$, $\kappa_{11}(q, p; a, b)$ and $\kappa_{12}(q, p; b, a)$ are given by (2.12)-(2.15).

VIII. For $\lambda = \frac{1}{2}$, Theorem 2.5 reduces to the following result.

Corollary 3.8. *Let $I = [a, b] \subset \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-harmonic convex function on I, $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| \frac{1}{4}\left[f(a) + 2f\left(\frac{2a^p b^p}{a^p + b^p}\right)^{\frac{1}{p}} + f(b) \right] - \frac{p(a^p b^p)}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1+p}} \mathrm{d}x \right| \\
\leq\ & \frac{(b^p - a^p)}{p(a^p b^p)} \times \left(\frac{2}{4^{p+1}(r+1)}\right)^{\frac{1}{r}} \Big[\Big(\kappa_9(q, p; a, b)|f'(a)|^q + \kappa_{11}(q, p; a, b)|f'(b)|^q \Big)^{\frac{1}{q}} \\
+\ & \Big(\kappa_{12}(q, p; b, a)|f'(a)|^q + \kappa_{10}(q, p; b, a)|f'(b)|^q \Big)^{\frac{1}{q}} \Big],
\end{aligned}$$

where $\kappa_9(q, p; a, b)$, $\kappa_{10}(q, p; b, a)$, $\kappa_{11}(q, p; a, b)$ and $\kappa_{12}(q, p; b, a)$ are given by (2.12)-(2.15).

Remark 3.9. *For appropriate and suitable choice of p and q, one can obtain several new and known results as special cases for various classes of convex functions and their variant forms. Authors have extended these results for different classes of p-harmonic convex functions including η-harmonic convex functions and p-beta-harmonic convex functions. It is expected that the ideas and technique of this paper may be starting point for further research in this dynamic area.*

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