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Holomorphically Projective Mappings Between Generalized m-Parabolic Kähler Manifolds

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Abstract. Generalized m-parabolic Kähler manifolds are defined and holomorphically projective mappings between such manifolds have been considered. Two non-linear systems of PDE's in covariant derivatives of the first and second kind for the existence of such mappings are given. Also, relations between five linearly independent curvature tensors of generalized m-parabolic Kähler manifolds with respect to these mappings are examined.

1. Introduction

Parabolic Kähler spaces were first considered by V. V. Vishnevskij as a parabolically analogue of A-spaces. An n-dimensional Riemannian manifold (M, g) is called an m-parabolic Kähler manifold if beside the metric tensor g there exists a $(1, 1)$ tensor field F on M such that $\operatorname{rank}(F) = m \leq \frac{n}{2}$ and the following conditions hold [7]

$$\begin{aligned} F^2 &= 0, \\ g(X, FX) &= 0 \\ \nabla F &= 0, \end{aligned}$$

where ∇ is the Levi-Civita connection corresponding to the metric g and X is an arbitrary tangent vector field on M.

Holomorphically projective mappings and their generalizations have been widely studied in the last decades, see for instance [3, 5, 6, 8, 9, 21]. Holomorphically projective mappings between parabolic Kähler manifolds are thoroughly studied by M. Shiha and J. Mikeš [21]. Many results on holomorphically projective mappings between parabolic Kähler manifolds and their generalizations are included in the excellent book [7].

A. Einstein [1] had aim to unite the gravitation theory and the theory of electromagnetism. Firstly, he had tried with a complex basic tensor, with symmetric real part and anti-symmetric imaginary part.

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Afterwards, he saw that it was appropriate to use real but non-symmetric basic tensor, whose symmetric part corresponds to gravitation and the anti-symmetric one to electromagnetism. In 1951. L. P. Eisenhart [2] introduced generalized Riemannian spaces as differentiable manifolds equipped with a non-symmetric basic tensor. These spaces play an important role in J. W. Moffat's non-symmetric gravitational field theory, see for instance [4, 10]. Some of the significant contributions to the study of linear connections and curvatures of generalized Riemannian spaces were given by M. Prvanović [20] and S. M. Minčić [11–17, 26].

S. M. Minčić in collaboration with M. S. Stanković and Lj. S. Velimirović proposed a generalized classical (elliptic) Kähler space and considered holomorphically projective mappings between such spaces [15, 22–25]. Recently, generalized hyperbolic Kähler spaces are defined and holomorphically projective mappings between such spaces were considered in [18].

In this paper, we define generalized m-parabolic Kähler manifolds as a particular case of generalized Riemannian spaces and consider holomorphically projective mappings between such manifolds. We examine necessary and sufficient conditions for the existence of holomorphically projective mappings in terms of the symmetric part of a non-symmetric metric and its covariant derivative. Also, we present some relations between curvature tensors of generalized m-parabolic Kähler manifolds with respect to holomorphically projective mappings.

2. Holomorphically Projective Mappings of Generalized Kähler Manifolds

Let $\left(U, u\right)$, $u = (u^1, \dots, u^n)$ be a local chart at the point $p \in M$. The set of vectors at p is the vector space with basis

$$\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}.$$

We shall use the following notation

$$X = \frac{\partial}{\partial u^i},\ \ Y = \frac{\partial}{\partial u^j},\ \ Z = \frac{\partial}{\partial u^k}.$$

and abbreviate $\frac{\partial}{\partial u^i}$ by the notation ∂_i.

L. P. Eisenhart in [2] introduced a generalized Riemannian space as a differentiable manifold M equipped with a non-symmetric metric g. Therefore the metric g is represented by

$$g(X, Y) = \underline{g}(X, Y) + \underset{\vee}{g}(X, Y),$$

where

$$\underline{g}(X, Y) = \frac{1}{2}(g(X, Y) + g(Y, X)) \ \text{ and } \ \underset{\vee}{g}(X, Y) = \frac{1}{2}(g(X, Y) - g(Y, X)).$$

The non-symmetric linear connection $\underset{1}{\nabla}$ of a generalized Riemannian space with the metric g is explicitly defined by

$$g(\underset{1}{\nabla}_X Y, Z) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(Y, X)), \tag{1}$$

or in local coordinates

$$\Gamma_{i.jk} = g_{\underline{ip}}\Gamma^p_{jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}).$$

Here the functions $\Gamma_{i.jk}$ and Γ^i_{jk} are called generalized Christoffel symbols of the first kind and the second kind, respectively.

As is well-known on the manifold M with the non-symmetric linear connection $\underset{1}{\nabla}$ another non-symmetric linear connection $\underset{2}{\nabla}$ can be defined by [20]

$$\underset{2}{\nabla}_X Y = \underset{1}{\nabla}_Y X.$$

In local coordinates with respect to a local chart (U, u), $u = (u^1, \dots, u^n)$ we have

$$\underset{1}{\nabla}_{\partial_i}\partial_j = \Gamma^h_{ij}\partial_h, \quad \underset{2}{\nabla}_{\partial_i}\partial_j = \Gamma^h_{ji}\partial_h, \quad \nabla_{\partial_i}\partial_j = \Gamma^h_{\underline{ij}}\partial_h,$$

where $\underline{ij}$ signifies a symmetrization with division, i.e., $\Gamma^h_{\underline{ij}} = \frac{1}{2}(\Gamma^h_{ij} + \Gamma^h_{ji})$.

Covariant derivatives of tensors with respect to the linear connections $\underset{1}{\nabla}$, $\underset{2}{\nabla}$ and ∇ are respectively given by:

$$\underset{1}{\nabla}_m a^i_j \equiv a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_p,$$
$$\underset{2}{\nabla}_m a^i_j \equiv a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{mj} a^i_p,$$
$$\nabla_m a^i_j \equiv a^i_{j;m} = a^i_{j,m} + \Gamma^i_{\underline{mp}} a^p_j - \Gamma^p_{\underline{mj}} a^i_p,$$

where $a^i_{j,m}$ denotes the partial derivative of a tensor a^i_j with respect to x^m.

Generalized classical (elliptic) and hyperbolic Kähler spaces are defined in [15] and [18], respectively. According to Definition 13.7 in [7] we propose a definition of a generalized m-parabolic Kähler manifold in generalized Riemannian settings.

Definition 2.1. *A generalized Riemannian manifold* (M, g) *of even dimension* n $(n > 2)$ *is called a* generalized m-parabolic Kähler manifold *if there exists a tensor field* F *on* M *of type* $(1, 1)$ *such that* $\operatorname{rank}(F) = m \leq \frac{n}{2}$ *and the following conditions hold*

$$F^2 = 0, \tag{2}$$
$$\underline{g}(X, FX) = 0 \tag{3}$$
$$\nabla F = 0, \tag{4}$$

where ∇ *denotes the Levi-Civita connection corresponding to the symmetric part* $\underline{g}$ *of the metric* g *and* X *is an arbitrary tangent vector field on* M*. In the case when* $\operatorname{rank}(F) = m = \frac{n}{2}$ *the manifold* (M, g) *is called a* generalized parabolic Kähler manifold.

Definition 2.2. *A curve* $l : I \to M$ *on a generalized* m*-parabolic Kähler manifold* M *with a metric* g *satisfying the regularity condition* $\lambda(t) = \frac{dl(t)}{dt} \neq 0, t \in I$*, is called a* holomorphically planar curve *if for some functions* ρ_1 *and* ρ_2 *of a parameter* t *the following equation holds*

$$\nabla_{\lambda(t)}\lambda(t) = \rho_1(t)\lambda(t) + \rho_2(t)F\lambda(t),$$

where ∇ *denotes the Levi-Civita connection corresponding to the symmetric part* $\underline{g}$ *of the metric* g.

Let M and $\overline{M}$ be two generalized m-parabolic Kähler manifolds of dimension n $(n > 2)$, with the metrics g and $\overline{g}$, respectively. We can consider these manifolds in the *common coordinate system with respect to the diffeomorphism* $f : M \to \overline{M}$. In this coordinate system the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates. Therefore we can suppose $M \equiv \overline{M}$ and we can put

$$\underset{1}{P} = \underset{1}{\overline{\nabla}} - \underset{1}{\nabla},$$

where $\underset{1}{P}$ is a tensor field of type $(1, 2)$, called the *deformation tensor field of linear connections* $\underset{1}{\overline{\nabla}}$ *and* $\underset{1}{\nabla}$ *with respect to the mapping* f.

Theorem 2.1. *A necessary and sufficient condition for a holomorphically projective mapping $f : M \to \overline{M}$ of generalized m-parabolic Kähler manifolds M and $\overline{M}$ is given by*

$$\underset{1}{P}(X,Y) = \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX + \xi(X,Y), \tag{5}$$

where φ is a linear form, ψ is a gradient-like form such that $\psi(X) = \varphi(FX)$ and ξ is an anti-symmetric tensor field of type $(1,2)$.

Proof. Let $f : M \to \overline{M}$ be a holomorphically projective mapping between generalized m-parabolic Kähler manifolds M and $\overline{M}$. It means that the linear connections ∇ and $\overline{\nabla}$ have all holomorphically planar curves in common. Therefore [9]

$$\overline{\nabla}_X Y - \nabla_X Y = \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX. \tag{6}$$

where φ is one-form and ψ is a gradient-like form such that $\psi(X) = \varphi(FX)$.

Since

$$\overline{\nabla}_X Y = \underset{1}{\overline{\nabla}}_X Y - \underset{1}{\overline{T}}(X,Y)$$

and

$$\nabla_X Y = \underset{1}{\nabla}_X Y - \underset{1}{T}(X,Y),$$

from relation (6) we can conclude that (5) holds for the anti-symmetric tensor field ξ defined by

$$\xi(X,Y) = \underset{1}{\overline{T}}(X,Y) - \underset{1}{T}(X,Y),$$

which proves the direct part of this theorem. The proof of the converse part is a simple verification and it is left to the reader. □

Our aim is to reformulate condition (5) in terms of the symmetric part $\underline{\overline{g}}$ of the metric g and the covariant derivative of the first kind with respect to the metric g.

Theorem 2.2. *A generalized m-parabolic Kähler manifold M with a metric g admits a holomorphically projective mapping onto a generalized m-parabolic Kähler manifold $\overline{M}$ with a metric $\overline{g}$ if and only if*

$$\begin{aligned}\underset{1}{\nabla}_Z \underline{\overline{g}}(X,Y) = 2\psi(Z)\underline{\overline{g}}(X,Y) + \sum_{CS(X,Y)} \Big(&\psi(X)\underline{\overline{g}}(Y,Z) + \varphi(X)\underline{\overline{g}}(Y,FZ) \\ &+ \underline{\overline{g}}(\xi(X,Z),Y)\Big),\end{aligned} \tag{7}$$

where $\underline{\overline{g}}$ denotes the symmetric part of the metric $\overline{g}$, φ is a linear form, ψ is a gradient-like form such that $\psi(X) = \varphi(FX)$, and ξ is an anti-symmetric tensor field of type $(1,2)$.

Proof. Since the metric $\underline{\overline{g}}$ is covariantly constant with respect to the connection $\underset{1}{\overline{\nabla}}$, it is not difficult to verify the relation

$$\underset{1}{\nabla}_Z \underline{\overline{g}}(X,Y) = \underline{\overline{g}}(\underset{1}{P}(X,Z),Y) + \underline{\overline{g}}(X,\underset{1}{P}(Y,Z)). \tag{8}$$

After changing the expression for $\underset{1}{P}(X,Y)$ given by (5) in the previous equation and by using (3) we obtain (7).

Let us prove the converse part. Analogous to the proof of Theorem 3.1 in [9] we introduce an auxiliary tensor field $\underset{1}{Q}$ of type $(1,3)$ by

$$\underset{1}{Q}(X,Y,Z) = \underline{\overline{g}}\Big(Z, \underset{1}{P}(X,Y) - \sum_{CS(X,Y)} \big(\psi(X)Y + \varphi(X)FY\big) - \xi(X,Y)\Big). \tag{9}$$

Since the metric $\underline{\overline{g}}$ is symmetric, relation (7) takes the following form

$$\begin{aligned} \underset{1}{\nabla}_Z \underline{\overline{g}}(X,Y) = \sum_{CS(X,Y)} \Big(&\psi(Z)\underline{\overline{g}}(X,Y) + \psi(X)\underline{\overline{g}}(Z,Y) - \varphi(X)\underline{\overline{g}}(FZ,Y) \\ &+ \underline{\overline{g}}(\xi(X,Z),Y)\Big), \end{aligned} \tag{10}$$

and relation (8) becomes

$$\underset{1}{\nabla}_Z \underline{\overline{g}}(X,Y) = \sum_{CS(X,Y)} \underline{\overline{g}}(\underset{1}{P}(X,Z),Y). \tag{11}$$

From (10) and (11) we obtain that

$$\sum_{CS(X,Y)} \underset{1}{Q}(X,Z,Y) = 0,$$

i.e., the tensor field $\underset{1}{Q}$ is anti-symmetric with respect to the first and the third argument. The tensor field $\underset{1}{Q}$ is evidently symmetric with respect to the first and the second argument. These facts ensure the validity of the following sequence of equalities

$$\begin{aligned} \underset{1}{Q}(X,Y,Z) &= \underset{1}{Q}(Y,X,Z) = -\underset{1}{Q}(Z,X,Y) = -\underset{1}{Q}(X,Z,Y) \\ &= \underset{1}{Q}(Y,Z,X) = \underset{1}{Q}(Z,Y,X) = -\underset{1}{Q}(X,Y,Z), \end{aligned}$$

which further implies $\underset{1}{Q}(X,Y,Z) = 0$.

The metric $\underline{\overline{g}}$ is regular and the tensor field $\underset{1}{Q}$ defined by (9) vanishes identically, so we conclude

$$\underset{1}{P}(X,Y) - \sum_{CS(X,Y)} \big(\psi(X)Y + \varphi(X)FY\big) - \xi(X,Y) = 0,$$

which completes the proof. □

Taking into account the covariant derivative of the second kind we can prove the following theorem.

Theorem 2.3. *A generalized m-parabolic Kähler manifold M with a metric g admits a holomorphically projective mapping onto a generalized m-parabolic Kähler manifold $\overline{M}$ with a metric $\overline{g}$ if and only if*

$$\begin{aligned} \underset{2}{\nabla}_Z \underline{\overline{g}}(X,Y) = 2\psi(Z)\underline{\overline{g}}(X,Y) + \sum_{CS(X,Y)} \Big(&\psi(X)\underline{\overline{g}}(Y,Z) + \varphi(X)\underline{\overline{g}}(Y,FZ) \\ &- \underline{\overline{g}}(\xi(X,Z),Y)\Big), \end{aligned} \tag{12}$$

where $\underline{\overline{g}}$ denotes the symmetric part of the metric $\overline{g}$, φ is a linear form, ψ is a gradient-like form such that $\psi(X) = \varphi(FX)$, and ξ is an anti-symmetric tensor field of type $(1,2)$.

Remark 2.1. *It is important to state that any of the conditions* (7) *and* (12) *is equivalent with the condition* [9]

$$\nabla_Z \underline{\overline{g}}(X,Y) = 2\psi(Z)\underline{\overline{g}}(X,Y) + \sum_{CS(X,Y)} \Big(\psi(X)\underline{\overline{g}}(Y,Z) + \varphi(X)\underline{\overline{g}}(Y,FZ)\Big), \tag{13}$$

where ∇ *denotes the symmetric part of the non-symmetric linear connections* $\underset{1}{\nabla}$ *and* $\underset{2}{\nabla}$.

Problem 2.1. *Can the non-linear systems* (7) *and* (12) *be transformed into linear systems of PDE's in covariant derivatives of the first and second kind? The same issue was raised in the case of the non-linear system* (13) *and the answer was affirmative.*

3. Some Relations Between Curvature Tensors With Respect to Holomorphically Projective Mappings

On manifolds with a non-symmetric linear connection one can define five independent curvature tensors [14]:

$$\begin{aligned}
\underset{\theta}{R}(X,Y)Z =& \underset{\theta}{\nabla}_X \underset{\theta}{\nabla}_Y Z - \underset{\theta}{\nabla}_Y \underset{\theta}{\nabla}_X Z - \underset{\theta}{\nabla}_{[X,Y]} Z, \quad \theta = 1,2,\\
\underset{3}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z,\\
\underset{4}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z,\\
\underset{5}{R}(X,Y)Z =& \frac{1}{2}\Big(\underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{1}{\nabla}_X Z + \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z\\
&+ \underset{1}{\nabla}_{[Y,X]} Z + \underset{2}{\nabla}_{[Y,X]} Z\Big).
\end{aligned} \tag{14}$$

The relations between the curvature tensors $\underset{\theta}{R}\,(\theta = 1,\dots,5)$ and the Riemannian curvature tensor R corresponding to the symmetric linear connection $\nabla_X Y = \frac{1}{2}\Big(\underset{1}{\nabla}_X Y + \underset{1}{\nabla}_Y X\Big)$ are examined in [12]:

$$\begin{aligned}
\underset{1}{R}(X,Y)Z =& R(X,Y)Z + \frac{1}{2}\nabla_X \underset{1}{T}(Z,Y) - \frac{1}{2}\nabla_Y \underset{1}{T}(Z,X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&- \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y),\\
\underset{2}{R}(X,Y)Z =& R(X,Y)Z - \frac{1}{2}\nabla_X \underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y \underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&+ \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y),\\
\underset{3}{R}(X,Y)Z =& R(X,Y)Z + \frac{1}{2}\nabla_X \underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y \underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&+ \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y) - \frac{1}{2}\underset{1}{T}(\underset{1}{T}(Y,X),Z),\\
\underset{4}{R}(X,Y)Z =& R(X,Y)Z + \frac{1}{2}\nabla_X \underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y \underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X)\\
&+ \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y) + \frac{1}{2}\underset{1}{T}(\underset{1}{T}(Y,X),Z),\\
\underset{5}{R}(X,Y)Z =& R(X,Y)Z + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X) + \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y).
\end{aligned}$$

For an arbitrary tensor field B we will use the symbol $\sum_{CA(\cdot,\cdot)}$ to denote

$$\sum_{CA(Y,Z)} B(X,Y,Z) = B(X,Y,Z) - B(X,Z,Y),$$

and by $\sum_{CS(\cdot,\cdot)}$ we will denote

$$\sum_{CS(Y,Z)} B(X,Y,Z) = B(X,Y,Z) + B(X,Z,Y).$$

Relations between the curvature tensors $\underset{\theta}{R}$ and $\underset{\theta}{\overline{R}}$ ($\theta = 1,\ldots,5$) of the generalized m-parabolic Kähler manifolds M and $\overline{M}$, respectively, with respect to holomorphically projective mappings are given in what follows.

Theorem 3.1. *Let $f : M \to \overline{M}$ be a holomorphically projective mapping and let $\underset{\theta}{R}$ and $\underset{\theta}{\overline{R}}$ are θ-kind ($\theta = 1,\ldots,5$) curvature tensors of the generalized m-parabolic Kähler manifolds M and $\overline{M}$, respectively. Then the following relations are valid*

$$\begin{aligned}\underset{1}{\overline{R}}(X,Y)Z =&\underset{1}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\ &+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ\\ &+ \frac{1}{2}\sum_{CA(X,Y)}\big(\overline{\nabla}_X\underset{1}{\overline{T}}(Z,Y) - \nabla_X\underset{1}{T}(Z,Y)\big)\\ &+ \frac{1}{4}\sum_{CA(X,Y)}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) - \underset{1}{T}(\underset{1}{T}(Z,Y),X)\big),\end{aligned} \tag{15}$$

$$\begin{aligned}\underset{2}{\overline{R}}(X,Y)Z =&\underset{2}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\ &+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ\\ &- \frac{1}{2}\sum_{CA(X,Y)}\big(\overline{\nabla}_X\underset{1}{\overline{T}}(Z,Y) - \nabla_X\underset{1}{T}(Z,Y)\big)\\ &+ \frac{1}{4}\sum_{CA(X,Y)}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) - \underset{1}{T}(\underset{1}{T}(Z,Y),X)\big),\end{aligned} \tag{16}$$

$$\begin{aligned}\underset{3}{\overline{R}}(X,Y)Z =&\underset{3}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\ &+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ\\ &+ \frac{1}{2}\sum_{CS(X,Y)}\big(\overline{\nabla}_X\underset{1}{\overline{T}}(Z,Y) - \nabla_X\underset{1}{T}(Z,Y)\big)\\ &- \frac{1}{4}\sum_{CA(X,Y)}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) - \underset{1}{T}(\underset{1}{T}(Z,Y),X)\big)\\ &- \frac{1}{2}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Y,X),Z) - \underset{1}{T}(\underset{1}{T}(Y,X),Z)\big),\end{aligned} \tag{17}$$

$$\begin{aligned}\underset{4}{\overline{R}}(X,Y)Z =&\underset{4}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\ &+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ\\ &+ \frac{1}{2}\sum_{CS(X,Y)}\big(\overline{\nabla}_X\underset{1}{\overline{T}}(Z,Y) - \nabla_X\underset{1}{T}(Z,Y)\big)\\ &- \frac{1}{4}\sum_{CA(X,Y)}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) - \underset{1}{T}(\underset{1}{T}(Z,Y),X)\big)\\ &+ \frac{1}{2}\big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Y,X),Z) - \underset{1}{T}(\underset{1}{T}(Y,X),Z)\big),\end{aligned} \tag{18}$$

$$
\begin{aligned}
\underset{5}{\overline{R}}(X,Y)Z =&\underset{5}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX \\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ \\
&- \frac{1}{4}\sum_{CA(X,Y)} \big(\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) - \underset{1}{T}(\underset{1}{T}(Z,Y),X)\big),
\end{aligned}
\tag{19}
$$

where $\varphi(X,Y)$ is defined by

$$
\varphi(X,Y) =\nabla_Y\varphi(X) - \psi(X)\varphi(Y) - \varphi(X)\psi(Y), \tag{20}
$$

and $\psi(X,Y)$ is defined by

$$
\psi(X,Y) = \varphi(FX,Y) = \nabla_Y\psi(X) - \psi(X)\psi(Y). \tag{21}
$$

Proof. The curvature tensors R and $\overline{R}$ of the symmetric linear connections ∇ and $\overline{\nabla}$, respectively, satisfy the well-known relation [9]

$$
\begin{aligned}
\overline{R}(X,Y)Z =&R(X,Y)Z + \nabla_XP(Z,Y) - \nabla_YP(Z,X) + P(P(Z,Y),X) \\
&- P(P(Z,X),Y),
\end{aligned}
\tag{22}
$$

where $P(X,Y) = \overline{\nabla}_XY - \nabla_XY$.

Since the deformation tensor field P is given by

$$
\begin{aligned}
P(X,Y) &= \frac{1}{2}\big(\underset{1}{P}(X,Y) + \underset{1}{P}(Y,X)\big) \\
&= \varphi(FX)Y + \varphi(FY)X + \varphi(X)FY + \varphi(Y)FX,
\end{aligned}
$$

relation (22) becomes [9]

$$
\begin{aligned}
\overline{R}(X,Y)Z =&R(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX \\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,
\end{aligned}
\tag{23}
$$

where $\varphi(X,Y)$ and $\psi(X,Y)$ are defined by (20) and (21), respectively.

The curvature tensors R and $\underset{1}{R}$ satisfy the relation

$$
\begin{aligned}
R(X,Y)Z =&\underset{1}{R}(X,Y)Z - \frac{1}{2}\nabla_X\underset{1}{T}(Z,Y) + \frac{1}{2}\nabla_Y\underset{1}{T}(Z,X) - \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,Y),X) \\
&+ \frac{1}{4}\underset{1}{T}(\underset{1}{T}(Z,X),Y),
\end{aligned}
$$

and the same relation is valid for the curvature tensors $\overline{R}$ and $\underset{1}{\overline{R}}$

$$
\begin{aligned}
\overline{R}(X,Y)Z =&\underset{1}{\overline{R}}(X,Y)Z - \frac{1}{2}\overline{\nabla}_X\underset{1}{\overline{T}}(Z,Y) + \frac{1}{2}\overline{\nabla}_Y\underset{1}{\overline{T}}(Z,X) - \frac{1}{4}\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,Y),X) \\
&+ \frac{1}{4}\underset{1}{\overline{T}}(\underset{1}{\overline{T}}(Z,X),Y).
\end{aligned}
$$

By plugging the last two relations in (23) we obtain (15). The proofs of relations (16)–(19) are analogous. □

Remark 3.1. *We should note that relations* (15)–(19) *can be obtained directly by plugging* (5) *in relations between curvature tensors $\underset{\theta}{R}$ and $\underset{\theta}{\overline{R}}$, $\theta = 1,\dots,5$, which are analogous to relation* (22) *for usual curvature tensors.*

4. Conclusion

In this paper generalized m-parabolic Kähler manifolds are defined and holomorphically projective mappings between such manifolds are considered. Two equivalent non-linear systems of PDE's for the existence of holomorphically projective mappings of generalized m-parabolic Kähler manifolds are given. Also, the relations between curvature tensors of m-parabolic Kähler manifolds with respect to holomorphically projective mappings are examined. The techniques used in this paper are different than those in the case of generalized classical (elliptic) and hyperbolic Kähler spaces. We hope that this paper will open possibilities for further extension of results from usual parabolic Kähler manifolds to generalized m-parabolic Kähler manifolds. Holomorphically projective mappings of generalized parabolic Kähler manifolds are a particular case of canonical almost geodesic mappings of the second type between generalized parabolic Kähler manifolds that were considered in [19].

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