A Note on the System of Linear Recurrence Equations

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Abstract. We will find a solution to a system of $2d$ linear recurrence equations. Each equation is of the form $x_{2k}(n+1) = x_k(n)$ or $x_{2k+1}(n+1) = x_k(n) + x_{2d-1,k}(n)$. This kind of system is connected with counting restricted permutations.

1. Introduction

The study of restricted permutations has a long history. Probably the most well known example is derangement problem or “le Problème des Rencontres” (see [4]). "Today, most of the restricted permutations considered in the literature deal with pattern avoidance. For an exhaustive survey of such studies, see [5]. For a related topic of pattern avoidance in compositions and words see [3] and in set partitions see [7]. Study of permutation patterns has applications in counting different combinatorial structures, computer science, statistical mechanics and computational biology [5, Ch.2,3].

Another type of restricted permutations is a generalization of the derangement problem. Detailed historical introduction to restricted permutations of this kind can be found in [1, 2]. Let $p$ be a permutation of the set $\mathbb{N}_n = \{1,2,\ldots,n\}$. So, $p(i)$ refers to the value taken by the function $p$ when evaluated at a point $i$. Mendelsohn, Lagrange, Lehmer, Tomescu and Stanley studied particular types of strongly restricted permutations satisfying the condition $|p(i) - i| \leq d$, where $d$ is 1, 2, or 3 (more information on their work can be found in [1]). In [1] we pursue more general, asymmetric cases and we end up with asymmetric cases with more forbidden positions.

In [1] we developed a technique for counting restricted permutations of $\mathbb{N}_n$ satisfying the conditions $-k \leq p(i) - i \leq r$ (for arbitrary natural numbers $k$ and $r$) and $p(i) - i \notin I$ (for some set $I$). For a given $k$, $r$ and $I$ the technique produces a system of linear recurrence equations. When trying to determine the reduced system in a particular case, we get the following system of linear recurrence equations:

\[
\begin{align*}
x_{2k}(n+1) &= x_k(n) \\
x_{2k+1}(n+1) &= x_k(n) + x_{2d-1,k}(n)
\end{align*}
\]

for $k = 0,1,\ldots,2^{d-1} - 1$.

Purpose of this note is to solve the system (*) (for other type of systems, we refer the reader to [6]). Solution of a special case of (*) is given in Lemma 2.1, which is used to solve the general case of (*) in Theorem 2.2.

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Now we will introduce some notations.
The number \( n < 2^d \) in binary form is represented by \( n_2 = (b_{d-1}b_{d-2}\ldots b_1b_0)_2 \), where \( b_i \in \{0, 1\} \) and \( n = \sum_{i=0}^{d} b_i \cdot 2^i \).

Binary operation \( \oplus \) is given by \( x \oplus y = x + y \mod d \).

For each position \( s, 0 \leq s \leq d - 1 \), let us introduce the function

\[
 f_s(a, b, r) = \begin{cases} 
 1, & a_s = b_s \\
 0, & a_s = 1, b_s = 0 \\
 t, & a_s = 0, b_s = 1, s < d - r \\
 t + 1, & a_s = 0, b_s = 1, s \geq d - r.
\end{cases}
\]

2. Main Results

Lemma 2.1. Suppose we have a system of \( 2^d \) linear recurrence equations which are of the form \( x_{2k}(n + 1) = x_k(n) \) and \( x_{2k+1}(n + 1) = x_k(n) + x_{2^{k+1}}(n) \) for \( k = 0, 1, \ldots, 2^{d-1} - 1 \), where \( 0 \leq a, b \leq 2^d - 1 \) and for some \( a, x_a(0) = 1 \), while for all \( b \neq a, x_b(0) = 0 \).

Then for \( n = d \cdot t + r, 0 \leq r \leq d - 1 \), the following equality is true:

\[
x_b(n) = \prod_{s=0}^{d-1} f_s(a, b, r).
\]

Proof. Let us prove that this solution satisfies the initial conditions and the equations of the first type and the second type of a given system.

1° The initial conditions

For \( n = 0 = d \cdot 0 + 0 \) we have that \( r = 0, \) so \( s \oplus r = s \oplus 0 = s. \)

If \( b = a \) for each position applies \( a_s = b_s, \) therefore \( f_s(a, a, 0) = 1, \) which implies that

\[
x_a(0) = \prod_{s=0}^{d-1} 1 = 1.
\]

If \( b \neq a, \) then there is a position \( s \) from where the binary forms \( a_2 \) and \( b_2 \) differ.

If \( a_s = 1 \) and \( b_s = 0, \) it immediately follows that \( f_s(a, b, 0) = 0. \)

If \( a_s = 0 \) and \( b_s = 1, \) as is true for \( s < d = d - r, \) we have that \( f_s(a, b, 0) = t = 0. \)

When \( b \neq a \) in both cases we get that \( f_s(a, b, 0) = 0 \) for some \( s, \) which implies that \( x_a(0) = 0. \)

2° \( x_{2k}(n + 1) = x_k(n) \)

Let \( b = 2k, \) for \( k < 2^{d-1}. \)

For binary forms

\[
 (k)_2 = (b_{d-1}b_{d-2}\ldots b_1b_0) \quad \text{and} \quad (2k)_2 = (b'_{d-1}b'_{d-2}\ldots b'_1b'_0)
\]

we have that \( (2k)_2 \) is obtained from \( (k)_2 \) with cyclic shift to the left by one position, i.e. \( b'_{0\oplus 1} = b_c. \) Also, with increasing \( n \) to \( n + 1 \) we have that the remainder of the division with \( d \) increases by 1 modulo \( d, \) i.e. \( r' = r \oplus 1. \)

Now we will prove the equality \( x_{2k}(n + 1) = x_k(n) \) by considering the following cases:

- If \( a_s = b_s \Rightarrow a_s = b_s \Rightarrow b'_{(s\oplus r)\oplus 1} = b'_{(s\oplus r)\oplus 1} \Rightarrow f_s(a, k, r) = f_s(a, 2k, r \oplus 1) = 1. \)

- If \( a_s = 1, b_s = 0 \Rightarrow a_s = 1, 0 = b_s \Rightarrow b'_{(s\oplus r)\oplus 1} = b'_{(s\oplus r)\oplus 1} \Rightarrow f_s(a, k, r) = f_s(a, 2k, r \oplus 1) = 0. \)
Let it is true that $x_k(n) = x_k(n) = x_k(n)$.

If $r = d - 1$, then $r \oplus 1 = 0$, and also previous equality holds, but with different reasoning:

If $a_s = 0, b_{\text{str}} = 1, s \geq d - r - 1 \Rightarrow a_s = 0, 1 = b_{\text{str}} = b_{\text{sr}) \oplus 1}, s \leq d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2k, r \oplus 1) = t$.

If $r = d - 1$, then $r \oplus 1 = 0$, and also previous equality holds, but with different reasoning:

If $a_s = 0, b_{\text{str}} = 1, s \geq d - r - 1 \Rightarrow a_s = 0, 1 = b_{\text{str}} = b_{\text{sr}) \oplus 1}, s \geq d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2k, r \oplus 1) = t + 1$.

If $r = d - 1$, then $r \oplus 1 = 0$, and also previous equality holds, but with different reasoning:

If $a_s = 0, b_{\text{str}} = 1, s \geq d - r - 1 \Rightarrow a_s = 0, 1 = b_{\text{str}} = b_{\text{sr}) \oplus 1}, s \geq d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2k, r \oplus 1) = t + 1$.

As in all cases, we get that $f_s(a, k, r) = f_s(a, 2k, r \oplus 1)$, which entails that

$$x_k(n) = \prod_{s=0}^{d-1} f_s(a, k, r) = \prod_{s=0}^{d-1} f_s(a, 2k, r \oplus 1) = x_{2k}(n+1).$$

Let $b = 2k + 1$, for $k < 2^{d-1}$. For binary forms $(k)_2 = (b_{d-1}, b_{d-2}, \ldots, b_0)$,

$$((2^{d-1} + k)_2 = (b''_{d-1}, b''_{d-2}, \ldots, b''_0), \quad (2k + 1)_2 = (b'_{d-1}, b'_{d-2}, \ldots, b'_0)$$

it is true that $b_{d-1} = 0, b'_{d-1} = 1, b_s = b'_s = 0$ for $s < d - 1$, while $(2k + 1)_2$ is obtained from $(2^{d-1} + k)_2$ with cyclic shift to the left by one position, i.e. $b''_s = b'_s$. The same as before we have that $r' = r \oplus 1$. Now we will prove the equality $x_{2k+1}(n+1) = x_k(n) + x_{2^{d-1}+k}(n)$ by considering the following cases:

If $a_s = b_{\text{str}}, r \neq d - 1 \Rightarrow a_s = b_{\text{str}} = b'_{\text{str}) \oplus 1}, s < d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2^{d-1} + k, r) = f_s(a, 2k + 1, r + 1) = 1$.

If $s = d - r - 1, a_s = 1, then we have

If $a_s = 0, b_{\text{str}} = 1, s < d - r - 1 \Rightarrow a_s = 0, 1 = b_{\text{str}} = b'_{\text{str}) \oplus 1}, s < d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2^{d-1} + k, r) = f_s(a, 2k + 1, r + 1) = t$.

If $a_s = 0, b_{\text{str}} = 1, s \geq d - r - 1 \Rightarrow a_s = 0, 1 = b_{\text{str}} = b'_{\text{str}) \oplus 1}, s \geq d - (r \oplus 1) \Rightarrow f_s(a, k, r) = f_s(a, 2^{d-1} + k, r) = f_s(a, 2k + 1, r + 1) = t + 1$.

If $s = d - r - 1 < d - r, a_s = 0$, then we have

If $a_s = 0, b_{\text{str}} = 0, 0 \Rightarrow f_s(a, k, r) = 1$;

If $s = d - r - 1 < d - r, a_s = 0$, then we have

If $a_s = 0, b_{\text{str}} = 0, 0 \Rightarrow f_s(a, k, r) = 1$;

If $s = d - r - 1 < d - r, a_s = 0$, then we have

If $a_s = 0, 1 = b_{\text{str}} = b'_{\text{str}) \oplus 1}, s \leq d - 0 \Rightarrow f_s(a, 2k + 1, r \oplus 1) = t' = t + 1$, because $n' = n + 1 = d \cdot t + (d - 1) + 1 = d \cdot (t + 1) + 0$. 
For $s = d - r - 1$ and $a_{d-r-1} = 1$ we get $f_s(a, k, r) = 0 \Rightarrow x_k(n) = 0$, and since $f_s(a, 2^{d-1} + k, r) = f_s(a, 2k + 1, r \oplus 1)$ for all positions $s \neq d - r - 1$, we have that $x_{2k+1}(n+1) = x_{2k+1}(n)$, which entails equality $x_{2k+1}(n+1) = x_{2k+1}(n) + x_{2k-1+4k}(n)$. For $s = d - r - 1$ and $a_{d-r-1} = 0$ we get $f_s(a, k, r) = 1$, $f_s(a, 2^{d-1} + k, r) = t$, $f_s(a, 2k + 1, r \oplus 1) = t + 1$, while $f_s(a, k, r) = f_s(a, 2^{d-1} + k, r) = f_s(a, 2k + 1, r \oplus 1)$ for all positions $s \neq d - r - 1$, and we have that

$$x_k(n) + x_{2k-1+4k}(n) = \prod_{s=0}^{d-1} f_s(a, k, r) + \prod_{s=0}^{d-1} f_s(a, 2^{d-1} + k, r)$$

$$= f_{d-r-1}(a, k, r) \cdot \prod_{0 \leq s \leq d-1 \atop s \neq d-r-1} f_s(a, k, r) + f_{d-r-1}(a, 2^{d-1} + k, r) \cdot \prod_{0 \leq s \leq d-1 \atop s \neq d-r-1} f_s(a, 2^{d-1} + k, r)$$

$$= 1 \cdot \prod_{0 \leq s \leq d-1 \atop s \neq d-r-1} f_s(a, 2k + 1, r \oplus 1) + t \cdot \prod_{0 \leq s \leq d-1 \atop s \neq d-r-1} f_s(a, 2k + 1, r \oplus 1)$$

$$= (1 + t) \cdot \prod_{0 \leq s \leq d-1 \atop s \neq d-r-1} f_s(a, 2k + 1, r \oplus 1) = \prod_{s=0}^{d-1} f_s(a, 2k + 1, r \oplus 1) = x_{2k+1}(n+1).$$

In both cases we get that $x_{2k+1}(n+1) = x_k(n) + x_{2k-1+4k}(n)$. □

We will illustrate this Theorem later, in Example 3.1. Now, we move on to the general case of the system (4).

**Theorem 2.2.** Suppose we have a system of $2^d$ linear recurrence equations of the form $x_{2k}(n + 1) = x_k(n)$ and $x_{2k+1}(n + 1) = x_k(n) + x_{2k-1+4k}(n)$ for $k = 0, 1, \ldots, 2^{d-1} - 1$, with initial conditions $x_0(0) = y_0$, $x_1(0) = y_1$, $\ldots$, $x_{2^{d-1}-1}(0) = y_{2^{d-1}}$, for arbitrary real numbers $y_0, y_1, \ldots, y_{2^{d-1}}$.

Then for $n = d \cdot t + r$, $0 \leq r \leq d - 1$, the following equality is true:

$$x_0(n) = \sum_{a=0}^{2^{d-1}-1} \left( y_a \cdot \prod_{s=0}^{d-1} f_s(a, b, r) \right).$$

**Proof.** This result is a direct consequence of Lemma 2.1 and the basic properties of the system of linear recurrence equations. □

### 3. Examples

**Example 3.1.** We will now illustrate Lemma 2.1, for the case $d = 3$ and $a = 4$. Then we have the system

$$
\begin{align*}
x_0(n + 1) &= x_0(n), \\
x_1(n + 1) &= x_0(n) + x_4(n), \\
x_2(n + 1) &= x_1(n), \\
x_3(n + 1) &= x_1(n) + x_5(n), \\
x_4(n + 1) &= x_2(n), \\
x_5(n + 1) &= x_2(n) + x_6(n), \\
x_6(n + 1) &= x_3(n), \\
x_7(n + 1) &= x_3(n) + x_7(n),
\end{align*}
$$

with initial conditions $x_4(0) = 1$ and $x_0(0) = 0$ for $b \neq 4$, $0 \leq b \leq 2^d - 1 = 7$.

**Solution.** We will take case analysis on all values of $b$.

- For $a = 4$ and $b = 0$ binary form $0_2 = 000$ has more zeros than binary form $4_2 = 100$, so by the Pigeonhole principle at least one position will be $a_n = 1$ and $b_{a_n} = 0$. Then for each $n$ the equality $x_0(n) = 0$ is satisfied. These conclusions are valid whenever the binary form $(b)_2$ has more zeros than $(a)_2$. 

• For \( a = 4_2 = 100 \) and \( b = 1_2 = 001 \) when \( r = 0 \) or \( r = 2 \) we will have a position \( s \) such that \( a_s = 1 \) and \( b_{s+1} = 0 \) (for \( r = 0 \), i.e. when there is no movement, \( a_2 = 1 \) and \( b_2 = 0 \); for \( r = 2 \), i.e. when moving to the right by two positions, \( a_2 = 1 \) and \( b_{2+2} = b_1 = 0 \)). Then for \( n \equiv 0 \pmod{3} \) and \( n \equiv 2 \pmod{3} \) it is true that \( x_0(n) = 0 \).

When \( r = 1 \) we have that

\[
\begin{align*}
  a_0 &= b_{0+1} = b_1 = 0 \quad \Rightarrow \quad f_0(a, b, 1) = 1, \\
  a_1 &= b_{1+1} = b_2 = 0 \quad \Rightarrow \quad f_1(a, b, 1) = 1, \\
  a_2 &= b_{2+1} = b_0 = 1 \quad \Rightarrow \quad f_2(a, b, 1) = 1
\end{align*}
\]

and we have that \( x_1(n) = x_3(n) = f_0(a, b, 1) \cdot f_1(a, b, 1) \cdot f_2(a, b, 1) = 1 \cdot 1 \cdot 1 = 1 \) for \( n \equiv 1 \pmod{3} \).

Thus, we have shown that:

\[
x_1(n) = \begin{cases} 0, & n = 3t \\ 1, & n = 3t + 1 \\ 0, & n = 3t + 2. \end{cases}
\]

• For \( a = 4_2 = 100 \) and \( b = 3_2 = 011 \) when \( r = 0 \) we have that \( a_2 = 1 \) and \( b_{2+0} = b_2 = 0 \).

Then for \( n \equiv 0 \pmod{3} \) it is true that \( x_3(n) = 0 \).

When \( r = 1 \) we have that

\[
\begin{align*}
  a_0 &= 0, \quad b_{0+1} = b_1 = 1 \quad \text{and} \quad s = 0 < 3 - 1 = d - r \quad \Rightarrow \quad f_0(a, b, 1) = t, \\
  a_1 &= b_{1+1} = b_2 = 0 \quad \Rightarrow \quad f_1(a, b, 1) = 1, \\
  a_2 &= 1, \quad b_{2+1} = b_0 = 1 \quad \Rightarrow \quad f_2(a, b, 1) = 1
\end{align*}
\]

and we have that \( x_3(n) = x_4(n) = f_0(a, b, 1) \cdot f_1(a, b, 1) \cdot f_2(a, b, 1) = t \cdot 1 \cdot 1 = t \) for \( n \equiv 1 \pmod{3} \).

When \( r = 2 \) we have that

\[
\begin{align*}
  a_0 &= 0, \quad b_{0+2} = b_2 = 0 \quad \Rightarrow \quad f_0(a, b, 1) = 1, \\
  a_1 &= 0, \quad b_{1+2} = b_0 = 1 \quad \Rightarrow \quad f_1(a, b, 1) = t + 1, \\
  a_2 &= 1, \quad b_{2+1} = b_0 = 1 \quad \Rightarrow \quad f_2(a, b, 1) = 1
\end{align*}
\]

and we have that \( x_3(n) = x_3(n) = 1 \cdot (t + 1) \cdot 1 = t + 1 \) for \( n \equiv 2 \pmod{3} \).

Thus, we have shown that:

\[
x_3(n) = \begin{cases} 0, & n = 3t \\ t, & n = 3t + 1 \\ t + 1, & n = 3t + 2. \end{cases}
\]

• For \( a = 4_2 = 100 \) and \( b = 7_2 = 111 \) when \( r = 0 \) we have \( x_7(n) = t \cdot t \cdot 1 = t^2 \) for \( n \equiv 1 \pmod{3} \).

When \( r = 1 \) we have that \( x_7(n) = t \cdot t \cdot 1 = t^2 \) for \( n \equiv 1 \pmod{3} \).

When \( r = 2 \) we have that \( x_7(n) = t \cdot (t + 1) \cdot 1 = t(t + 1) \) for \( n \equiv 2 \pmod{3} \).

Thus, we have shown that:

\[
x_7(n) = \begin{cases} t^2, & n = 3t \\ t^2, & n = 3t + 1 \\ t(t + 1), & n = 3t + 2. \end{cases}
\]

• Analogously we obtain:

\[
x_2(n) = \begin{cases} 0, & n = 3t \\ 0, & n = 3t + 1 \\ 1, & n = 3t + 2, \end{cases} \quad x_4(n) = \begin{cases} 1, & n = 3t \\ 0, & n = 3t + 1 \\ 0, & n = 3t + 2, \end{cases}
\]

\[
x_5(n) = \begin{cases} t, & n = 3t \\ t, & n = 3t + 1 \\ 0, & n = 3t + 2, \end{cases} \quad x_6(n) = \begin{cases} t, & n = 3t \\ 0, & n = 3t + 1 \\ t, & n = 3t + 2. \end{cases}
\]
All these sequences can be found in [8]: $x_0$ is A000004, $x_1$ is shifted A079978, $x_2$ and $x_4$ are A079978, $x_3$ is A087509, $x_5$ is shifted A087508, $x_6$ is shifted A087509, $x_7$ is A008133.

This particular example can be solved by using generating functions, such as in [6]. Although generating functions and then Cramers method can be used to solve the system (♣) in general, we think that results in Lemma 2.1 and Theorem 2.2 are more straightforward.

The following discussion illustrates the connection between the system considered in the paper, and restricted permutations from [1].

Let $C_{md+1−q}$ denote the number of combinations where the smallest element is equal to $md + 1 − q$, for $q = 0, 1, \ldots, md$, and can be obtained from the initial combination $(r + 1, r + 2, \ldots, r + k + 1)$ using techniques developed in [1] (those techniques count the number of permutations that satisfy $p(i) − i \in S$ and $S = \{−d, −d + 1, \ldots, md\} \setminus \{−d, 0, md\}$). Then, $C_{md+1−q}$ is equal to

$$C_{md+1−q} = \sum_{b=0}^{2^d−1} x_b(q) = \sum_{b=0}^{2^d−1} \prod_{s=0}^{d−1} f_s(2^d−1, b, r),$$

where $q = d \cdot t + r$.

Example 3.2. Let us illustrate these considerations for the case $d = 3$ and $m = 2$ (when $k = d = 3$ and $r = md = 6$).

Solution. Then we deal with the permutations that satisfy $p(i) − i \in S$, $S = \{−3, −2, \ldots, 5, 6\} \setminus \{−3, 0, 6\} = \{−2, −1, 1, 2, 3, 4\}$, i.e. $I = \{−3, 0, 6\}$ and $r + 1 − l = 7 − l = \{10, 7, 1\}$. The number of such permutations is given in sequence A224810 at [8].

The set $C$ consists of all combinations of the set $\mathbb{N}_{k+r+1} = \{1, 2, \ldots, 10\}$, with $k + 1 = 4$ elements and containing a number $k + r + 1 = 10$. The set $C$ has $|C| = \binom{10}{4} = 84$ elements, but most of them are not relevant to the technique developed in [1], because they cannot be generated starting from the initial combination $(7, 8, 9, 10)$.

In Example 3.1 we get the values of all sequences $x_b$ that occur in the previous theorem.

For $q = 3t$ we have that

$$C_{md+1−q} = x_0(q) + x_1(q) + \ldots + x_{6}(q) + x_7(q) + x_8(q) + x_9(q) = 0 + 0 + 0 + 0 + 1 + t + t + t^2 = (t + 1)^2,$$

for $q = 3t + 1$ we have that

$$C_{md+1−q} = 0 + 1 + 0 + t + 0 + t + 0 + t^2 = (t + 1)^2,$$

for $q = 3t + 2$ we have that

$$C_{md+1−q} = 0 + 0 + 1 + (t + 1) + 0 + 0 + t + t + 1 = (t + 1)(t + 2).$$

Thus, we find that for combinations starting with $md + 1 − q$ the following equality is satisfied:

$$C_{md+1−q} = \begin{cases} (t + 1)^2, & q = 3t \\ (t + 1)^2, & q = 3t + 1 \\ (t + 1)(t + 2), & q = 3t + 2. \end{cases}$$

This sequence is A008133 at [8].

For $q = 0$ we have $(0 + 1)^2 = 1$ combination that begins with $md + 1 − q = 7$. This is the initial combination $(7, 8, 9, 10)$.

For $q = 1$ we have $(0 + 1)^2 = 1$ combination that begins with 6: $(6, 7, 8, 10)$. 

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For $q = 2$ we have $(0+1) \cdot (0+2) = 2$ combinations starting with 5: $(5, 7, 9, 10), (5, 6, 7, 10)$.
For $q = 3$ we have $(1+1)^2 = 4$ combinations starting with 4: $(4, 8, 9, 10), (4, 6, 8, 10), (4, 5, 9, 10), (4, 5, 6, 10)$.
For $q = 4$ we have $(1+1)^2 = 4$ combinations starting with 3: $(3, 7, 8, 10), (3, 5, 7, 10), (3, 4, 8, 10), (3, 4, 5, 10)$.
For $q = 5$ we have $(1+1) \cdot (1+2) = 6$ combinations starting with 2: $(2, 7, 9, 10), (2, 6, 7, 10), (2, 4, 9, 10), (2, 4, 6, 10), (2, 3, 7, 10), (2, 3, 4, 10)$.
For $q = 6$ we have $(2+1)^2 = 9$ combinations starting with 1: $(1, 8, 9, 10), (1, 6, 8, 10), (1, 5, 9, 10), (1, 5, 6, 10), (1, 3, 8, 10), (1, 3, 5, 10), (1, 2, 9, 10), (1, 2, 6, 10), (1, 2, 3, 10)$.

Altogether we have

$$1 + 1 + 2 + 4 + 4 + 6 + 9 = 27 = (m + 1)^d$$

combinations which occur in the technique developed in [1]. We get a $(m+1)^d \times (m+1)^d$ matrix as the matrix of the reduced system of linear recurrence equations. Furthermore, the generating function corresponding to the restricted permutations is a rational function $P(z)/Q(z)$. Also, the denominator $Q(z)$ is of degree less than or equal to $(m+1)^d$, i.e. $\deg Q(z) \leq (m+1)^d$, which is significantly less than $|C| = \binom{(m+1)d}{d}$, the total number of combinations that occur in technique developed in [1].

In this particular case, we have that $\deg Q(z) = 24 \leq 27 = (m + 1)^d$, because

$$A(z) = \frac{1 + z^3 - z^4 - z^5 + z^6 - 2z^7 - z^8 - 2z^{10} - z^{12} - z^{13} - z^{15}}{1 - z + z^3 - 2z^4 + 2z^6 - 4z^7 - 2z^9 - 2z^{10} - 4z^{12} + 2z^{13} - 2z^{15} + 4z^{16} + 2z^{18} + 2z^{19} + z^{21} + z^{22} + z^{24}}.$$  

The denominator of $A(z)$ is $(z-1)(z^2 + z + 1)(z^3 + z - 1)(z^4 + 3z^5 + 4z^6 + 3z^7 + 7z^8 + 9z^9 + 7z^{10} + z^{11} + 2z^{12} + 2z^{13} + 3z^{14} + z^{15})$ and the numerator is $2 - (z+1)(z^2 - z + 1)z^{15} + z^{10} + z^7 + z^6 + z^5 + z^4 - 2z^3 + 1$. It is significantly less than $|C| = \binom{(m+1)d}{d} = \binom{9}{d} = 84$.
The sequence corresponding to $A(z)$ is $A224810$ in [8].

4. Acknowledgment

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References