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Fuzzy Uniform Structures

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Abstract. The concept of fuzzy uniform structure was introduced in [7] as a fuzzy counterpart of the concept of gauge associated with a uniformity. In fact, the category of fuzzy uniform structures is isomorphic to that of uniform spaces. Here, we introduce two other concepts of fuzzy uniform structures which allow to establish two categories isomorphic to the categories of probabilistic uniform spaces and Lowen uniform spaces, respectively. This sheds light on the relationship between these fuzzy uniformities and classical uniformities. Furthermore, we obtain a factorization of Lowen's adjoint functors ω_* and ι which establish a relationship between the categories of uniform spaces and Lowen uniform spaces.

1. Introduction

In [7] (see also [8]) the authors introduced the concept of fuzzy uniform structure which can be considered as the fuzzy approach to uniformities by means of a family of pseudometrics (a *gauge*). In this way, the authors defined a fuzzy uniform structure as a pair $(\mathcal{M}, *)$ where $*$ is a continuous t-norm and $\mathcal{M}$ is a family of fuzzy pseudometrics (in the sense of Kramosil and Michalek) on a nonempty set X satisfying certain properties. Then they considered the category $\mathsf{FUnif}(*)$ of fuzzy uniform structures with respect to a t-norm $*$ and proved that it is isomorphic to the category Unif of uniform spaces.

On the other hand, probabilistic uniformities and Lowen uniformities were introduced in [9] and [16] as a fuzzy counterpart of the concept of uniformity (see, for example, [21] for a discussion about several notions of fuzzy uniformity). These spaces joint with the uniformly continuous functions form two categories which will be denoted by PUnif and LUnif respectively. In [16], Lowen introduced two adjoint functors ω_* and ι between the categories of uniform spaces and Lowen uniform spaces which show that Lowen uniformity is a suitable concept for uniformity in the fuzzy area. These functors also work considering the category of probabilistic uniform spaces (recall that Lowen uniformities are saturated probabilistic uniformities [21]).

Furthermore, in [10] it is proved that a probabilistic uniformity is probabilistic pseudometrizable if and only if it has a countable base (the corresponding result for Lowen uniformities was proved in [12]). As a consequence of these results and as it was pointed out in [21], Lowen and probabilistic uniformities can be described as a collection of probabilistic pseudometrics which is the counterpart of the classical concept of a gauge.

Nevertheless, probabilistic pseudometrics are very related with the most frequent notions of fuzzy pseudometrics [3, 14] (see also [17]) in such a way that it is usual that the same ideas and techniques can be used for proving similar results for both structures.

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Consequently, it is natural to wonder whether every probabilistic uniformity (resp. Lowen uniformity) can be obtained by means of a family of fuzzy pseudometrics which, in general, will be called a fuzzy uniform structure. Here, we show explicitly that this is true by establishing two category isomorphisms theorems (see Theorems 4.9 and 4.10). Furthermore, the study of fuzzy uniformities by their corresponding fuzzy uniform structures provide a more understandable way of establishing the relationships between these uniformities. Concretely, we show that uniform spaces are, categorically speaking, included in Lowen uniform spaces which in turn are included in probabilistic uniform spaces (Theorem 4.12). We also give a factorization of Lowen's functors ω_* and ι (Corollaries 4.8 and 4.13).

2. Uniformities and Fuzzy Uniform Structures

We start recalling some well-known facts about uniformities that will be useful later on. Our basic references are [1, 13].

Definition 2.1. *A **uniformity** on a nonempty set X is a filter $\mathcal{U}$ on $X \times X$ such that:*

(U1) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$;

(U2) $\Delta \subseteq U$ for all $U \in \mathcal{U}$ where $\Delta = \{(x, x) : x \in X\}$;

(U3) given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$ where $V^2 = \{(x, y) \in X \times X :$ there exists $z \in X$ such that $(x, z), (z, y) \in V\}$.

The pair $(X, \mathcal{U})$ is said to be a uniform space.

We will denote by Unif the topological category whose objects are the uniform spaces and whose morphisms are the uniformly continuous functions (a function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is said to be uniformly continuous if $(f \times f)^{-1}(V) \in \mathcal{U}$, for all $V \in \mathcal{V}$).

Uniformities can be defined alternatively by means of a family of pseudometrics called gauge or uniform structure.

Definition 2.2. *Let X be a nonempty set. A gauge or a **uniform structure** on X is a nonempty family $\mathcal{D}$ of pseudometrics on X such that:*

(G1) if $d, q \in \mathcal{D}$ then $d \vee q \in \mathcal{D}$;

(G2) if e is a pseudometric on X and for each $\varepsilon > 0$ there exist $d \in \mathcal{D}$ and $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$ for all $x, y \in X$, then $e \in \mathcal{D}$.

Remark 2.3. *Recall that if d is a pseudometric on a nonempty set X then it generates a uniformity $\mathcal{U}_d$ on X having as a subbase the family $\{U_{d,\varepsilon} : \varepsilon > 0\}$, where $U_{d,\varepsilon} = \{(x, y) \in X \times X : d(x, y) < \varepsilon)$. Hence, if $\mathcal{D}$ is a uniform structure on X, condition (G2) is equivalent to $e \in \mathcal{D}$ whenever $\mathcal{U}_e \subseteq \mathcal{U}_d$ for some $d \in \mathcal{D}$.*

Definition 2.4. *A **base for a uniform structure** on a nonempty set X is a nonempty family $\mathcal{B}$ of pseudometrics on X satisfying (G1).*

A base for a uniform structure $\mathcal{B}$ generates a uniform structure on X given by all the pseudometrics e on X such that for each $\varepsilon > 0$ there exist $d \in \mathcal{B}$ and $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$ for all $x, y \in X$.

If we say that a function $f : (X, \mathcal{D}) \to (Y, \mathcal{Q})$ between two spaces endowed with a uniform structure is uniformly continuous whenever given $q \in \mathcal{Q}$ we can find $d \in \mathcal{D}$ such that $f : (X, \mathcal{U}_d) \to (Y, \mathcal{U}_q)$ is uniformly continuous, then we can consider the category SUnif whose objects are the spaces endowed with a uniform structure and whose morphisms are the uniformly continuous functions. It is well-known that Unif and SUnif are isomorphic categories as the next theorem shows.

Theorem 2.5. *Let $\mathcal{U}$ and $\mathcal{D}$ be a uniformity and a uniform structure on a nonempty set X respectively. Define:*

- $\mathcal{D}_{\mathcal{U}}$ *as the family of all pseudometrics d on X such that* $\mathcal{U}_d \subseteq \mathcal{U}$;
- $\mathcal{U}_{\mathcal{D}}$ *as the uniformity* $\bigvee_{d\in\mathcal{D}} \mathcal{U}_d$.

Then the mappings:

- $\Delta : \mathsf{Unif} \to \mathsf{SUnif}$ *given by* $\Delta((X,\mathcal{U})) = (X,\mathcal{D}_{\mathcal{U}})$;
- $\Lambda : \mathsf{SUnif} \to \mathsf{Unif}$ *given by* $\Lambda((X,\mathcal{D})) = (X,\mathcal{U}_{\mathcal{D}})$;

which leave morphisms unchanged are covariant functors such that $\Delta \circ \Lambda = 1_{\mathsf{SUnif}}$ *and* $\Lambda \circ \Delta = 1_{\mathsf{Unif}}$.

In [7] the authors studied a fuzzy notion of the concept of uniform structure giving a new category isomorphic to Unif. We recall the necessary notions to establish this isomorphism.

In the sequel we wilI use the following notation: $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. Furthermore, by abuse of notation, we will use the same symbol to denote a function between two spaces when the base sets of the domain and codomain are the same although we change the structures associated with the sets.

Definition 2.6 ([19]). *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is called a* ***continuous t-norm*** *if* $([0,1],*)$ *is an Abelian topological monoid with unit 1, such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *with* $a,b,c,d \in [0,1]$.

Three distinguished examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm) which are defined as $a \wedge b = \min\{a,b\}$, $a \cdot b = ab$ and $a *_L b = \max\{a+b-1,0\}$ for all $a,b \in [0,1]$, respectively. It is well-known and easy to see that $* \leq \wedge$ for each continuous t-norm $*$.

Definition 2.7 ([7]). *A* ***fuzzy pseudometric*** *(in the sense of Kramosil and Michalek) on a nonempty set X is a pair* $(M,*)$ *such that* $*$ *is a continuous t-norm and M is a fuzzy set in* $X \times X \times [0,+\infty)$ *such that*

(FM1) $M(x,y,0) = 0$;

(FM2) $M(x,x,t) = 1$;

(FM3) $M(x,y,t) = M(y,x,t)$;

(FM4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(FM5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ *is left continuous;*

for every $x,y,z \in X$ *and* $t,s > 0$.
If the fuzzy pseudometric $(M,*)$ *also satisfies:*

(F2′) $M(x,y,t) = 1$ *for all* $t > 0$ *if and only if* $x = y$

then $(M,*)$ *is said to be a* ***fuzzy metric*** *on X [14].*

A ***fuzzy (pseudo)metric*** *space is a triple* $(X,M,*)$ *such that X is a nonempty set and* $(M,*)$ *is a fuzzy (pseudo)metric on X.*

Remark 2.8. *Fuzzy metrics are very related with probabilistic metrics as considered in [9, 10]. In fact, if* $(X,M,*)$ *is a fuzzy metric space, given* $x,y \in X$ *then* $M(x,y,\cdot)$ *is a distribution function on* $[0,+\infty)$ *whenever* $\lim_{t\to\infty} M(x,x,t) = 1$. *In this way, the modern definition of fuzzy metric in the sense of Kramosil and Michalek [14] that we have presented above, is a translation of the axioms of the probabilistic metrics in terms of fuzzy sets instead of distribution functions. The only significative difference is that probabilistic metrics require that* $\lim_{t\to\infty} M(x,x,t) = 1$ *for every* $(x,y) \in X\times X$. *Consequently, every probabilistic metric is a fuzzy metric.*

Remark 2.9. *Every fuzzy pseudometric* $(M,*)$ *on a nonempty set X generates a topology* τ_M *on X which has as a base the family* $\{B_M(x,\varepsilon,t) : x \in X, \varepsilon \in (0,1), t > 0\}$ *where* $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$. *Furthermore (cf. [5]) every fuzzy metric space* $(X,M,*)$ *is metrizable and it possesses a compatible uniformity* $\mathcal{U}_M$ *with a countable base given by*

$$U_n^M = \{(x,y) \in X \times X : M(x,y,1/n) > 1 - 1/n\}$$

(we will omit the superscript M if no confusion arises).

Definition 2.10 ([4]). *A function $f : (X, M, *) \to (Y, N, \star)$ between two fuzzy metric spaces is said to be* ***uniformly continuous*** *if for every $\varepsilon \in (0, 1)$ and $t > 0$ there exist $\delta \in (0, 1)$ and $s > 0$ such that*

$$\text{if } M(x, y, s) > 1 - \delta \text{ then } N(f(x), f(y), t) > 1 - \varepsilon$$

where $x, y \in X$.

This is equivalent to assert that $f : (X, \mathcal{U}_M) \to (Y, \mathcal{U}_N)$ is uniformly continuous.

We will denote by FMet the category whose objects are the fuzzy pseudometric spaces and whose morphisms are the uniformly continuous functions. Furthermore, FMet($*$) indicates the full subcategory of FMet made up of the fuzzy pseudometric spaces with respect to a fixed continuous t-norm $*$.

Example 2.11 ([7], cf. [3]). *Let (X, d) be a pseudometric space. Let M_d be the fuzzy set on $X \times X \times [0, \infty)$ given by*

$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d, *)$ is a fuzzy pseudometric on X which is called the standard fuzzy pseudometric induced by d.

Furthermore, we notice that $\mathcal{U}_d = \mathcal{U}_{M_d}$ (cf. [6, Lemma 5]) where $\mathcal{U}_d$ is the uniformity generated by d.

Denoting by Met *the category of pseudometric spaces endowed with the uniformly continuous functions, then for every continuous t-norm $*$ we can define a fully faithful covariant functor $\mathfrak{F}_* :$ Met $\to$ FMet($*$) such that $\mathfrak{F}_*((X, d)) = (X, M_d, *)$ and leaving morphisms unchanged (see [6, Lemmas 1 and 5]).*

Remark 2.12. *If $(M, *)$ is a fuzzy (pseudo)metric on X we will denote by M_t the function on $X \times X$ given by $M_t(x, y) = M(x, y, t)$ for all $t > 0$.*

Definition 2.13. *A* ***base of fuzzy pseudometrics*** *on a nonempty set X is a pair $(\mathcal{B}, *)$ where $*$ is a continuous t-norm and $\mathcal{B}$ is family of fuzzy pseudometrics on X with respect to the t-norm $*$ closed under finite infimum.*

If no confusion arises, we will write $M \in \mathcal{B}$ whenever $(M, *) \in \mathcal{B}$.

We introduce some operators applicable to a base of fuzzy pseudometrics which will be useful later on.

Definition 2.14. *Let $(\mathcal{B}, *)$ be a base of fuzzy pseudometrics on a nonempty set X. We define:*

- $\langle \mathcal{B} \rangle = \{(N, *) \in$ FMet($*$) : *for all $t > 0$ there exists $M \in \mathcal{B}$ and $s > 0$ such that $M_s \leq N_t$*$\}$.
- $\widetilde{\mathcal{B}} = \{(N, *) \in$ FMet($*$) : *for all $\varepsilon \in I_0$ and $t > 0$ there exist $s > 0, M \in \mathcal{B}$ such that $M_s - \varepsilon \leq N_t$*$\}$.
- $\widehat{\mathcal{B}} = \{(N, *) \in$ FMet($*$) : *for all $\varepsilon \in I_0$ and $t > 0$ there exist $\delta \in I_0, s > 0, M \in \mathcal{B}$ such that $M(x, y, s) > 1 - \delta$ implies $N(x, y, t) > 1 - \varepsilon$*$\}$.

Lemma 2.15. *Let $(\mathcal{B}, *)$ be a base of fuzzy pseudometrics on a nonempty set X. Then:*

$$\mathcal{B} \subseteq \langle \mathcal{B} \rangle \subseteq \widetilde{\mathcal{B}} \subseteq \widehat{\mathcal{B}}.$$

Furthermore, all these operators are idempotent.

Proof. The two first inclusions are obvious. For the third one, let $(N, *) \in \widetilde{\mathcal{B}}$, $\varepsilon \in I_0$ and $t > 0$. Given $0 < \delta < \varepsilon$ we can find $s > 0$ and $M \in \mathcal{B}$ such that $M_s - \delta/2 \leq N_t$. If $1 - \delta/2 < M(x, y, s)$ then

$$1 - \varepsilon < 1 - \delta = 1 - \frac{\delta}{2} - \frac{\delta}{2} < M(x, y, s) - \frac{\delta}{2} < N(x, y, t)$$

which proves the last inclusion.

On the other hand, it is obvious that $\langle\langle \mathcal{B} \rangle\rangle = \langle \mathcal{B} \rangle$ and an easy computation shows $\widehat{\widehat{\mathcal{B}}} = \widehat{\mathcal{B}}$. Lowen [16, Proposition 1.3] proved that $\widetilde{\widetilde{\mathcal{B}}} = \widetilde{\mathcal{B}}$. $\square$

Definition 2.16 ([7]). *Let X be a nonempty set and let $*$ be a continuous t-norm. A* ***fuzzy uniform structure*** *for $*$ is base of fuzzy pseudometrics $(\mathcal{M}, *)$ on X such that:*

$$\widehat{\mathcal{M}} = \mathcal{M}.$$

A ***fuzzy uniform space*** *is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.*

We will also call a base of fuzzy pseudometrics $(\mathcal{B}, *)$ on X as ***base for a fuzzy uniform structure*** *on X since $(\widehat{\mathcal{B}}, *) = (\mathcal{M}_{\mathcal{B}}, *)$ is a fuzzy uniform structure on X.*

This concept of fuzzy uniform structure must not be confused with that considered in [2], which is another notion of uniformity in the fuzzy context.

Remark 2.17. *Observe that if $(\mathcal{B}, *)$ is a base for a fuzzy uniform structure on X then*

$$\mathcal{M}_{\mathcal{B}} = \{(N, *) \in \mathsf{FMet}(*) : \mathcal{U}_N \subseteq \bigvee_{M \in \mathcal{B}} \mathcal{U}_M\}$$

so

$$\bigvee_{M \in \mathcal{M}_{\mathcal{B}}} \mathcal{U}_M = \bigvee_{M \in \widehat{\mathcal{B}}} \mathcal{U}_M = \bigvee_{M \in \mathcal{B}} \mathcal{U}_M.$$

Consequently, and as it was pointed out in [7, Proposition 3.4], every fuzzy uniform structure $(\mathcal{M}, *)$ on a nonempty set X induces a uniformity $\mathcal{U}_{\mathcal{M}}$ on X given by

$$\mathcal{U}_{\mathcal{M}} = \bigvee_{M \in \mathcal{M}} \mathcal{U}_M.$$

This uniformity has as a base the family $\{U_{M,\varepsilon,t} : (M, *) \in \mathcal{M}, \varepsilon \in (0,1), t > 0\}$ where $U_{M,\varepsilon,t} = \{(x, y) \in X \times X : M(x, y, t) > 1 - \varepsilon\}$.

The following definition was given in [7] under the name *fuzzy uniformly continuous function*. Nevertheless, we change this terminology by reasons that will be clarified later.

Definition 2.18 ([7]). *Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be* ***uniformly continuous*** *if for each $N \in \mathcal{N}$, $\varepsilon \in (0,1)$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in (0,1)$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$.*

Remark 2.19. *Notice that a function $f : (X, \mathcal{M}, *) \to (Y, \mathcal{N}, \star)$ between two fuzzy uniform spaces is uniformly continuous if and only if $f : (X, \mathcal{U}_{\mathcal{M}}) \to (Y, \mathcal{U}_{\mathcal{N}})$ so is. Furthermore, if $(\mathcal{B}, *)$ is a base for a fuzzy uniform structure on X then $(\mathcal{M}_{\mathcal{B}}, *)$ is the largest fuzzy uniform structure on X such that* id $: (X, \mathcal{U}_{\mathcal{B}}) \to (X, \mathcal{U}_{\mathcal{M}_{\mathcal{B}}})$ *is uniformly continuous.*

Then we can consider the category FUnif whose objects are the fuzzy uniform spaces and whose morphisms are the uniformly continuous functions. Besides, if $*$ is a continuous t-norm, we denote by FUnif$(*)$ the full subcategory of FUnif whose objects are the fuzzy uniform spaces of the form $(X, \mathcal{M}, *)$. This is a topological category [7, Corollary 3.15]. In fact, if $\{(X_i, \mathcal{M}_i, *) : i \in I\}$ is a family of uniform spaces and X is a nonempty set, given a family of functions $f_i : X \to X_i$, we can endow X with the uniform structure generated by the family of fuzzy pseudometrics $\{(M_{f_i}, *) : (M, *) \in \mathcal{M}_i, i \in I\}$ where $M_{f_i}(x, y, t) = M(f_i(x), f_i(y), t)$ for all $x, y \in X$ and all $t \geq 0$.

In [7] it is proved that the category FUnif$(*)$ is isomorphic to Unif as follows:

Theorem 2.20 ([7]). *Let $(X, \mathcal{U})$ be a uniform space and $(X, \mathcal{M}, *)$ be a fuzzy uniform space. Let us consider:*

- *$(\varphi_*(\mathcal{D}_{\mathcal{U}}), *)$ the fuzzy uniform structure on X which has as a base the family $\{(M_d, *) : d \in \mathcal{D}_{\mathcal{U}}\}$ where $\mathcal{D}_{\mathcal{U}}$ is the gauge of $\mathcal{U}$, i. e. $\varphi_*(\mathcal{D}_{\mathcal{U}}) = \{(M, *) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \mathcal{U}\}$;*

- *$\psi(\mathcal{M})$ is the family of all pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}_{\mathcal{M}}$.*

Then:

(i) $\Phi_* : \mathsf{Unif} \to \mathsf{FUnif}(*)$ *is a covariant functor sending each* $(X, \mathcal{U})$ *to* $(X, \varphi_*(\mathcal{D}_{\mathcal{U}}), *)$*;*

(ii) $\Psi : \mathsf{FUnif}(*) \to \mathsf{Unif}$ *is a covariant functor sending each* $(X, \mathcal{M}, *)$ *to* $(X, \mathcal{U}_{\mathcal{M}}) = (X, \mathcal{U}_{\psi(\mathcal{M})})$*;*

(iii) $\Phi_* \circ \Psi = 1_{\mathsf{FUnif}(*)}$ *and* $\Psi \circ \Phi_* = 1_{\mathsf{Unif}}$.

Remark 2.21. *Observe that* $\Phi_* = \widehat{\mathfrak{F}_* \circ \Delta}$.

3. Probabilistic Uniform Structures

Probabilistic uniformities were first considered by Höhle and Katsaras [9, 11] as a fuzzy counterpart of uniformities. Lowen introduced in [16] for the t-norm $\wedge$ a different type of fuzzy uniformities, now called Lowen uniformities or Lowen-Höhle uniformities [21], which were also studied by Höhle [10] for an arbitrary t-norm. Lowen [16] (see Theorem 3.11) provided a pair of adjoint functors ω_* and ι between the category Unif and the category PUnif($*$) of probabilistic uniform spaces with respect to a fixed continuous t-norm $*$. The functor ω_* is injective on objects so, in some sense, we can consider Unif as a subcategory of PUnif($*$). We will clarify this assertion by introducing two new kinds of uniform structures in the fuzzy context: the probabilistic uniform structures and the Lowen uniform structures. These structures contain properly all the fuzzy uniform structures as defined in the above section. Furthermore, as we will see in the next section, the category of probabilistic uniformities (resp. Lowen uniformities) is isomorphic to the category of probabilistic uniform structures (resp. Lowen uniform structures).

We begin with some definitions.

Definition 3.1 ([15, 16]). *Let X be a nonempty set.*

- *A **prefilter** $\mathcal{F}$ on X is a filter on the lattice I^X.*
- *A **prefilter base** $\mathcal{B}$ on X is filter base on the lattice I^X. We denote (cf. Definition 2.14)*

$$\langle B\rangle = \{F \in I^X : B \leq F \text{ for some } B \in \mathcal{B}\}.$$

- *A prefilter $\mathcal{F}$ on X is said to be **saturated** if for every $\{F_\varepsilon : \varepsilon \in I_0\} \subseteq \mathcal{F}$ we have that $\sup_{\varepsilon\in I_0}(F_\varepsilon - \varepsilon) \in \mathcal{F}$.*
- *Given a prefilter $\mathcal{F}$ on X we define (cf. Definition 2.14)*

$$\widetilde{\mathcal{F}} = \{\sup_{\varepsilon\in I_0}(F_\varepsilon - \varepsilon) : (F_\varepsilon)_{\varepsilon\in I_0} \in \mathcal{F}^{I_0}\}.$$

*It is straightforward to see that $\widetilde{\mathcal{F}}$ is a saturated prefilter called the **saturation** of $\mathcal{F}$.*

Definition 3.2 ([9, Definition 2.1],[11]). *A **probabilistic $*$-uniformity** on a nonempty set X is a pair $(\mathfrak{U}, *)$, where $*$ is a continuous t-norm and $\mathfrak{U}$ is a prefilter on $X \times X$ such that:*

(PU1) $U(x, x) = 1$ *for all* $U \in \mathfrak{U}$*;*

(PU2) *if* $U \in \mathfrak{U}$ *then* $U^{-1} \in \mathfrak{U}$ *where* $U^{-1}(x, y) = U(y, x)$*;*

(PU3) *for each* $U \in \mathfrak{U}$ *there exists* $V \in \mathfrak{U}$ *such that*

$$V^2 \leq U$$

where $V^2(x, y) = \sup_{z\in X} V(x, z) * V(z, y)$.

In this case we say that $(X, \mathcal{U}, *)$ is a ***probabilistic $*$-uniform space****. In general, we will not make reference to the t-norm $*$ if no confusion arises.*

Probabilistic uniformities are called Höhle-Katsaras uniformities in [21].

Definition 3.3. *A pair $(\mathcal{B}, *)$ is said to be a* ***probabilistic $*$-uniform base*** *on a nonempty set X if $\mathcal{B}$ is a prefilter base and:*

(BPU1) $U(x, x) = 1$ for all $U \in \mathcal{B}$ and all $x \in X$;

(BPU2) for all $U \in \mathcal{B}$ we can find $V \in \mathcal{B}$ such that $V^2 \leq U^{-1}$.

The probabilistic uniformity $(\mathcal{U}_{\mathcal{B}}, *)$ generated by $(\mathcal{B}, *)$ is defined as

$$\mathcal{U}_{\mathcal{B}} = \langle \mathcal{B} \rangle = \{U \in I^{X \times X} : B \leq U \text{ for some } B \in \mathcal{B}\}.$$

Definition 3.4. *A function $f : (X, \mathcal{U}, *) \to (Y, \mathcal{V}, \star)$ between two probabilistic uniform spaces is said to be* ***fuzzy uniformly continuous*** *if $(f \times f)^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$, i. e. for every $V \in \mathcal{V}$ we can find $U \in \mathcal{U}$ such that*

$$U(x, y) \leq V(f(x), f(y)) \text{ for all } x, y \in X.$$

Then we can consider the category PUnif whose objects are the probabilistic uniform spaces and whose morphisms are the fuzzy uniformly continuous functions. For a fixed continuous t-norm $*$, PUnif($*$) is the full subcategory of PUnif whose objects are the probabilistic uniform spaces with respect to the continuous t-norm $*$.

In 1981, Lowen introduced, for the t-norm $\wedge$, the following notion of fuzzy uniformity which is very related with that of probabilistic uniformity.

Definition 3.5 ([16], cf. [10, Definition 2.3]). *A* ***Lowen $*$-uniformity*** *on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:*

(LU1) $U(x, x) = 1$ for all $U \in \mathcal{U}$;

(LU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x, y) = U(y, x)$;

(LU3) for each $U \in \mathcal{U}$ and each $\varepsilon \in I_0$ there exists $V \in \mathcal{U}$ such that

$$V^2 - \varepsilon \leq U,$$

where $V^2(x, y) = \sup_{z \in X} V(x, z) * V(z, y)$;

(LU4) $\mathcal{U}$ is saturated, i. e. $\widetilde{\mathcal{U}} = \mathcal{U}$.

In this case we say that $(X, \mathcal{U}, *)$ is a ***Lowen $*$-uniform space*** *(we will omit the t-norm $*$ if no confusion arises).*

Definition 3.6. *A* ***Lowen $*$-uniform base*** *on a nonempty set X is a pair $(\mathcal{B}, *)$, where $*$ is a continuous t-norm and $\mathcal{B}$ is a prefilter base on $X \times X$ such that:*

(BLU1) $U(x, x) = 1$ for all $U \in \mathcal{B}$ and all $x \in X$;

(BLU2) for all $U \in \mathcal{B}$ and all $\varepsilon \in I_0$ we can find $V \in \mathcal{B}$ such that $V^2 - \varepsilon \leq U^{-1}$.

The Lowen uniformity $(\widetilde{\mathcal{U}_{\mathcal{B}}}, *)$ generated by $(\mathcal{B}, *)$ is defined as

$$\widetilde{\mathcal{U}_{\mathcal{B}}} = \widetilde{\langle \mathcal{B} \rangle} = \{U \in I^{X \times X} : \text{ for all } \varepsilon \in I_0 \text{ there exists } V_\varepsilon \in \mathcal{B} \text{ such that } V_\varepsilon - \varepsilon \leq U\}.$$

As in the case of probabilistic uniform spaces, we can consider the category LUnif whose objects are the Lowen uniform spaces and whose morphisms are the uniformly continuous functions as defined in the obvious way (see [16, Definition 2.4]). For a fixed continuous t-norm $*$, LUnif($*$) is the full subcategory of LUnif whose objects are the Lowen $*$-uniform spaces.

Remark 3.7. *It is obvious that every saturated probabilistic uniformity is a Lowen uniformity. In [10, Lemma 2.4, Remark 2.5] it is proved that a Lowen $\wedge$-uniformity is a probabilistic $\wedge$-uniformity. Later on, Katsaras [12, Corollary 3.5] showed that every Lowen $\wedge$-uniformity has a probabilistic uniform base. Consequently, Lowen $\wedge$-uniformities are precisely the saturated probabilistic $\wedge$-uniformities and this can be extended to an arbitrary continuous t-norm $*$ (see [21]). In fact,* LUnif *is a coreflective full subcategory of* PUnif *and the coreflector is the functor $\mathcal{S}$ which assigns to every probabilistic uniformity $(\mathcal{U},*)$ its saturation $(\widetilde{\mathcal{U}},*)$ and which leaves morphisms unchanged [21, Corollary 4.5]. Observe that this functor is well-constructed since if $f:(X,\mathcal{U},*)\to(Y,\mathcal{V},\star)$ is a fuzzy uniformly continuous function between two probabilistic uniform spaces and $V\in\widetilde{\mathcal{V}}$, then for each $\varepsilon\in I_0$ we can find $V_\varepsilon\in\mathcal{V}$ such that $V_\varepsilon-\varepsilon\le V$. By hypothesis, there exists $U_\varepsilon\in\mathcal{U}$ with $U_\varepsilon\le(f\times f)^{-1}(V_\varepsilon)$ for all $\varepsilon\in I_0$. Hence $U_\varepsilon-\varepsilon\le(f\times f)^{-1}(V_\varepsilon)-\varepsilon\le(f\times f)^{-1}(V)$ for all $\varepsilon\in I_0$ so $(f\times f)^{-1}(V)\in\widetilde{\mathcal{U}}$. Consequently, $f:(X,\widetilde{\mathcal{U}},*)\to(Y,\widetilde{\mathcal{V}},\star)$ is fuzzy uniformly continuous.*

Remark 3.8. *If $(X,M,*)$ is a fuzzy pseudometric space then the family $\{M_t:t>0\}$, is a base for a probabilistic uniformity $(\mathcal{U}_M,*)$ on X [9, Theorem 3.3] and for a Lowen uniformity $(\widetilde{\mathcal{U}_M},*)$ [10, Theorem 2.6] which is obviously the saturation of $(\mathcal{U}_M,*)$.*

Furthermore, if $(\mathcal{B},*)$ is a base of fuzzy pseudometrics then

$$\langle\mathcal{B}\rangle=\{(M,*)\in\mathsf{FMet}(*):\mathcal{U}_M\subseteq\bigvee_{(N,*)\in\mathcal{B}}\mathcal{U}_N\},$$

$$\widetilde{\mathcal{B}}=\{(M,*)\in\mathsf{FMet}(*):\mathcal{U}_M\subseteq\widetilde{\left(\bigvee_{(N,*)\in\mathcal{B}}\mathcal{U}_N\right)}\}$$

$$=\{(M,*)\in\mathsf{FMet}(*):\widetilde{\mathcal{U}_M}\subseteq\widetilde{\left(\bigvee_{(N,*)\in\mathcal{B}}\mathcal{U}_N\right)}\}.$$

The following result is a translation of [9, Theorem 3.5], proved for probabilistic pseudometrics, to the language of fuzzy pseudometrics.

Proposition 3.9 ([9, Theorem 3.5],[12, Theorem 3.4]). *Let $(X,\mathcal{U},*)$ be a probabilistic uniform space and let $U\in\mathcal{U}$. Then there exists a fuzzy pseudometric $(M_U,*)$ on X such that:*

(1) $M_{U,t}\in\mathcal{U}$ *for all* $t>0$;

(2) $M_{U,\frac{1}{4}}\le U$.

On his behalf, Katsaras [12, Theorem 3.4] proved a similar result for Lowen $\wedge$-uniformities which can be extended to an arbitrary continuous t-norm [21]. It can be reformulated in terms of fuzzy pseudometrics as follows.

Proposition 3.10 ([12, Theorem 3.4], [21, Theorem 4.4]). *Given a Lowen uniformity $(\mathcal{U},*)$ on a nonempty set X and $U\in\mathcal{U}$ there exists a fuzzy pseudometric $(M,*)$ on X such that $U\in\widetilde{\mathcal{U}_M}\subseteq\mathcal{U}$.*

In [16], Lowen defined two functors to establish a relation between classical uniformities and Lowen uniformities as follows:

Theorem 3.11 ([16]). *Let X be a nonempty set, $\mathcal{U}$ be a uniformity on X and $(\mathfrak{U}, *)$ be a Lowen uniformity on X. Define*

$$\omega(\mathcal{U}) = \{F \in I^{X\times X} : F^{-1}((\varepsilon, 1]) \in \mathcal{U} \text{ for all } \varepsilon \in I_1\}$$

and

$$\iota(\mathfrak{U}) = \{U^{-1}((\varepsilon, 1]) : U \in \mathfrak{U}, \varepsilon \in I_1\}.$$

Then:

1. *$(\omega(\mathcal{U}), *)$ is a Lowen uniformity on X;*
2. *$\iota(\mathfrak{U})$ is a uniformity on X;*
3. *$\iota(\omega(\mathcal{U})) = \mathcal{U}$;*
4. *$(\omega(\iota(\mathfrak{U})), *)$ is the coarsest Lowen uniformity generated by a uniformity and which is finer than $\mathfrak{U}$.*

Furthermore, if we consider the mappings $\omega_* : \mathsf{Unif} \to \mathsf{LUnif}(*)$ and $\iota : \mathsf{LUnif} \to \mathsf{Unif}$, which leave morphisms unchanged, and $\omega_*((X, \mathcal{U})) = (X, \omega(\mathcal{U}), *)$ and $\iota((X, \mathfrak{U}, *)) = (X, \iota(\mathfrak{U}))$ then they are fully faithful and faithful functors respectively. Therefore Unif *is isomorphic to a full subcategory of* LUnif(∗).

Remark 3.12. *We observe that ω as well as ι are join-preserving [16, Lemma 4.6, Theorem 4.7, Theorem 4.8].*

If $(X, M, *)$ is a fuzzy pseudometric space then $(\omega(\mathcal{U}_M), *)$ is a Lowen uniformity. On the other hand, $(\widetilde{\mathfrak{U}_M}, *)$ (see Remark 3.8) is another Lowen uniformity associated with the fuzzy pseudometric $(M, *)$. It is natural to wonder whether $\omega(\mathcal{U}_M) = \widetilde{\mathfrak{U}_M}$. Obviously $\widetilde{\mathfrak{U}_M} \subseteq \omega(\mathcal{U}_M)$ but in general the inclusion is strict as the next example shows.

Example 3.13 (cf. [18]). *Let us consider on $X = [3, +\infty)$ the fuzzy metric $(M, *_L)$ given by*

$$M(x, y, t) = \begin{cases} 1 & \text{if } x = y, t > 0 \\ \frac{1}{x} + \frac{1}{y} & \text{if } x \neq y, t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$ and $t \geq 0$.

It is straightforward to check that $\tau(M)$ is the discrete topology since $B_M(x, \varepsilon, t) = \{x\}$ whenever $0 < \varepsilon < \frac{1}{3}$. Let us consider $F \in I^{X\times X}$ given by $F(x, y) = 1_\Delta$ where Δ is the diagonal of X. Obviously $F \in \omega(\mathcal{U}_M)$. Nevertheless, $F \notin \widetilde{\mathfrak{U}_M}$. In fact, $M(3, 4, t) = \frac{7}{12}$ for all $t > 0$ so given $0 < \varepsilon < \frac{7}{12}$ we have that $M_t(3, 4) - \varepsilon > 0$ but $F(3, 4) = 0$. Hence $M_t - \varepsilon \nleq F$ for all $t > 0$ and $F \notin \widetilde{\mathfrak{U}_M}$.

However, by using the next lemma, we can prove that $\omega(\mathcal{U}_d) = \widetilde{\mathfrak{U}_{M_d}}$ for a pseudometric d.

Lemma 3.14. *Let $(X, \mathcal{U})$ be a uniform space. Given $F \in \omega(\mathcal{U})$ and $\mathcal{B}_\mathcal{U}$ a base for the uniform structure $\mathcal{D}_\mathcal{U}$ of $\mathcal{U}$, there exist a family of pseudometrics $\{d_\varepsilon : \varepsilon \in I_0\} \subseteq \mathcal{B}_\mathcal{U}$ and a family of positive real numbers $\{t_\varepsilon : \varepsilon \in I_0\}$ such that*

$$\sup_{\varepsilon \in I_0} M_{d_\varepsilon, t_\varepsilon} - \varepsilon \leq F.$$

Proof. Given $F \in \omega(\mathcal{U})$ and $\varepsilon \in (0, 1)$, we have that $F^{-1}((\varepsilon, 1]) \in \mathcal{U}$. Hence there is $d_\varepsilon \in \mathcal{D}_\mathcal{U}$ and $\delta_\varepsilon > 0$ such that $d_\varepsilon(x, y) < \delta_\varepsilon$ implies $F(x, y) > \varepsilon$ for all $x, y \in X$. Let $t_\varepsilon = \frac{\varepsilon \delta_\varepsilon}{1-\varepsilon}$. An easy computation shows that if $M_{d_\varepsilon}(x, y, t_\varepsilon) > \varepsilon$ then $d_\varepsilon(x, y) < \delta_\varepsilon$. We assert that $M_{d_\varepsilon, t_\varepsilon} - \varepsilon \leq F$. Given $x, y \in X$, if $M_{d_\varepsilon, t_\varepsilon}(x, y) \leq \varepsilon$ then $M_{d_\varepsilon, t_\varepsilon}(x, y) - \varepsilon \leq 0 \leq F(x, y)$. If $M_{d, t_\varepsilon}(x, y) > \varepsilon$ then $M_{d_\varepsilon, t_\varepsilon}(x, y) - \varepsilon \leq 1 - \varepsilon < F(x, y)$ since $d_\varepsilon(x, y) < \delta_\varepsilon$.

If $\varepsilon = 1$ it is enough to take M_{d_1, t_1} as $M_{d_{\varepsilon_0}, t_{\varepsilon_0}}$ for a fixed $\varepsilon_0 \in (0, 1)$. □

Proposition 3.15. *Let (X, d) be a metric space. Then*

$$\omega(\mathcal{U}_d) = \widetilde{\mathfrak{U}_{M_d}}.$$

Proof. It is obvious that, by construction of $\mathcal{U}_d = \mathcal{U}_{M_d}$, $M_{d,t} \in \omega(\mathcal{U}_d)$ so $\widetilde{\mathfrak{U}_{M_d}} \subseteq \omega(\mathcal{U}_d)$ (see [16, Proposition 1.3]). Now, let us consider $F \in \omega(\mathcal{U}_d)$. Since $\{d\}$ is a base for $\mathcal{D}_{\mathcal{U}_d}$, applying the above lemma we immediately deduce that $F \in \widetilde{\mathfrak{U}_{M_d}}$. □

As we have observed in Remark 2.19, the notion of uniformly continuous function between fuzzy uniform structures is a translation to the fuzzy context of the notion of uniformly continuous function between uniform spaces. Consequently, in some sense, this definition is not fuzzy at all. In contrast with this definition we have that of fuzzy uniform continuity for probabilistic uniform spaces. We give a similar definition in the context of fuzzy pseudometric spaces.

Definition 3.16. *Let $(X, M, *)$ and $(Y, N, \star)$ be two fuzzy (pseudo)metric spaces. A mapping $f : X \to Y$ is said to be **fuzzy uniformly continuous** if for every $t > 0$ we can find $s > 0$ such that*

$$M(x, y, s) \leq N(f(x), f(y), t)$$

for all $x, y \in X$.

We observe that this definition appears in [20] under the name continuity for functions between fuzzy pseudometric spaces in the sense of George and Veeramani. On the other hand, it is obvious that every fuzzy uniformly continuous function between two fuzzy pseudometric spaces is uniformly continuous but the converse is not true as the following example shows:

Example 3.17. *Let e be the euclidean metric on $\mathbb{R}$. Let us consider the two following fuzzy metric spaces $(\mathbb{R}, N, \cdot)$ and $(\mathbb{R}, M_e, \cdot)$ where N is the fuzzy set (cf. [18, Proposition 3.2]) given by*

$$N(x, y, t) = \begin{cases} 0 & \text{if } t = 0 \\ \max\{M_e(x, y, t), 1/2\} & \text{if } t > 0 \end{cases}.$$

It is obvious that the identity function $\mathrm{id} : (\mathbb{R}, N, \cdot) \to (\mathbb{R}, M_e, \cdot)$ is uniformly continuous. In fact, let $\varepsilon \in (0, 1)$ and $t > 0$. Let $\delta > 0$ with $1 - \delta > \max\{1/2, 1 - \varepsilon\}$. Then if $N(x, y, t) > 1 - \delta$ we have that $M_e(x, y, t) > 1 - \delta \geq 1 - \varepsilon$ so the conclusion follows (we observe that, in fact, $\mathcal{U}_N = \mathcal{U}_{M_e} = \mathcal{U}_e$).

However, fix $t > 0$. Let $x, y \in \mathbb{R}$ such that $e(x, y) > t$. Hence

$$M_e(x, y, t) = \frac{t}{t + e(x, y)} < \frac{t}{t + t} = \frac{1}{2}.$$

Since $N(x, y, s) \geq \frac{1}{2}$ for all $s > 0$ we have that $N(x, y, s) \nleq M_e(\mathrm{id}(x), \mathrm{id}(y), t)$ for all $s > 0$ so id is not fuzzy uniformly continuous.

We next introduce two new fuzzy uniform structures which, as we will see in the next section, are very related with probabilistic and Lowen uniformities.

Definition 3.18. *Let X be a nonempty set and let $*$ be a continuous t-norm. A **probabilistic $*$-uniform structure** (resp. **Lowen $*$-uniform structure**) on X is base of fuzzy pseudometrics $(\mathcal{M}, *)$ on X such that*

$$\langle \mathcal{M} \rangle = \mathcal{M}$$

$$(\textit{resp. } \widetilde{\mathcal{M}} = \mathcal{M}).$$

A space with a probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) on X (the t-norm $*$ will be omitted if no confusion arises).

Remark 3.19. *It is obvious that every fuzzy uniform structure is a Lowen uniform structure which in turn is a probabilistic uniform structure (see Lemma 2.15).*

Definition 3.20. *Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two spaces endowed with two probabilistic uniform structures (resp. Lowen uniform structures). A mapping $f : X \to Y$ is said to be* ***fuzzy uniformly continuous*** *if for every $(N, \star) \in \mathcal{N}$ and $t > 0$ there exist $(M, *) \in \mathcal{M}$ and $s > 0$ such that $M(x, y, s) \le N(f(x), f(y), t)$ for all $x, y \in X$.*

In the sequel, we will denote by PSUnif (resp. LSUnif) the category whose objects are the spaces with a probabilistic uniform structure (resp. Lowen uniform structure) and whose morphisms are the fuzzy uniformly continuous functions. PSUnif($*$) (resp. LSUnif($*$)) will denote the full subcategory of PSUnif (resp. LSUnif) whose objects are the spaces with a probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) where $*$ is a fixed continuous t-norm $*$.

Definition 3.21. *A* ***base for a probabilistic $*$-uniform structure*** *(resp.* ***base for a Lowen $*$-uniform structure****) on a nonempty set X is simply a base of fuzzy pseudometrics $(\mathcal{B}, *)$ on X since it generates on X a probabilistic $*$-uniform structure $(\langle\mathcal{B}\rangle, *)$ (resp. Lowen $*$-uniform structure $(\widetilde{\mathcal{B}}, *)$) that we will denote by $(\mathcal{M}_{\mathcal{B}}, *)$ (resp. $(\widetilde{\mathcal{M}_{\mathcal{B}}}, *)$).*

Proposition 3.22. *If $(\mathcal{B}, *)$ is base of fuzzy pseudometrics on X then*

$$\mathcal{M}_{\mathcal{B}} \subseteq \widetilde{\mathcal{M}_{\mathcal{B}}} \subseteq \boldsymbol{\mathcal{M}}_{\mathcal{B}}.$$

Proof. This is a direct consequence of Lemma 2.15. □

The inclusions of the above proposition are in general strict as the next examples show.

Example 3.23. *Consider the fuzzy metrics of Example 3.17, and $\mathcal{B} = \{(N, \cdot)\}$. Then $(M_e, \cdot) \in (\boldsymbol{\mathcal{M}}_{\mathcal{B}}, \cdot)$ but $(M_e, \cdot) \notin (\widetilde{\mathcal{M}_{\mathcal{B}}}, \cdot)$. Hence $\boldsymbol{\mathcal{M}}_{\mathcal{B}} \nsubseteq \widetilde{\mathcal{M}_{\mathcal{B}}}$.*

Example 3.24. *Let us consider the real line $\mathbb{R}$ endowed with the euclidean metric e. For each $\varepsilon \in I_1$ let us define the fuzzy metric $(M_e^\varepsilon, \cdot)$ on $\mathbb{R}$ given by*

$$M_e^\varepsilon(x, y, t) = \begin{cases} \max\{M_e(x, y, t), \varepsilon\} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Then $\mathcal{B} = \{M_e^\varepsilon : \varepsilon \in I_1\}$ is a base of fuzzy pseudometrics on $\mathbb{R}$ (notice that $M_e^\varepsilon \wedge M_e^\delta = M_e^{\varepsilon\wedge\delta}$). It is clear that $M_e \notin \mathcal{M}_{\mathcal{B}}$ since given $t > 0$, for any $s > 0$ and $\varepsilon \in I_1$ we can find $x, y \in \mathbb{R}$ such that $M_e(x, y, t \vee s) < \varepsilon$. Consequently, $M_e^\varepsilon(x, y, s) = \varepsilon \nleq M_e(x, y, t) < M_e(x, y, t \vee s)$.

However, we next show that $M_e = \sup_{\varepsilon\in I_0}(M_e^{\frac{\varepsilon}{2}} - \varepsilon)$ which implies that $M_e \in \widetilde{\mathcal{M}_{\mathcal{B}}}$. Let $x, y \in X$ and $t > 0$. It is obvious that if $M_e(x, y, t) \ge \frac{1}{2}$ then

$$\sup_{\varepsilon\in I_0}(M_e^{\frac{\varepsilon}{2}}(x, y, t) - \varepsilon) = \sup_{\varepsilon\in I_0}(M_e(x, y, t) - \varepsilon) = M_e(x, y, t).$$

If $M_e(x, y, t) < \frac{1}{2}$ then we have that

$$M_e^{\frac{\varepsilon}{2}}(x, y, t) - \varepsilon = \begin{cases} -\frac{\varepsilon}{2} & \text{if } 2M_e(x, y, t) < \varepsilon \\ M_e(x, y, t) - \varepsilon & \text{if } 2M_e(x, y, t) \ge \varepsilon \end{cases}.$$

Hence

$$\sup_{\varepsilon\in I_0}(M_e^{\frac{\varepsilon}{2}}(x, y, t) - \varepsilon) = \sup\{M_e(x, y, t) - \varepsilon : \varepsilon \le 2M_e(x, y, t)\} = M_e(x, y, t).$$

Therefore, $M_e = \sup_{\varepsilon\in I_0}(M_e^{\frac{\varepsilon}{2}} - \varepsilon)$ so $\widetilde{\mathcal{M}_{\mathcal{B}}} \nsubseteq \mathcal{M}_{\mathcal{B}}$.

Proposition 3.25 (cf. Remark 3.7). LSUnif($*$) *is a coreflective subcategory of* PSUnif($*$) *whose coreflector is the functor $\mathcal{S}_s$:* PSUnif($*$) $\to$ LSUnif($*$) *given by $\mathcal{S}_s((X, \mathcal{M}, *)) = (X, \widetilde{\mathcal{M}}, *)$ and leaving morphisms unchanged.*

Proof. It is clear that if $(\mathcal{M},*)$ is a probabilistic uniform structure on a nonempty set X then $(\widetilde{\mathcal{M}},*)$ is a Lowen uniform structure on X since $\widetilde{\widetilde{\mathcal{M}}}=\widetilde{\mathcal{M}}$.

Now suppose that $f:(X,\mathcal{M},*)\to(Y,\mathcal{N},*)$ is a fuzzy uniformly continuous function between two spaces endowed with probabilistic uniform structures. We first notice that if $(N,*)\in\mathcal{N}$ then $(N_f,*)\in\mathcal{M}$ where $N_f(x,y,t)=N(f(x),f(y),t)$ for all $x,y\in X$, $t>0$. In fact, it is easy to see that $(N_f,*)$ is a fuzzy pseudometric on X. Furthermore given $t>0$ we can find $(M,*)\in\mathcal{M}$ and $s>0$ with $M(x,y,s)\leq N(f(x),f(y),t)=N_f(x,y,t)$ for all $x,y,\in X$. Hence $(N_f,*)\in\langle\mathcal{M}\rangle=\mathcal{M}$.

On the other hand, given $(N,*)\in\widetilde{\mathcal{N}}$, $\varepsilon\in I_0$ and $t>0$ there exist $(N^\varepsilon,*)\in\mathcal{N}$ and $t_\varepsilon>0$ such that $N^\varepsilon_{t_\varepsilon}-\varepsilon\leq N$. Since $(N^\varepsilon_f,*)\in\mathcal{M}$ and $N^\varepsilon_f(x,y,t_\varepsilon)-\varepsilon=N^\varepsilon(f(x),f(y),t_\varepsilon)-\varepsilon\leq N(f(x),f(y),t)$ for all $x,y,\in X$ then $(N_f,*)\in\widetilde{\mathcal{M}}$ so $f:(X,\widetilde{\mathcal{M}},*)\to(Y,\widetilde{\mathcal{N}},*)$ is fuzzy uniformly continuous. Therefore, $\mathcal{S}_s$ is a covariant functor.

Next, we prove that $\mathcal{S}_s$ is a right adjoint functor for the inclusion functor $i:\mathsf{LSUnif}(*)\to\mathsf{PSUnif}(*)$. Let X,Y be two nonempty sets, and $(\mathcal{M},*)$ and $(\mathcal{N},*)$ be a Lowen uniform structure and a probabilistic uniform structure on X and Y respectively. It is clear that if $f:(X,\mathcal{M},*)\to(Y,\widetilde{\mathcal{N}},*)$ is fuzzy uniformly continuous then $f:(X,\mathcal{M},*)\to(Y,\mathcal{N},*)$ so is since $\mathcal{N}\subseteq\widetilde{\mathcal{N}}$. On the other hand, if $f:(X,\mathcal{M},*)\to(Y,\mathcal{N},*)$ is fuzzy uniformly continuous then $\mathcal{S}_s(f)=f:(X,\widetilde{\mathcal{M}},*)\to(Y,\widetilde{\mathcal{N}},*)$ so is and since $\widetilde{\mathcal{M}}=\mathcal{M}$ the proof is finished. □

Proposition 3.26. $\mathsf{FUnif}(*)$ *is a coreflective subcategory of* $\mathsf{LSUnif}(*)$ *whose coreflector is the functor* $\iota_s:\mathsf{LSUnif}(*)\to\mathsf{FUnif}(*)$ *given by* $\iota_s((X,\mathcal{M},*))=(X,\widehat{\mathcal{M}},*)$ *and leaving morphisms unchanged.*

Proof. It is obvious that if $(X,\mathcal{M},*)$ is a set endowed with a Lowen uniform structure then $(X,\widehat{\mathcal{M}},*)$ is a fuzzy uniform space.

Now, let us consider a fuzzy uniformly continuous function $f:(X,\mathcal{M},*)\to(Y,\mathcal{N},*)$ between two spaces endowed with Lowen uniform structures. Given $(N,*)\in\widehat{\mathcal{N}}$, $\varepsilon\in(0,1)$ and $t>0$ there exist $(N',*)\in\mathcal{N}$, $\delta\in(0,1)$ and $s>0$ such that $N(x,y,t)>1-\varepsilon$ whenever $N'(x,y,s)>1-\delta$ for all $x,y\in Y$. By assumption we can find $(M,*)\in\mathcal{M}$ and $p>0$ verifying $M(x,y,p)\leq N'(f(x),f(y),s)$ for all $x,y\in X$. Hence $N(x,y,t)>1-\varepsilon$ whenever $M(x,y,p)>1-\delta$ so $f:(X,\widehat{\mathcal{M}},*)\to(Y,\widehat{\mathcal{N}},*)$ is uniformly continuous. Consequently ι_s is a covariant functor.

Finally, we show that ι_s is a right adjoint functor for the inclusion functor $i:\mathsf{FUnif}(*)\to\mathsf{LSUnif}(*)$. Let $(X,\boldsymbol{\mathcal{M}},*)$ be a fuzzy uniform space and $(Y,\mathcal{N},*)$ a space endowed with a Lowen uniform structure. If $f:(X,\boldsymbol{\mathcal{M}},*)\to(Y,\mathcal{N},*)$ is fuzzy uniformly continuous then $\iota_s(f)=f:(X,\boldsymbol{\mathcal{M}},*)\to(Y,\widehat{\mathcal{N}},*)$ is uniformly continuous. On the other hand, consider that $g:(X,\boldsymbol{\mathcal{M}},*)\to(Y,\widehat{\mathcal{N}},*)$ is uniformly continuous. Then, by Remark 2.19, $g:(X,\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{M}}})\to(Y,\boldsymbol{\mathcal{U}}_{\widehat{\mathcal{N}}})$ is uniformly continuous. Let $(N,*)\in\mathcal{N}$ and $t>0$. Since $N_t\in\omega(\boldsymbol{\mathcal{U}}_N)\subseteq\omega(\boldsymbol{\mathcal{U}}_{\widehat{\mathcal{N}}})$, then $N_{g,t}=N_t\circ(g\times g)\in\omega(\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{M}}})$. By Lemma 3.14, there exist a family of pseudometrics $\{d_\varepsilon:\varepsilon\in I_0\}\subseteq\mathcal{D}_{\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{M}}}}$ and a family of positive real numbers $\{t_\varepsilon:\varepsilon\in I_0\}$ such that $\sup_{\varepsilon\in I_0}M_{d_\varepsilon,t_\varepsilon}-\varepsilon\leq N_{g,t}$. Since $M_{d_\varepsilon}\in\boldsymbol{\mathcal{M}}$ (notice that $\boldsymbol{\mathcal{U}}_{M_{d_\varepsilon}}=\boldsymbol{\mathcal{U}}_{d_\varepsilon}\subseteq\boldsymbol{\mathcal{U}}_{\boldsymbol{\mathcal{M}}}$ and $\widehat{\boldsymbol{\mathcal{M}}}=\boldsymbol{\mathcal{M}}$), we have that $N_{g,t}\in\widetilde{\boldsymbol{\mathcal{M}}}=\boldsymbol{\mathcal{M}}$. Consequently, $g:(X,\boldsymbol{\mathcal{M}},*)\to(Y,\mathcal{N},*)$ is fuzzy uniformly continuous. □

4. PSUnif and PUnif are Isomoprhic

In [7] it was proved that the categories $\mathsf{FUnif}(*)$ and Unif are isomorphic. Here we will show that the categories PSUnif (resp. LSUnif) and PUnif (resp. LUnif) are isomorphic.

Proposition 4.1. *Let us consider the map* $\Upsilon:\mathsf{PSUnif}\to\mathsf{PUnif}$ *given by*

$$\Upsilon((X,\mathcal{M},*))=(X,\upsilon(\mathcal{M}),*)=(X,\mathcal{U}_{\mathcal{M}},*)$$

where $(\mathcal{U}_{\mathcal{M}},*)$ *is the probabilistic uniformity which has as base the family* $\{M_t:t>0,(M,*)\in\mathcal{M}\}$ *and*

$$\Upsilon(f)=f$$

for every morphism f *in* PSUnif*. Then* Υ *is a fully faithful covariant functor.*

Proof. Let us suppose that $(X, \mathcal{M}, *)$ is a space with a probabilistic uniform structure. We show that $\{M_t : t > 0, (M, *) \in \mathcal{M}\}$ is a base for a probabilistic uniformity $(\mathcal{U}_{\mathcal{M}}, *)$ on X.

It is obvious that it is a filter base on $I^{X\times X}$ since given $t, s \in (0, +\infty)$ and $(M, *), (N, *) \in \mathcal{M}$ we have that $(M \wedge N)_{s\wedge t} \leq M_t \wedge N_s$ and $(M \wedge N, *) \in \mathcal{M}$.

Furthermore, it is obvious that $M_t(x, x) = M(x, x, t) = 1$ for all $(M, *) \in \mathcal{M}$ and all $t > 0$. Finally, since M_t is symmetric and $M^2_{\frac{t}{2}} \leq M_t$ for all $t > 0$ we have that $\{M_t : t > 0, (M, *) \in \mathcal{M}\}$ is a base for a probabilistic uniformity on X.

On the other hand, suppose that $f : (X, \mathcal{M}, *) \to (Y, \mathcal{N}, \star)$ is a fuzzy uniformly continuous function between two spaces endowed with probabilistic uniform structures. Let $U \in \mathcal{U}_{\mathcal{N}}$. Then we can find $(N, \star) \in \mathcal{N}$ and $t > 0$ such that $N_t \leq U$. By assumption there exist $(M, *) \in \mathcal{M}$ and $s > 0$ such that $M_s(x, y) \leq N_t(f(x), f(y))$ for all $x, y \in X$. Since $M_s \in \mathcal{U}_{\mathcal{M}}$ we deduce that $f : (X, \mathcal{U}_{\mathcal{M}}, *) \to (Y, \mathcal{U}_{\mathcal{N}}, \star)$ is fuzzy uniformly continuous. In a similar way it can be shown that if $f : (X, \mathcal{U}_{\mathcal{M}}, *) \to (Y, \mathcal{U}_{\mathcal{N}}, \star)$ is fuzzy uniformly continuous then $f : (X, \mathcal{M}, *) \to (Y, \mathcal{N}, \star)$ so is. □

Remark 4.2. *We have already observed that if $(M, *)$ is a fuzzy pseudometric on a nonempty set X then the family $\{M_t : t > 0\}$ is a base for a probabilistic uniformity $(\mathcal{U}_M, *)$ on X (see [9, Theorem 3.3]). Then if $(\mathcal{M}, *)$ is a probabilistic uniform structure on X it is clear that*

$$\mathcal{U}_{\mathcal{M}} = \bigvee_{(M,*)\in\mathcal{M}} \mathcal{U}_M.$$

Consequently if $(\mathcal{B}, *)$ is a base for a probabilistic uniform structure $(\mathcal{M}_{\mathcal{B}}, *)$ on X we have that

$$\mathcal{U}_{\mathcal{B}} := \bigvee_{(M,*)\in(\mathcal{B},*)} \mathcal{U}_M = \mathcal{U}_{\mathcal{M}_{\mathcal{B}}} = \bigvee_{(M,*)\in(\mathcal{M}_{\mathcal{B}},*)} \mathcal{U}_M.$$

Hence, $(\mathcal{U}_{\mathcal{M}_{\mathcal{B}}}, *)$ is the largest probabilistic uniformity on X such that id $: (X, \mathcal{U}_{\mathcal{B}}, *) \to (X, \mathcal{U}_{\mathcal{M}_{\mathcal{B}}}, *)$ *is fuzzy uniformly continuous (cf. Remark 2.19).*

Proposition 4.3. *The mappping Υ :* LSUnif → LUnif *which is the restriction of the functor Υ to the coreflective subcategory* LSUnif *of* PSUnif *is a covariant fully faithful functor.*

Proof. It is enough to prove that if $(X, \mathcal{M}, *)$ is a space endowed with a Lowen uniform structure then $(\upsilon(\mathcal{M}), *) = (\mathcal{U}_{\mathcal{M}}, *)$ is a Lowen uniformity on X. Let $U \in \widetilde{\mathcal{U}_{\mathcal{M}}}$ which is also a probabilistic uniformity. By Proposition 3.9, there exists a fuzzy pseudometric $(M, *)$ on X such that $M_t \in \widetilde{\mathcal{U}_M}$ for all $t > 0$ and $M_{\frac{1}{4}} \leq U$. Then since it is clear that $M \in \widetilde{\mathcal{M}} = \mathcal{M}$ we have that $U \in \mathcal{U}_{\mathcal{M}}$. □

Proposition 4.4. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathsf{PSUnif}(*) & \xrightarrow{\quad\Upsilon\quad} & \mathsf{PUnif}(*) \\
\downarrow{\scriptstyle \mathcal{S}_s} & & \downarrow{\scriptstyle \mathcal{S}} \\
\mathsf{LSUnif}(*) & \xrightarrow{\quad\Upsilon\quad} & \mathsf{LUnif}(*) \\
\downarrow{\scriptstyle \iota_s} & & \downarrow{\scriptstyle \iota} \\
\mathsf{FUnif}(*) & \xrightarrow{\quad\Psi\quad} & \mathsf{Unif}
\end{array}$$

Proof. Since all functors leave morphisms unchanged, we restrict ourselves to prove the commutativity on objects.

Let $(\mathcal{M},*)$ be a probabilistic uniform structure on a nonempty set X. Denote $\widetilde{\mathcal{U}_{\widetilde{\mathcal{M}}}} := \upsilon(\widetilde{\mathcal{M}})$ and $\widetilde{\mathcal{U}_{\mathcal{M}}} := \widetilde{\upsilon(\mathcal{M})}$. We must show that $\widetilde{\mathcal{U}_{\mathcal{M}}} = \widetilde{\mathcal{U}_{\widetilde{\mathcal{M}}}}$. It is clear that $\widetilde{\mathcal{U}_{\mathcal{M}}} \subseteq \widetilde{\mathcal{U}_{\widetilde{\mathcal{M}}}}$. Let $U \in \widetilde{\mathcal{U}_{\widetilde{\mathcal{M}}}}$. Given $\varepsilon \in I_0$ there exists $(M^\varepsilon,*) \in \widetilde{\mathcal{M}}$ and $t_\varepsilon > 0$ such that $M^\varepsilon_{t_\varepsilon} - t_\varepsilon \leq U$. Since $(M^\varepsilon,*) \in \widetilde{\mathcal{M}}$ then, by definition of $\widetilde{\mathcal{M}}$, we have that $M^\varepsilon_t \in \widetilde{\mathcal{U}_{\mathcal{M}}}$ for all $t > 0$. Hence $U \in \widetilde{\widetilde{\mathcal{U}_{\mathcal{M}}}} = \widetilde{\mathcal{U}_{\mathcal{M}}}$ which proves $\Upsilon \circ \mathcal{S}_s = \mathcal{S} \circ \Upsilon$.

Now, let $(\mathcal{M},*)$ be a Lowen uniform structure on a nonempty set X. We will prove that $\boldsymbol{\mathcal{U}}_{\widehat{\mathcal{M}}} = \iota(\widetilde{\mathcal{U}_{\mathcal{M}}})$. Let $U \in \boldsymbol{\mathcal{U}}_{\widehat{\mathcal{M}}}$. Then there exists $(M,*) \in \widehat{M}$, $\varepsilon \in (0,1)$ and $t > 0$ such that if $M(x,y,t) > \varepsilon$ then $(x,y) \in U$, i. e. $M_t^{-1}((\varepsilon,1]) \subseteq U$. Since $(M,*) \in \widehat{M}$ there exists $(N,*) \in \mathcal{M}$ and $s > 0$ such that $N(x,y,s) > \delta$ implies $M(x,y,t) > \varepsilon$. Due to the fact that $N_s \in \widetilde{\mathcal{U}_{\mathcal{M}}}$ then $N_s^{-1}((\delta,1]) \in \iota(\widetilde{\mathcal{U}_{\mathcal{M}}})$. Because $N_s^{-1}((\delta,1]) \subseteq M_t^{-1}((\varepsilon,1]) \subseteq U$ we have that $U \in \iota(\widetilde{\mathcal{U}_{\mathcal{M}}})$.

On the other hand, let $U \in \iota(\widetilde{\mathcal{U}_{\mathcal{M}}})$. By construction there exists $F \in \widetilde{\mathcal{U}_{\mathcal{M}}}$ and $\varepsilon \in I_1$ with $F^{-1}((\varepsilon,1]) \subseteq U$. Since $F \in \widetilde{\mathcal{U}_{\mathcal{M}}}$ then given $\delta \in I_0$ we can find $(M^\delta,*) \in \mathcal{M}$ and $t_\delta > 0$ such that $M^\delta_{t_\delta} - \delta \leq F$. In particular, given $\delta > 0$ such that $\varepsilon + \delta < 1$ then if $M^\delta(x,y,t_\delta) > \varepsilon + \delta$ we have that $F(x,y) > \varepsilon$ so $(x,y) \in U$. Hence $U \in \boldsymbol{\mathcal{U}}_{M^\delta} \subseteq \boldsymbol{\mathcal{U}}_{\widehat{\mathcal{M}}}$. Therefore $\Psi \circ \iota_s = \iota \circ \Upsilon$. □

Proposition 4.5. *Let us consider the map* $\mathfrak{S} : \mathsf{PUnif} \to \mathsf{PSUnif}$ *given by*

$$\mathfrak{S}((X,\mathcal{U},*)) = (X,\mathfrak{s}(\mathcal{U}),*) = (X,\mathcal{M}_{\mathcal{U}},*)$$

where $(\mathfrak{s}(\mathcal{U}),*) = (\mathcal{M}_{\mathcal{U}},*)$ *is the probabilistic uniform structure of all fuzzy pseudometrics* $(M,*)$ *on* X *such that* $M_t \in \mathcal{U}$ *for all* $t > 0$*, i. e.*

$$\mathcal{M}_{\mathcal{U}} = \{(M,*) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \mathcal{U}\}$$

and

$$\mathfrak{S}(f) = f$$

for every morphism f *in* PUnif*. Then* $\mathfrak{S}$ *is a covariant fully faithful functor.*

Proof. Let $(X,\mathcal{U},*)$ be a probabilistic uniform space. Let $(M,*),(N,*) \in \mathcal{M}_{\mathcal{U}}$. Then $\mathcal{U}_{M\wedge N} \subseteq \mathcal{U}_M \wedge \mathcal{U}_M \subseteq \mathcal{U}$ so $(M \wedge N,*) \in \mathcal{M}_{\mathcal{U}}$.

On the other hand, let $(M,*) \in \langle \mathcal{M}_{\mathcal{U}} \rangle$. Then $\mathcal{U}_M \subseteq \bigvee_{(N,*)\in\mathcal{M}_{\mathcal{U}}} \mathcal{U}_N \subseteq \mathcal{U}$ so $(M,*) \in \mathcal{M}_{\mathcal{U}}$. Therefore, $(\mathcal{M}_{\mathcal{U}},*)$ is a probabilistic uniform structure.

Furthermore, let $f : (X,\mathcal{U},*) \to (Y,\mathcal{V},\star)$ be a fuzzy uniformly continuous function between two probabilistic uniform spaces. Given $(N,\star) \in \mathcal{M}_{\mathcal{V}}$ and $t > 0$ then $N_t \in \mathcal{V}$ so we can find $U \in \mathcal{U}$ such that $U(x,y) \leq N(f(x),f(y),t)$ for all $x,y \in X$. By Proposition 3.9 there exists a fuzzy pseudometric $(M,*) \in \mathcal{M}_{\mathcal{U}}$ such that $M_{\frac{1}{4}} \leq U$. Hence $f : (X,\mathcal{M}_{\mathcal{U}},*) \to (Y,\mathcal{M}_{\mathcal{V}},\star)$ is fuzzy uniformly continuous. In a similar way it can be proved that fuzzy uniform continuity of $f : (X,\mathcal{M}_{\mathcal{U}},*) \to (Y,\mathcal{M}_{\mathcal{V}},\star)$ implies fuzzy uniform continuity of $f : (X,\mathcal{U},*) \to (Y,\mathcal{V},\star)$ which finishes the proof. □

Proposition 4.6. *The mapping* $\mathfrak{S} : \mathsf{LUnif} \to \mathsf{LSUnif}$*, which is the restriction of the functor* $\mathfrak{S}$ *to the coreflective subcategory* LUnif *of* PUnif *is a covariant fully faithful functor.*

Proof. Let $(X,\mathcal{U},*)$ be a Lowen uniform space. Observe that

$$\mathcal{M}_{\mathcal{U}} = \{(M,*) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \mathcal{U}\} = \{(M,*) \in \mathsf{FMet}(*) : \widetilde{\mathcal{U}_M} \subseteq \mathcal{U}\}$$

since $\mathcal{U}_{\mathcal{M}} \subseteq \widetilde{\mathcal{U}_{\mathcal{M}}}$ and $\widetilde{\mathcal{U}} = \mathcal{U}$.

We have already checked in the previous proof that $\mathcal{M}_{\mathcal{U}}$ is a base of fuzzy pseudometrics. Let $(M,*) \in \widetilde{\mathcal{M}_{\mathcal{U}}}$. Then

$$\mathcal{U}_M \subseteq \left(\widetilde{\bigvee_{(N,*)\in\mathcal{M}_{\mathcal{U}}} \mathcal{U}_N} \right) \subseteq \widetilde{\mathcal{U}} = \mathcal{U}.$$

Hence $(M,*) \in \mathcal{M}_{\mathcal{U}}$ so $(\mathcal{M}_{\mathcal{U}},*)$ is a Lowen uniform structure. □

Proposition 4.7. *The following diagram commutes:*

$$\begin{array}{ccc} \mathsf{PUnif}(*) & \xrightarrow{\quad \mathfrak{S} \quad} & \mathsf{PSUnif}(*) \\ \downarrow{\scriptstyle \mathcal{S}} & & \downarrow{\scriptstyle \mathcal{S}_s} \\ \mathsf{LUnif}(*) & \xrightarrow{\quad \mathfrak{S} \quad} & \mathsf{LSUnif}(*) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota_s} \\ \mathsf{Unif} & \xrightarrow{\quad \Phi_* \quad} & \mathsf{FUnif}(*) \end{array}$$

Proof. Observe that we only have to prove the commutation when the functors are restricted to objects since all of them leave morphisms unchanged.

Let $(\mathcal{U}, *)$ be a probabilisitic uniformity on a nonempty set X. Let $(M, *) \in \mathfrak{s}(\widetilde{\mathcal{U}}) = \mathcal{M}_{\widetilde{\mathcal{U}}}$. Given $t > 0$ then $M_t \in \widetilde{\mathcal{U}}$ so for each $\varepsilon \in I_0$ we can find $U_\varepsilon \in \mathcal{U}$ such that $U_\varepsilon - \varepsilon \leq M_t$. By Proposition 3.9 we can find $(N^\varepsilon, *) \in \mathcal{M}_\mathcal{U}$ with $N^\varepsilon_{\frac{1}{4}} \leq U_\varepsilon$. Hence $N^\varepsilon_{\frac{1}{4}} - \varepsilon \leq M_t$ for all $\varepsilon \in I_0$. Therefore $(M, *) \in \widetilde{\mathfrak{s}(\mathcal{U})} = \widetilde{\mathcal{M}_\mathcal{U}}$.

Conversely, given $(M, *) \in \widetilde{\mathcal{M}_\mathcal{U}}$ we have that for any $t > 0$ and $\varepsilon \in I_0$ there exist $(N^\varepsilon, *) \in \mathcal{M}_\mathcal{U}$ and $t_\varepsilon > 0$ such that $N^\varepsilon_{t_\varepsilon} - \varepsilon \leq M_t$. Since $N^\varepsilon_{t_\varepsilon} \in \mathcal{U}$ for all $\varepsilon \in I_0$ we have that $M_t \in \widetilde{\mathcal{U}}$ so $(M, *) \in \mathcal{M}_{\widetilde{\mathcal{U}}}$. This proves that $\widetilde{\mathcal{M}_\mathcal{U}} = \mathcal{M}_{\widetilde{\mathcal{U}}}$, i. e. $\mathfrak{S} \circ \mathcal{S}_s = \mathcal{S}_s \circ \mathfrak{S}$.

Let $(\mathcal{U}, *)$ be a Lowen uniformity on a nonempty set X. We shall prove that $(\varphi_* \circ \iota)(\mathcal{U}) = (\iota_s \circ \mathfrak{s})(\mathcal{U}) = \widehat{\mathcal{M}_\mathcal{U}}$. We first notice that by Theorem 2.20

$$(\varphi_* \circ \iota)(\mathcal{U}) = \{(M, *) \in \mathsf{FMet}(*) : \boldsymbol{\mathcal{U}}_M \subseteq \iota(\mathcal{U})\} \tag{1}$$

while

$$\widehat{\mathcal{M}_\mathcal{U}} = \{(M, *) \in \mathsf{FMet}(*) : \boldsymbol{\mathcal{U}}_M \subseteq \bigvee_{(N,*) \in \mathcal{M}_\mathcal{U}} \boldsymbol{\mathcal{U}}_N\}. \tag{2}$$

We assert that

$$\iota(\mathcal{U}) = \bigvee_{(N,*) \in \mathcal{M}_\mathcal{U}} \boldsymbol{\mathcal{U}}_N.$$

from which we deduce the equality of (1) and (2).

It is obvious that if $(N, *) \in \mathcal{M}_\mathcal{U}$ then $\boldsymbol{\mathcal{U}}_N \subseteq \iota(\mathcal{U})$ since a base for $\boldsymbol{\mathcal{U}}_N$ is the family $\{\{(x, y) \in X \times X : N(x, y, t) > 1 - \varepsilon\} : \varepsilon \in (0, 1), t > 0\} = \{N_t^{-1}((1 - \varepsilon, 1]) : \varepsilon \in (0, 1), t > 0\}$ which is obviously included in $\iota(\mathcal{U})$ because $N_t \in \mathcal{U}$ for all $t > 0$. On the other hand, given $U \in \mathcal{U}$, by Proposition 3.9, there exists a fuzzy pseudometric $(M, *) \in \mathcal{M}_\mathcal{U}$ such that $M_{\frac{1}{4}} \leq U$. Hence $U^{-1}((\varepsilon, 1]) \in \boldsymbol{\mathcal{U}}_M$ for all $\varepsilon \in I_1$ so $\iota(\mathcal{U}) \subseteq \bigvee_{(N,*) \in \mathcal{M}_\mathcal{U}} \boldsymbol{\mathcal{U}}_N$ and we obtain the equality. □

Since $\Psi \circ \Phi_* = 1_{\mathsf{Unif}}$ we have the following:

Corollary 4.8. *Lowen's functor ι factorizes as follows:*

$$\iota = \Psi \circ \iota_s \circ \mathfrak{S}.$$

Hence, if $(X, \mathcal{U}, *)$ is Lowen uniform space then then $\iota(\mathcal{U}) = \boldsymbol{\mathcal{U}}_{\widehat{\mathcal{M}_\mathcal{U}}}$.

Theorem 4.9. $\mathfrak{S} \circ \Upsilon = 1_{\mathsf{PSUnif}}$ *and* $\Upsilon \circ \mathfrak{S} = 1_{\mathsf{PUnif}}$ *so the categories* PSUnif *and* PUnif *are isomorphic.*

Proof. We first show that $\mathfrak{S}\circ\Upsilon = 1_{\mathsf{PSUnif}}$. Let $(X, \mathcal{M}, *)$ be a space with a probabilistic uniform structure. It is obvious that $\mathcal{M} \subseteq \mathcal{M}_{\mathcal{U}_{\mathcal{M}}}$ since given $(M, *) \in \mathcal{M}$ then $M_t \in \mathcal{U}_{\mathcal{M}}$ for all $t > 0$. Now, suppose that $(M, *)$ is a fuzzy pseudometric such that $M_t \in \mathcal{U}_{\mathcal{M}}$ for all $t > 0$. Since $\langle\mathcal{M}\rangle = \mathcal{M}$ we deduce that $(M, *) \in \mathcal{M}$.

Let us prove that $\Upsilon\circ\mathfrak{S} = 1_{\mathsf{PUnif}}$. Given a probabilistic uniform space $(X, \mathcal{U}, *)$, let $U \in \mathcal{U}_{\mathcal{M}_{\mathcal{U}}}$. Then we can find $M \in \mathcal{M}_{\mathcal{U}}$ and $t > 0$ such that $M_t \leq U$. Since $M_t \in \mathcal{U}$ then $U \in \mathcal{U}$.

On the other hand, given $U \in \mathcal{U}$ we have by Proposition 3.9 that there exists a fuzzy pseudometric $(M, *)$ on X such that $M_t \in \mathcal{U}$ for all $t > 0$ and $M_{\frac{1}{4}} \leq U$. Consequently, $U \in \mathcal{U}_{\mathcal{M}_{\mathcal{U}}}$ which proves the equality. □

Theorem 4.10. *$\mathfrak{S}\circ\Upsilon = 1_{\mathsf{LSUnif}}$ and $\Upsilon\circ\mathfrak{S} = 1_{\mathsf{LUnif}}$ so the categories* LSUnif *and* LUnif *are isomorphic.*

Theorem 4.11. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathsf{PUnif}(*) & \underset{\Upsilon}{\overset{\mathfrak{S}}{\rightleftarrows}} & \mathsf{PSUnif}(*) \\
\downarrow \mathcal{S} & & \downarrow \mathcal{S}_s \\
\mathsf{LUnif}(*) & \underset{\Upsilon}{\overset{\mathfrak{S}}{\rightleftarrows}} & \mathsf{LSUnif}(*) \\
\downarrow \iota & & \downarrow \iota_s \\
\mathsf{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathsf{FUnif}(*)
\end{array}$$

Theorem 4.12. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathsf{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathsf{FUnif}(*) \\
\downarrow \omega_* & & \downarrow i \\
\mathsf{LUnif}(*) & \underset{\mathfrak{S}}{\overset{\Upsilon}{\leftrightarrows}} & \mathsf{LSUnif}(*) \\
\downarrow i & & \downarrow i \\
\mathsf{PUnif}(*) & \underset{\mathfrak{S}}{\overset{\Upsilon}{\leftrightarrows}} & \mathsf{PSUnif}(*)
\end{array}$$

where i denotes the inclusion functor.

Proof. We only have to prove that $\mathfrak{S}\circ\omega_* = i\circ\Phi_*$ because the other compositions commute trivially. We have that if $(X, \boldsymbol{\mathcal{U}})$ is a uniform space then

$$\begin{aligned}
(\mathfrak{s}\circ\omega)(\boldsymbol{\mathcal{U}}) &= \{(M, *) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \omega(\boldsymbol{\mathcal{U}})\} = \{(M, *) \in \mathsf{FMet}(*) : \boldsymbol{\mathcal{U}}_M \subseteq \boldsymbol{\mathcal{U}}\} \\
&= \varphi_*(\boldsymbol{\mathcal{U}}) = (i\circ\varphi_*)(\boldsymbol{\mathcal{U}}).
\end{aligned}$$

□

Corollary 4.13. *Lowen's functor ω_* can be factorized as follows:*

$$\omega_* = \Upsilon\circ i\circ\Phi_*.$$

Hence, if $(X, \boldsymbol{\mathcal{U}})$ is a uniform space then $\omega(\boldsymbol{\mathcal{U}}) = \widetilde{\mathcal{U}}_{\varphi_*(\mathcal{D}_{\mathcal{U}})}$.

Remark 4.14. *We can also provide a direct proof of $\omega(\mathcal{U}) = \widetilde{\mathfrak{U}}_{\varphi_*(\mathcal{D}_\mathcal{U})}$ by using Proposition 4.7.*

We have that $\mathfrak{U}_{\varphi_*(\mathcal{D}_\mathcal{U})} = \bigvee_{d\in\mathcal{D}_\mathcal{U}} \mathfrak{U}_{M_d}$. By Proposition 3.15 $\omega(\mathfrak{U}_d) = \widetilde{\mathfrak{U}_d}$ so

$$\widetilde{\mathfrak{U}}_{\varphi_*(\mathcal{D}_\mathcal{U})} = \widetilde{\left(\bigvee_{d\in\mathcal{D}_\mathcal{U}} \mathfrak{U}_{M_d}\right)} = \bigvee_{d\in\mathcal{D}_\mathcal{U}} \widetilde{\mathfrak{U}_{M_d}} = \bigvee_{d\in\mathcal{D}_\mathcal{U}} \omega(\mathcal{U}_d) = \omega\left(\bigvee_{d\in\mathcal{D}_\mathcal{U}} \mathcal{U}_d\right) = \omega(\mathcal{U}).$$

Remark 4.15. *We observe that the above results clarify why $\iota \circ \omega_* = 1_{\mathsf{Unif}}$ but $\omega_* \circ \iota \neq 1_{\mathsf{LUnif}(*)}$ (see Theorem 3.11). Notice that $\iota_s \circ i = 1_{\mathsf{FUnif}(*)}$ where $i : \mathsf{FUnif}(*) \to \mathsf{LUnif}(*)$ is the inclusion functor. Consequently, by Theorems 2.20, 4.10 and 4.12, we have*

$$\iota \circ \omega_* = (\Psi \circ \iota_s \circ \mathfrak{S}) \circ (\Upsilon \circ i \circ \Phi_*) = (\Psi \circ \iota_s \circ i \circ \Phi_*) = \Psi \circ \Phi_* = 1_{\mathsf{Unif}}.$$

Nevertheless, the same procedure cannot be used for the composition $\omega_* \circ \iota$ because $i \circ \iota_s \neq 1_{\mathsf{LUnif}(*)}$.

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