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On the Roman Domination Number of Generalized Sierpiński Graphs

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Abstract. A map $f : V \to \{0, 1, 2\}$ is a Roman dominating function on a graph $G = (V, E)$ if for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex u, adjacent to v, such that $f(u) = 2$. The weight of a Roman dominating function is given by $f(V) = \sum_{u \in V} f(u)$. The minimum weight among all Roman dominating functions on G is called the Roman domination number of G. In this article we study the Roman domination number of Generalized Sierpiński graphs $S(G, t)$. More precisely, we obtain a general upper bound on the Roman domination number of $S(G, t)$ and discuss the tightness of this bound. In particular, we focus on the cases in which the base graph G is a path, a cycle, a complete graph or a graph having exactly one universal vertex.

1. Introduction

Let $G = (V, E)$ be a non-empty graph of order $n \geq 2$, and t a positive integer. We denote by V^t the set of words of length t on the alphabet V. The letters of a word u of length t are denoted by $u_1u_2...u_t$. The concatenation of two words u and v is denoted by uv. Klavžar and Milutinović introduced in [12] the graph $S(K_n, t)$, $t \geq 1$, whose vertex set is V^t, where $\{u, v\}$ is an edge if and only if there exists $i \in \{1, ..., t\}$ such that:

$$\text{(i) } u_j = v_j, \text{ if } j < i; \text{ (ii) } u_i \neq v_i; \text{ (iii) } u_j = v_i \text{ and } v_j = u_i \text{ if } j > i.$$

As noted in [10], in a compact form, the edge sets can be described as

$$\{\{wu_iu_j^{r-1}, wu_ju_i^{r-1}\} : u_i, u_j \in V, i \neq j; r \in \{1, ..., t\}; w \in V^{t-r}\}.$$

The graph $S(K_3, t)$ is isomorphic to the graph of the Tower of Hanoi with t disks [12]. Later, those graphs have been called Sierpiński graphs in [13] and they are studied by now from numerous points of view. For instance, the authors of [6] studied identifying codes, locating-dominating codes, and total-dominating codes in Sierpiński graphs. In [9] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex-disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpiński graphs. The authors of [13] proved that for any $n \geq 1$ and

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$t \geq 1$, the Sierpiński graph $S(K_n, t)$ has a unique 1-perfect code (or efficient dominating set) if t is even, and $S(K_n, t)$ has exactly n 1-perfect codes if t is odd. The Hamming dimension of a graph G was introduced in [14] as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. That paper gives an upper bound for the Hamming dimension of the Sierpiński graphs $S(K_n, t)$ for $n \geq 3$. It also showed that the Hamming dimension of $S(K_n, t)$ grows as 3^{t-3}. The idea of almost-extreme vertex of $S(K_n, t)$ was introduced in [15] as a vertex that is either adjacent to an extreme vertex of $S(K_n, t)$ or is incident to an edge between two subgraphs of $S(K_n, t)$ isomorphic to $S(K_n, t-1)$. The authors of [15] deduced explicit formulas for the distance in $S(K_n, t)$ between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of the metric dimension of a Sierpiński graph, which was independently obtained by Parreau in her Ph.D. thesis. For a general background on Sierpiński graph, the reader is invited to read the comprehensive survey [11] and references therein.

This construction was generalized in [7] for any graph $G = (V, E)$, by defining the t-th *generalized Sierpiński graph* of G, denoted by $S(G, t)$, as the graph with vertex set V^t and edge set defined as

$$\{\{wu_iu_j^{r-1}, wu_ju_i^{r-1}\} : \{u_i, u_j\} \in E; r \in \{1, ..., t\}; w \in V^{t-r}\}.$$

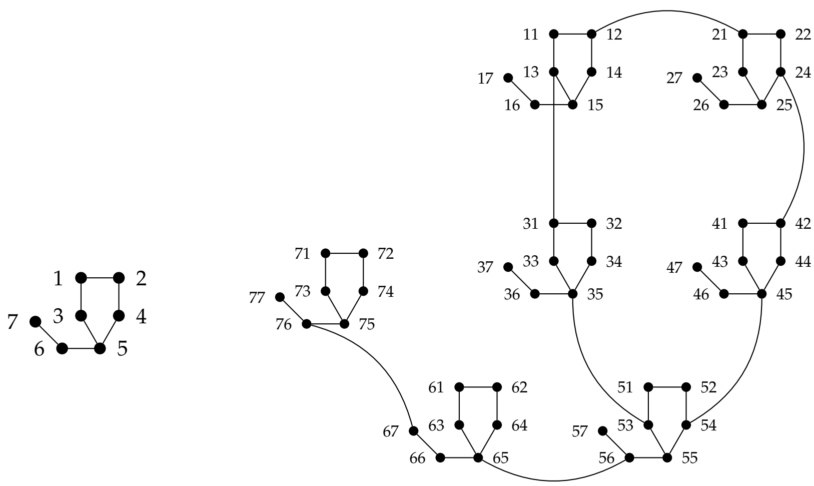

Figure 1: A graph G and the Sierpiński graph $S(G, 2)$.

Figure 1 shows a graph G and the generalized Sierpiński graph $S(G, 2)$, while Figure 2 shows the Sierpiński graph $S(G, 3)$.

Notice that if $\{u, v\}$ is an edge of $S(G, t)$, there is an edge $\{x, y\}$ of G and a word w such that $u = wxyy \ldots y$ and $v = wyxx \ldots x$. In general, $S(G, t)$ can be constructed recursively from G with the following process: $S(G, 1) = G$ and, for $t \geq 2$, we copy n times $S(G, t-1)$ and add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G, t-1)$ corresponding to x. Then for every edge $\{x, y\}$ of G, add an edge between vertex $xyy \ldots y$ and vertex $yxx \ldots x$. See, for instance, Figure 2. Vertices of the form $xx \ldots x$ are called *extreme vertices* of $S(G, t)$. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G, t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx \ldots x$ of $S(G, t)$ also has degree $d(x)$. Moreover, the degrees of two vertices $yxx \ldots x$ and $xyy \ldots y$, which connect two copies of $S(G, t-1)$, are equal to $d(x) + 1$ and $d(y) + 1$, respectively.

For any $w \in V^{t-1}$ and $t \geq 2$ the subgraph $\langle V_w \rangle$ of $S(G, t)$, induced by $V_w = \{wx : x \in V\}$, is isomorphic to G. Notice that there exists only one vertex $u \in V_w$ of the form $w'xx \ldots x$, where $w' \in V^r$ for some $r \leq t-2$. We will say that $w'xx \ldots x$ is *the extreme vertex* of $\langle V_w \rangle$, which is an extreme vertex in $S(G, t)$ whenever $r = 0$. By definition of $S(G, t)$ we deduce the following remark.

Remark 1.1. *Let $G = (V, E)$ be a graph, let $t \geq 2$ be an integer and $w \in V^{t-1}$. If $u \in V_w$ and $v \in V^t \setminus V_w$ are adjacent in $S(G, t)$, then either u is the extreme vertex of $\langle V_w \rangle$ or u is adjacent to the extreme vertex of $\langle V_w \rangle$.*

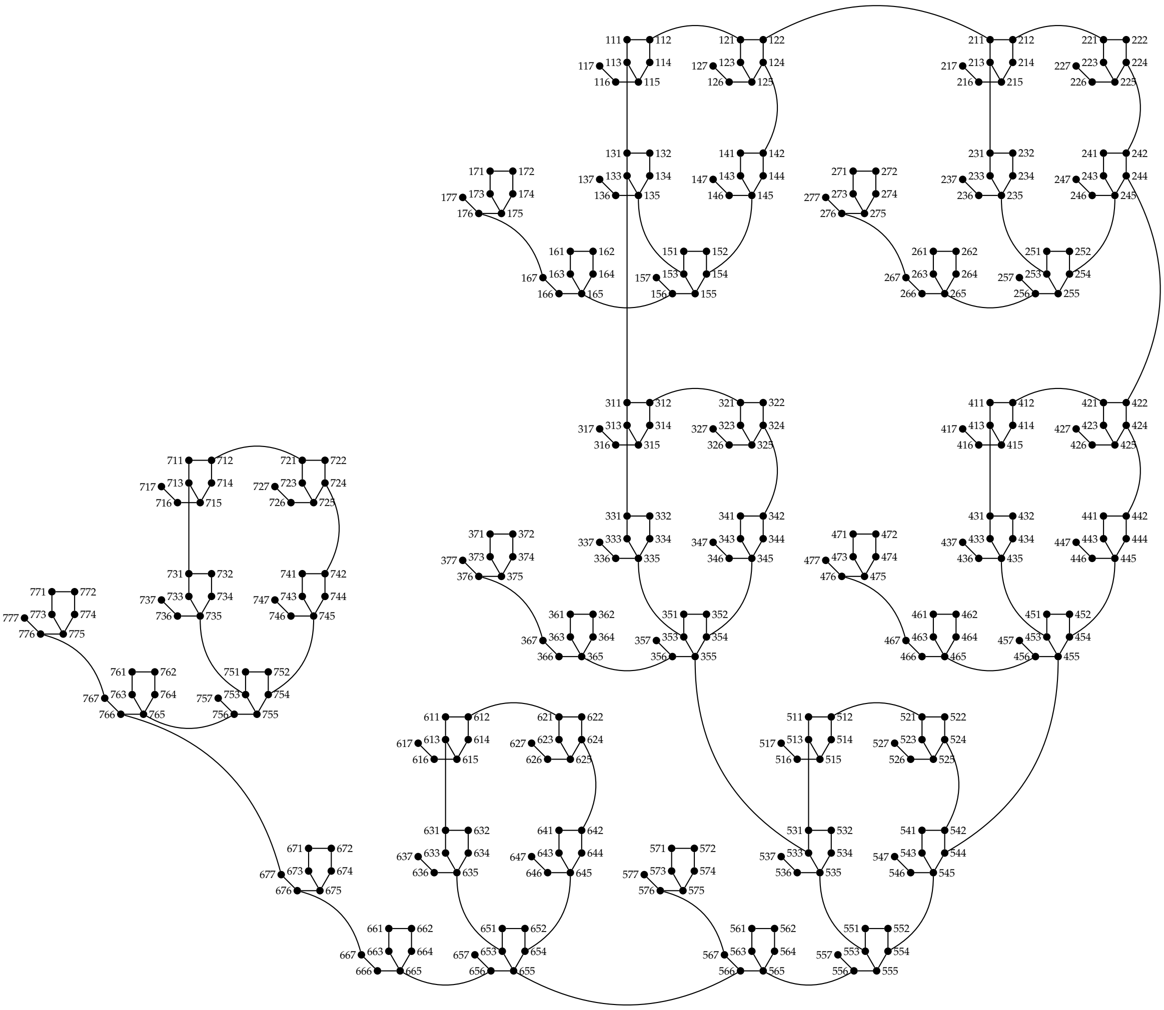

Figure 2: The Sierpiński graph $S(G, 3)$ for the graph G of Figure 1.

The authors of [7] announced some results about generalized Sierpiński graphs concerning their automorphism groups and perfect codes. These results definitely deserve to be published. Since then some papers have been published on various aspects of generalized Sierpiński graphs. For instance, in [17] their chromatic number, vertex cover number, clique number, and domination number, are investigated. The authors of [18] obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs, while in [4] this work was extended to the so-called generalized Randić index. Also, the total chromatic number of generalized Sierpiński graphs was studied in [5] and the strong metric dimension has recently been studied in [16]. In this paper we obtain closed formulae or bounds on the Roman domination number of generalized Sierpiński graphs $S(G, t)$ in terms of parameters of the base graph G.

We begin by establishing the principal terminology and notation which we will use throughout the article. Hereafter $G = (V, E)$ denotes a finite simple graph of order $n \geq 2$. The distance between two vertices $x, y \in V$ will be denoted by $d_G(x, y)$. For two adjacent vertices u and v of G we use the notation $u \sim v$. For a vertex v of G, $N_G(v) = \{u \in V : \ u \sim v\}$ denotes the set of neighbors that v has in G. $N_G(v)$ is called the *open neighborhood of* v and the *closed neighborhood of* v is defined as $N_G[v] = N_G(v) \cup \{v\}$. For a set $D \subseteq V$, the *open neighborhood* is $N_G(D) = \cup_{v \in D} N_G(v)$ and the *closed neighborhood* is $N_G[D] = N_G(D) \cup D$. A set D is a *dominating set* if $N_G[D] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets in G.

We say that a set S is a $\gamma(G)$-set if it is a dominating set and $|S| = \gamma(G)$. The subgraph induced by a subset S of vertices will be denoted by $\langle S\rangle$.

A map $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* on a graph G if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of a Roman dominating function is given by $f(V) = \sum_{u\in V} f(u)$. The minimum weight among all Roman dominating functions on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$.

Any Roman dominating function f on a graph G induces three sets B_0, B_1, B_2, where $B_i = \{v \in V : f(v) = i\}$. Thus, we will write $f = (B_0, B_1, B_2)$. It is clear that for any Roman dominating function $f = (B_0, B_1, B_2)$ on a graph $G = (V, E)$ of order n we have that $f(V) = \sum_{u\in V} f(u) = 2|B_2| + |B_1|$ and $|B_0| + |B_1| + |B_2| = n$. We say that a function $f = (B_0, B_1, B_2)$ is a *$\gamma_R(G)$-function* or a *γ_R-function on* G if it is a Roman dominating function and $f(V) = \gamma_R(G)$.

The Roman domination number was introduced by Cockayne et al. [3] in 2004 and since then about 100 papers have been published on various aspects of Roman domination in graphs (for examples, see [1, 2]). For instance, in [3, 8] was obtained the following result, which shows the relationship between the domination number and the Roman domination number of a graph.

Lemma 1.2. [3, 8] *For any graph G, $\gamma(G) \le \gamma_R(G) \le 2\gamma(G)$.*

As shown in [3], $\gamma(G) = \gamma_R(G)$ if and only if G is an empty graph. A graph G is said to be a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$. Several examples of Roman graphs are given in [3, 19, 20].

Theorem 1.3. [3] *A graph G is Roman if and only if it has a γ_R-function $f = (B_0, \emptyset, B_2)$.*

The following result, stated in [3], will be used as a tool to study the Roman domination number of $S(G, t)$ for the cases in which the base graph is a path or a cycle.

Theorem 1.4. [3] *For the classes of paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.*

Let $G = (V, E)$ be a graph, and $H = (V, E')$ a subgraph of G. Since any $\gamma_R(H)$-function is a Roman dominating function of G, we can state the following remark.

Remark 1.5. *Let $G = (V, E)$ be a graph, and $H = (V, E')$ a subgraph of G. Then $\gamma_R(G) \le \gamma_R(H)$.*

2. An Upper Bound on the Roman Domination Number of $S(G, t)$

Let $f = (B_0, B_1, B_2)$ be a γ_R-function on G and let D_i be the set of non-isolated vertices of $\langle B_i\rangle$ for $i \in \{0, 1, 2\}$. Also, let D_{12} be the set of non-isolated vertices of $\langle B_1 \cup B_2\rangle$. Notice that, if we take f such that $|B_1|$ is minimum, then B_1 is an independent set, which implies that $D_1 = \emptyset$ and $D_{1,2} = D_2$. With these notations in mind we state the following result.

Theorem 2.1. *Let G be a graph of order n. For any γ_R-function $f = (B_0, B_1, B_2)$ on G, and any integer $t \ge 2$,*

$$\gamma_R(S(G, t)) \le n^{t-2}(n\gamma_R(G) - |B_2| - |D_{12}| - \theta + \tfrac{1}{2}|D_1|),$$

where $\theta = |\{u \in B_1 \setminus D_1 : d_G(u, v) = 2$ for some $v \in B_2$ such that $|N_G(v) \cap B_0| = 2\}|$.

Proof. Let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G)$-function. For a given integer $t \ge 2$ we define $S_i = \{wx; \quad w \in V^{t-1}, x \in B_i\}$, for $i \in \{0, 1, 2\}$. Let $g : V^t \to \{0, 1, 2\}$ such that $g = (S_0, S_1, S_2)$. If $v \in V^t$ and $g(v) = 0$, then $v = wy$ where w is a word in V^{t-1} and $y \in B_0$. Since f is a γ_R-function on G, there exists $z \in B_2 \cap N_G(y)$. Hence, $wz \in S_2 \cap N_{S(G,t)}(wy)$. So g is a Roman dominating function on $S(G, t)$ and $\gamma_R(S(G, t)) \le \omega(g) = n^{t-1}(|B_1| + 2|B_2|) = n^{t-1}\gamma_R(G)$. Now we have four steps for reaching the result.

Step 1: Set $S'_2 = \{wuu : w \in V^{t-2}, u \in B_2\}$. We define $g_1 : V^t \to \{0, 1, 2\}$ such that $g_1 = (S_0, S_1 \cup S'_2, S_2 \setminus S'_2)$. Let $y \in S_0$. Then y has the form wuv_0 where $w \in V^{t-2}$, $v_0 \in B_0$ and $u \in V$. Since f is a $\gamma_R(G)$-function, there is

$v_2 \in B_2$ such that v_0 is adjacent to v_2 in G. So wuv_0 is adjacent to wuv_2. If $wuv_2 \in S_2 \setminus S_2'$, then we are done. Now, if $wuv_2 \in S_2'$, then $v_2 = u$ and, since v_0 is adjacent to v_2, we can conclude that $y = wv_2v_0$ is adjacent to wv_0v_2. Hence, g_1 is a Roman dominating function on $S(G,t)$. Therefore $\gamma_R(S(G,t)) \le \omega(g_1) = n^{t-2}(n\gamma_R(G) - |B_2|)$.

Step 2: Set $S_2'' = \{wvv : w \in V^{t-2}, v \in D_2\}$. We define $g_2 : V^t \to \{0,1,2\}$ where

$$g_2(x) = \begin{cases} 0, & x \in S_2''; \\ g_1(x), & \text{otherwise.} \end{cases}$$

Let $x \in V^t$ such that $g_2(x) = 0$. In this case, $g_1(x) = 0$ or $x \in S_2''$.

Suppose that $g_1(x) = 0$. Since x must belong to S_0, it is of the form $x = wuv_0$, where $w \in V^{t-2}$, $u \in V$ and $v_0 \in B_0$. If $N_{S(G,t)}(x) \cap S_2'' = \emptyset$, then there exists $y \in N_{S(G,t)}(x) \cap (S_2 \setminus S_2')$. On the other side, if $z \in N_{S(G,t)}(x) \cap S_2''$, then $z = wv_2v_2$, where $v_2 \in D_2$ and $u = v_2$, and so $v_2 \sim v_0$, which implies that $x = wv_2v_0 \sim wv_0v_2$, and we know that $g_2(wv_0v_2) = g_1(wv_0v_2) = g(wv_0v_2) = 2$.

Now, if $x \in S_2''$, then there exists $w \in V^{t-2}$ and $v \in D_2$ such that $x = wvv$. So, by definition of D_2, x must be adjacent to wvu for some $u \in D_2 \setminus \{v\}$. Hence, $g_2(wvu) = g_1(wvu) = g(wvu) = f(u) = 2$.

Therefore, g_2 is a Roman dominating function on $S(G,t)$, and so $\gamma_R(S(G,t)) \le n^{t-2}(n\gamma_R(G) - |B_2| - |D_2|)$.

Step 3: We know that the maximum degree on $\langle B_1 \rangle$ is one. Since D_1 is the set of non-isolated vertices of $\langle B_1 \rangle$, $\langle D_1 \rangle \cong \cup_{i=1}^{k} P_2$, where $k = \frac{1}{2}|D_1|$. Suppose that $\{v_1, u_1, v_2, u_2, \ldots, v_k, u_k\}$ is the vertex set of $\langle D_1 \rangle$, where $v_i \sim u_i$ for $1 \le i \le k$. Set $S_1' = \{wv_iv_i : w \in V^{t-2}, v_i \in D_1 \text{ and } 1 \le i \le k\}$, $S_1'' = \{wu_iv_i : w \in V^{t-2}, v_i, u_i \in D_1 \text{ and } 1 \le i \le k\}$ and $S_1''' = \{wv_iu_i : wu_iv_i \in S_1''\}$. We define $g_3 : V^t \to \{0,1,2\}$ such that

$$g_3(x) = \begin{cases} 0, & x \in S_1' \cup S_1''; \\ 2, & x \in S_1'''; \\ g_2(x), & \textit{otherwise.} \end{cases}$$

Notice that S_1''' dominates every vertex in $S_1' \cup S_2''$. So g_3 is a Roman dominating function on $S(G,t)$. Also $\omega(g_3) = \omega(g_2) - |S_1'| - |S_1''| + |S_1'''|$ and $|S_1'| = \frac{n^{t-2}}{2}|D_1|$. Hence, $\gamma_R(S(G,t)) \le n^{t-2}(n\gamma_R(G) - |B_2| - |D_2| - \frac{1}{2}|D_1|)$. We know that there are not any edges between B_1 and B_2. So $|D_{12}| = |D_1| + |D_2|$. Hence, $\gamma_R(S(G,t)) \le n^{t-2}(n\gamma_R(G) - |B_2| - |D_{12}| + \frac{1}{2}|D_1|)$.

Step 4: Let $B_2' = \{v \in B_2 : |N_G(v) \cap B_0| = 2 \text{ and } d_G(v,u) = 2 \text{ for some } u \in B_1 \setminus D_1\}$. Let Π be the set of paths v_0, w_2, w_0, w_1 in G such that $w_2 \in B_2'$, $v_0, w_0 \in B_0$ and $w_1 \in B_1 \setminus D_1$. Given two vertices $x, y \in V$, we say that $\mu(x,y) = (i,j)$ if there exist a path v_0, w_2, w_0, w_1 in Π such that x and y are (from the left) in position i and j, respectively. We define the following sets.

$$A_1 = \{wxy : w \in V^{t-2} \text{ and } \mu(x,y) = (3,4)\},$$

$$A_2 = \{wxy : w \in V^{t-2} \text{ and } \mu(x,y) = (4,4)\},$$

$$A_3 = \{wxy : w \in V^{t-2} \text{ and } \mu(x,y) = (4,2)\},$$

$$A_4 = \{wxy : w \in V^{t-2} \text{ and } \mu(x,y) = (4,1)\},$$

$$A_5 = \{wxy : w \in V^{t-2} \text{ and } \mu(x,y) = (4,3)\}.$$

Notice that $|A_2| = \theta$ and $A_i \cap A_j = \emptyset$, for all $i \neq j$, $i, j \in \{1, \ldots, 5\}$. Also, since the weight of f is minimum, for every $w_2 \in B_2'$ there exists exactly one vertex $w_1 \in B_1 \setminus D_1$ such that $d_G(w_2, w_1) = 2$. Hence, $|A_3| = |B_2'|$. Furthermore, since $|N_G(w_2) \cap B_0| = 2$, we can conclude that $|A_1| = |A_4| = |A_5|$. On the other hand, suppose that there are two different paths v_0, w_2, w_0, w_1 and v_0, w_2', w_0', w_1 in Π. In such a case, the weight of the cycle $v_0, w_2, w_0, w_1, w_0', w_2', v_0$ equals 5 and we can find a Roman dominating function with weight equal to 4, as we can consider that v_0 and w_1 have label 2 and the remaining vertices have label 0, which is a contradiction with the minimality of f. Hence, $|A_4| = |B_2'|$. Now, define the function $g_4 : V^t \to \{0,1,2\}$ such that

$$g_4(v) = \begin{cases} 0, & v \in A_1 \cup A_2 \cup A_3; \\ 1, & v \in A_4; \\ 2, & v \in A_5; \\ g_3(v), & \text{otherwise.} \end{cases}$$

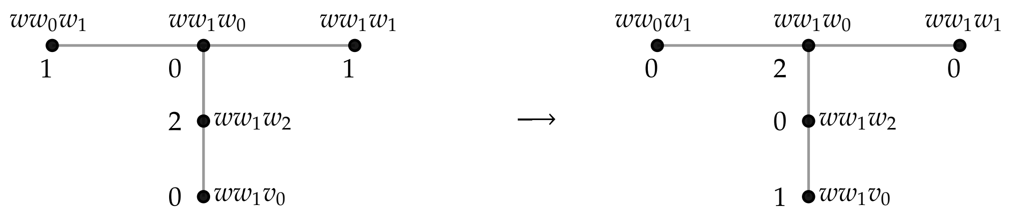

Figure 3: This figure shows how the labels imposed by function g_3 are transformed by function g_4.

Notice that A_5 is a dominating set for $A_1 \cup A_2 \cup A_3$. So g_4 is a Roman dominating function on $S(G,t)$ (See Figure 3). Then

$$\begin{aligned} \omega(g_4) &= 2|A_5| + |A_4| + \omega(g_3) - |A_1| - |A_2| - 2|A_3| \\ &= \omega(g_3) - \theta \\ &\le n^{t-2}\left(n\gamma_R(G) - |B_2| - |D_{12}| - \theta + \frac{|D_1|}{2}\right), \end{aligned}$$

as required. $\square$

As we can see in Theorems 3.1 and 4.3 the bound above is achieved for any Sierpiński graph whose base graph has one universal vertex or is a path P_n, where $n \equiv 0, 1 \pmod 3$.

Since any Roman graph has a γ_R-function $f = (B_0, \emptyset, B_2)$, we can state the following particular case of Theorem 2.1.

Corollary 2.2. *For any Roman graph G of order n and any integer $t \ge 2$,*

$$\gamma_R(S(G,t)) \le \gamma(G)n^{t-2}(2n-1).$$

3. Graphs Having Exactly One Universal Vertex

Theorem 3.1. *If G is a graph of order $n \ge 4$ having exactly one vertex of degree $n-1$, then for any integer $t \ge 2$, $\gamma_R(S(G,t)) = n^{t-2}(2n-1)$.*

Proof. By Theorem 2.1 we deduce that $\gamma_R(S(G,t)) \le n^{t-2}(2n-1)$. We will show that for any $\gamma_R(S(G,t))$-function $f = (B_0, B_1, B_2)$, $\omega(f) \ge n^{t-2}(2n-1)$. Let $V = \{0, 1, \ldots, n-1\}$ such that $deg(0) = n-1$. We would point out that for any $w \in V^{t-2}$, $i \in V$ and $t \ge 3$, the subgraph $\langle V_{wi} \rangle$ of $S(G,t)$, induced by $V_{wi} = \{wij : j \in V\}$, is isomorphic to G. Let $\lambda(V_{wi}) = |\{wij \in V_{wi} : deg(wij) \ne deg(j)\}|$. There are two general cases.
Case I. $i \ne 0$. In this case $1 \le \lambda(V_{wi}) \le n-1$. So there exists $wij \in V_{wi}$ such that $deg(wij) = deg(j)$ for $1 \le j \le n-1$. If $B_2 \cap V_{wi} \ne \emptyset$, then $\omega(V_{wi}) \ge 2$. Otherwise, $wij \in B_1$ and $\omega(V_{wi}) \ge 1$. If $\omega(V_{wi}) = 1$, then $f(wik) = 0$ for $k \in V \setminus \{j\}$. Let $l \in V \setminus \{0, i, j\}$. Then $wil \in N(wli)$ where $wli \in B_2$. Since $l \ne 0$, $\lambda(V_{wl}) \le n-1$, and so there exists $wll' \in V_{wl} \cap (B_1 \cup B_2)$ such that $l' \ne i$. Hence, $\omega(V_{wl}) \ge 3$. This shows that $\omega(V_{wi}) + \omega(V_{wl}) \ge 4$. Therefore, for every copy of G of weight 1 there is another copy of G of weight at least 3. Since there are $n^{t-2}(n-1)$ copies of G of this type in $S(G,t)$, the contribution of these copies of G to $\omega(f)$ equals $\sum_{w \in V^{t-2}} \sum_{i=1}^{n-1} \omega(V_{wi}) \ge 2n^{t-2}(n-1)$.

Case II. $i = 0$. Then $n-1 \le \lambda(V_{w0}) \le n$. If $V_{w0} \nsubseteq B_0$, then $\omega(V_{w0}) \ge 1$. Suppose that $\omega(V_{w0}) = 0$. Hence, $\lambda(V_{w0}) = n$ and so $w00 \in N(w'jj)$ for $w' \in V^{t-2}$ and $j \neq 0$. Since f is a γ_R-function, $w'jj \in B_2$. Also $deg(j) < n-1$, so there exists $z \in V \setminus \{0, j\}$ such that $w'jz \notin N(w'jj)$ and $deg(w'jz) = deg(z)$. Hence, $f(w'jz) \in \{0, 1\}$. If $f(w'jz) = 1$, then we can move the label 2 from $w'jj$ to $w'00$ and the label 1 from $w'jz$ to $w00$. The function obtained in this manner is a γ_R-function on $S(G,t)$, and so we can assume that f is such a function, *i.e.*, $\omega(V_{w0}) = 1$. Now, If $f(w'jz) = 0$, then we have two possibilities. Either $f(w'j0) = 2$ or $f(w'jl) = 2$, for some $l \in N(z)$. The case $f(w'j0) = 2$ is impossible, as we can put the label 1 to $w00$ and the label 0 to $w'jj$, and the function obtained is a Roman dominating function of weight less than f, which is a contradiction. Finally, if $f(w'jl) = 2$, then we can modify the following weights: we put label 2 to $w'j0$, label 0 to $w'jl$, label 1 to $w00$, label 0 to $w'jj$ and, if $l \in N(j)$, then we put label 1 to $w'lj$. The function obtained in this manner is a γ_R-function on $S(G,t)$, and so we can assume that f is such a function, *i.e.*, $\omega(V_{w0}) = 1$. So $\sum_{w \in V^{t-2}} \omega(V_{w0}) \ge n^{t-2}$.

Therefore, $\gamma_R(S(G,t)) = \omega(f) \ge n^{t-2} + 2n^{t-2}(n-1) = n^{t-2}(2n-1)$. The proof is completed. □

Since any graph of order n having at most one vertex of degree greater than or equal to $n-2$ is a subgraph of a graph of order n having exactly one vertex of degree $n-1$, Remark 1.5 and Theorem 3.1 lead to the following result.

Theorem 3.2. *If G is a graph of order $n \ge 4$ having at most one vertex of degree greater than or equal to $n-2$, then for any integer $t \ge 2$, $\gamma_R(S(G,t)) \ge n^{t-2}(2n-1)$.*

4. The Particular Case of Paths

Notice that $S(P_2,t) \cong P_{2^t}$ and so $\gamma_R(S(P_2,t)) = \left\lceil \frac{2^{t+1}}{3} \right\rceil$. From now on we assume that $n \ge 3$. Let $V = \{1, 2, \dots, n\}$ be the vertex set of P_n, and $\langle V_{wu} \rangle$ a copy of P_n in $S(P_n,t)$ for $w \in V^{t-2}$ and $u \in V$. Set

$$A_{wu} = \begin{cases} \{wui \in V_{wu} : i < u-1\}, & 3 \le u \le n\,; \\ \emptyset, & u = 1, 2. \end{cases}$$

$$B_{wu} = \begin{cases} \{wuj \in V_{wu} : j > u+1\}, & 1 \le u \le n-2\,; \\ \emptyset, & u = n-1, n. \end{cases}$$

Also, let

$$D_i = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + i \right\}, \text{ for } i \in \{0, 1\}$$

and

$$D_2 = \left\{ \langle V_{wu} \rangle : \omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + j, \text{ for some } j \ge 2 \right\},$$

where the weight $\omega(V_{wu})$ corresponds to a labelling defined by a γ_R-function on $S(P_n,t)$. Also set $\Lambda = \{\langle V_{wu} \rangle : deg(wuu) \neq deg(u) \text{ for } 1 \le u \le n\}$. With these notations in mind we will prove the following Lemmas.

Lemma 4.1. *Let $f = (B_0, B_1, B_2)$ be a γ_R-function on $S(P_n,t)$, where $n \ge 3$. For any $w \in V^{t-2}$ and $u \in V$ there exists $i \ge 0$ such that $\langle V_{wu} \rangle \in D_i$, and $i \ge 1$ whenever $V_{wu} \notin \Lambda$.*

Proof. Let $P_r = \langle A_{wu} \rangle$ and $P_{r'} = \langle B_{wu} \rangle$. Notice that Theorem 1.4 leads to $\gamma_R(\langle A_{wu} \rangle) = \lceil \frac{2r}{3} \rceil$ and $\gamma_R(\langle B_{wu} \rangle) = \lceil \frac{2r'}{3} \rceil$. If $V_{wu} \notin \Lambda$, then $deg(wuu) = deg(u) \le 2$. Since

$$\omega(V_{wu}) = \omega(A_{wu}) + \sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) + \omega(B_{wu}),$$

$\omega(V_{wu}) \geq \omega(A_{wu}) + \omega(B_{wu}) + 1$. If $\omega(A_{wu}) \geq \lceil \frac{2r}{3} \rceil$ or $\omega(B_{wu}) \geq \lceil \frac{2r'}{3} \rceil$, then we are done. If $A_{wu} \neq \emptyset$ and $\omega(A_{wu}) < \lceil \frac{2r}{3} \rceil$, then $f(wu(u-2)) = 0$, $f(wu(u-3)) \leq 1$, and so $f(wu(u-1)) = 2$. Hence, $\omega(A_{wu}) + f(wu(u-1)) = \lceil \frac{2(r-2)}{3} \rceil + 1 + 2 \geq \lceil \frac{2r}{3} \rceil + 1$. By analogy, if $B_{wu} \neq \emptyset$ and $\omega(B_{wu}) < \lceil \frac{2r'}{3} \rceil$, then $\omega(B_{wu}) + f(wu(u+1)) \geq \lceil \frac{2r'}{3} \rceil + 1$. Therefore, in any case,

$$\omega(V_{wu}) \geq \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + 1.$$

Let $V_{wu} \in \Lambda$. Then $wuu \in N(w'vv)$ where $w' \in V^{t-2}$ and $v \in V$. Thus, as above,

$$\omega(V_{wu}) = \omega(A_{wu}) + \sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) + \omega(B_{wu}) \geq \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil.$$

□

Lemma 4.2. *Let V be the vertex set of P_n, $n \geq 3$, and t a positive integer. If for some $w \in V^{t-2}$ and $u \in V$ we have that $\langle V_{wu} \rangle \in D_0$, then there exists $w' \in V^{t-2}$ and $v \in N_G(u)$ such that $\langle V_{w'v} \rangle \in D_2$.*

Proof. Let $f = (B_0, B_1, B_2)$ be γ_R-function on $S(P_n, t)$, and $\langle V_{wu} \rangle \in D_0$. Then $\sum_{wui \notin A_{wu} \cup B_{wu}} f(wui) = 0$. Thus, $wuu \in N(w'vv)$ where $w'vv \in V^{t-2} \cap B_2$ for $w' \in V^{t-2}$ and $v \in V$. Hence, $\langle V_{w'v} \rangle \in \Lambda$ and $\omega(V_{w'v}) \geq \left\lceil \frac{2|A_{w'v}|}{3} \right\rceil + \left\lceil \frac{2|B_{w'v}|}{3} \right\rceil + 2$. So, $\langle V_{w'v} \rangle \in D_2$. □

Theorem 4.3. *For any integers $n \geq 3$ and $t \geq 2$,*

$$\gamma_R(S(P_n, t)) = \begin{cases} n^{t-2}\left(n\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil\right), & n \equiv 0, 1 \pmod 3; \\ n^{t-2}\left(n\lceil \frac{2n}{3} \rceil - 2\lceil \frac{n}{3} \rceil + 1\right), & n \equiv 2 \pmod 3. \end{cases}$$

Proof. We first proceed to deduce the lower bound $\gamma_R(S(P_n, t)) \geq n^{t-2}(n\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{3} \rceil)$. Let $V = \{1, 2, \ldots, n\}$, and $f = (B_0, B_1, B_2)$ a γ_R-function on $S(P_n, t)$. Let $\langle V_{wu} \rangle$ be a copy of P_n in $S(P_n, t)$ for $w \in V^{t-2}$ and $u \in V$. Since

$$\gamma_R(S(P_n, t)) = \sum_{w \in V^{t-2}, u \in V} \omega(V_{wu}),$$

we will obtain a lower bound on $\omega(V_{wu})$ in terms of n. Before doing it, notice that

$$\gamma_R(S(P_n, t)) = \sum_{\langle V_{wu} \rangle \in D_0} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_1} \omega(V_{wu}) + \sum_{\langle V_{wu} \rangle \in D_2} \omega(V_{wu})$$

and by Lemma 4.2, there exists an injective application $\psi : D_0 \longrightarrow D_2$, so that we emphasize that if $\langle V_{wu} \rangle \in D_0$, then the contribution of $\omega(V_{wu}) + \omega(\psi(\langle V_{wu} \rangle))$ to $\gamma_R(S(P_n, t))$ is greater than or equal to its contribution when both $\langle V_{wu} \rangle$ and $\psi(\langle V_{wu} \rangle)$ belong to D_1. With this observation in mind we continue the proof.

By Lemma 4.1, $\omega(V_{wu}) = \left\lceil \frac{2|A_{wu}|}{3} \right\rceil + \left\lceil \frac{2|B_{wu}|}{3} \right\rceil + i$, for some $i \geq 0$. Hence, we now proceed to express $\left\lceil \frac{|A_{wu}|}{3} \right\rceil$ and $\left\lceil \frac{2|B_{wu}|}{3} \right\rceil$ in terms of n. To this end, we consider the set $S = \{x \in V : x \equiv 2 \pmod 3\}$ and differentiate three cases.

Case 1: $n = 3k$ for some positive integer k. So S is a $\gamma(P_n)$-set. If $u \in S$, then $|A_{wu}|, |B_{wu}| \in \{3k' : 0 \leq k' \leq k-1\}$ and, as $|A_{wu} \cup B_{wu}| = n - 3$, we have

$$\omega(V_{wu}) = 2\frac{n-3}{3} + i = \frac{2n}{3} + i - 2. \tag{1}$$

If $u \in N(S) \setminus \{1, n\}$, then $|A_{wu}| \in \{l : l \equiv 1 (\mathrm{mod}\, 3)\}$ and $|B_{wu}| \in \{l : l \equiv 2 (\mathrm{mod}\, 3)\}$ or vice versa. Hence, $\omega(V_{wu}) = \frac{2n}{3} + i - 1$. Notice that if $u = 1$, then $A_{wu} = \emptyset$ and $|B_{wu}| \equiv 1 \pmod 3$, which implies that $\omega(V_{wu}) = \frac{2n}{3} + i - 1$. The case $u = n$ is analogous to the previous one. Therefore,

$$
\begin{aligned}
\gamma_{R}(S(P_n, t)) &= \sum_{w \in V^{t-2}} \sum_{u \in V} \omega(V_{wu}) \\
&\geq n^{t-2} \left(\left(\frac{2n}{3} - 1 \right) \gamma(P_n) + \frac{2n}{3} \left(n - \gamma(P_n) \right) \right) \\
&= n^{t-2} \left(n \left\lceil \frac{2n}{3} \right\rceil - \left\lceil \frac{n}{3} \right\rceil \right).
\end{aligned}
$$

Case 2: $n = 3k + 1$ for some positive integer k. In this case, $S' = S \cup \{n - 1\}$ is a γ-set of P_n. If $\langle V_{wd} \rangle$ is a copy of P_n for some $d \in S'$, $|A_{wd}| \in \{l : l \equiv 0 \pmod 3\}$ and $|B_{wd}| \in \{l : l \equiv 1 \pmod 3\}$ or vice versa. Hence,

$$
\omega(V_{wd}) = \left\lceil \frac{2|A_{wd}|}{3} \right\rceil + \left\lceil \frac{2|B_{wd}|}{3} \right\rceil + i = 2 \left\lfloor \frac{n}{3} \right\rfloor + i - 1. \tag{2}
$$

Let V_{wu} where $u \in N(S') \setminus \{1, n\}$. Hence, we have two possibilities, $|A_{wu}|, |B_{wu}| \in \{l : l \equiv 2 \pmod 3\}$ or $|A_{wu}|, |B_{wu}| \in \{l : l \equiv 0, 1 \pmod 3\}$ where $|A_{wu}| \not\equiv |B_{wu}| \pmod 3$. In the first case, $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + i$ and, in the second one, $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + i - 1$.

Suppose that $\omega(V_{wv}) = 2\lfloor \frac{n}{3} \rfloor + i - 1$ for $w \in V^{t-2}$ and $v \in V$. Then $\omega(V_{w(v-1)}) > 2\lfloor \frac{n}{3} \rfloor + i - 1$ where $v - 1 \in S$. Therefore $\omega(V_{wu})$ is equal to $2\lfloor \frac{n}{3} \rfloor + i - 1$ at most for $\gamma(P_n)$ copies of P_n, and for other copies it is more than $2\lfloor \frac{n}{3} \rfloor + i - 1$. Hence,

$$
\gamma_{R}(S(P_n, t)) \geq n^{t-2} \left(2\gamma(P_n) \left\lfloor \frac{n}{3} \right\rfloor + (n - \gamma(P_n)) \left(2 \left\lfloor \frac{n}{3} \right\rfloor + 1 \right) \right) = n^{t-2} \left(n \left\lceil \frac{2n}{3} \right\rceil - \left\lceil \frac{n}{3} \right\rceil \right).
$$

Case 3: $n = 3k + 2$ for some positive integer k. We discuss first words of the form wu where $2 \leq u \leq n - 1$ and $w \in V^{t-2}$. If $wuu \in B_2 \cup B_1$, then $\omega(V_{wu}) \geq \lceil \frac{2(u-2)}{3} \rceil + \lceil \frac{2(n-u-1)}{3} \rceil + 1$. Hence, $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$ for $u \equiv 0 (\mathrm{mod}\, 3)$ and $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ for others. Now, suppose that $wuu \in B_0$ and $\langle V_{wu} \rangle \notin D_0$. In this case $wu(u-1) \in B_2$ or $wu(u+1) \in B_2$, say $wu(u+1) \in B_2$. Hence, $\omega(V_{wu}) \geq \lceil \frac{2(u-2)}{3} \rceil + \lceil \frac{2(n-u-2)}{3} \rceil + 2$, which implies that $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$ for $u \in \{3k', 3k' + 2 : 0 \leq k' \leq k - 1\}$ and $\omega(V_{w'}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ for others. In summary, $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 1$ for $u \equiv 0, 2 \pmod 3$ and $\omega(V_{wu}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ for $u \equiv 1 \pmod 3$.

Now, let $u \in \{1, n\}$. Suppose that $u = 1$ (for $u = n$, the proof is likewise). If $\langle V_{w1} \rangle \in D_2$, then $\omega(V_{w1}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$. Now, if $\langle V_{w1} \rangle \in D_1$, then $f(w11) = 1$ or $f(w11) = 0$. In the first case, $f(w21) = 2$, as $f(w13) = 2$ implies that $\langle V_{w1} \rangle \in D_2$, which is a contradiction. In the second case, there exists $w' \in V^{t-2}$ such that $f(w'22) = 2$ and $w11 \in N(w'22)$. As a consequence, $\omega(V_{w1}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$ or for some $w' \in V^{t-2}$, $\omega(V_{w'2}) \geq 2\lfloor \frac{n}{3} \rfloor + 2$. In summary, we can collect the lower bounds for the weight of the copies of P_n in $S(P_n, 2)$ in a table.

	$u =$	$3k'$	$3k' + 1$	$3k' + 2$
	$u \neq 1, n, \omega(V_{wu}) \geq$	$2\lfloor \frac{n}{3} \rfloor + 1$	$2\lfloor \frac{n}{3} \rfloor + 2$	$2\lfloor \frac{n}{3} \rfloor + 1$
$\langle V_{w1} \rangle \in D_0$	$\omega(V_{w1}) \geq$		$2\lfloor \frac{n}{3} \rfloor$	
	$\omega(V_{w2}) \geq$			$2\lfloor \frac{n}{3} \rfloor + 2$
	and			
	$\exists w' \in V^{t-2}: \ \omega(V_{w'2}) \geq$			$2\lfloor \frac{n}{3} \rfloor + 2$
$\langle V_{w1} \rangle \in D_1$	$\omega(V_{w1}) \geq$		$2\lfloor \frac{n}{3} \rfloor + 1$	
	$\omega(V_{w2}) \geq$			$2\lfloor \frac{n}{3} \rfloor + 2$
	or			
	$\exists w' \in V^{t-2}: \ \omega(V_{w'2}) \geq$			$2\lfloor \frac{n}{3} \rfloor + 2$
$\langle V_{w1} \rangle \in D_2$	$\omega(V_{w1}) \geq$		$2\lfloor \frac{n}{3} \rfloor + 2$	

Therefore,

$$\begin{aligned}\gamma_R(S(P_n,t)) &= \sum_{w\in V^{t-2}}\sum_{u\in V}\omega(V_{wu})\\ &\geq n^{t-2}\left(\left(2\left\lfloor\frac{n}{3}\right\rfloor+1\right)\left(2\left\lfloor\frac{n}{3}\right\rfloor+1\right)+\left\lceil\frac{n}{3}\right\rceil\left(2\left\lfloor\frac{n}{3}\right\rfloor+2\right)\right)\\ &= n^{t-2}\left(n\left\lceil\frac{2n}{3}\right\rceil-2\left\lceil\frac{n}{3}\right\rceil+1\right).\end{aligned}$$

and the proof of the lower bound is complete.

For $n \equiv 0,1 \pmod 3$, the upper bound $\gamma_R(S(P_n,t)) \leq n^{t-2}(n\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{3}\rceil)$ is obtained from Theorem 2.1. Thus we consider the case $n = 3k+2$ for some positive integer k.

As above, consider the set $S = \{x \in V \setminus \{n\} : x \equiv 2 \pmod 3\}$. In order to construct a Roman dominating function we introduce the following sets.

$$\begin{aligned}A_1 &= \{wis : w \in V^{t-2},\ s \in S,\ i \geq s+2\},\\ A_2 &= \{wi(n-1) : w \in V^{t-2},\ i \in \{1,n\}\},\\ A_3 &= \{wij : w \in V^{t-2},\ 1 \leq i \leq n-2,\ j = i+1+3k',\ 0 \leq k' \leq k-1\},\\ C_1 &= \{win : w \in V^{t-2},\ i \in S\},\\ C_2 &= \{w(s+1)(s-1) : w \in V^{t-2},\ s \in S\},\\ C_3 &= \{w(n-1)(n-1) : w \in V^{t-2}\}.\end{aligned}$$

Define $g : V^t \to \{0,1,2\}$ such that

$$g(wij) = \begin{cases} 2, & wij \in \bigcup_{i=1}^{3} A_i;\\ 1, & wij \in \bigcup_{i=1}^{3} C_i;\\ 0, & \text{otherwise.}\end{cases}$$

Suppose that $g(wij) = 0$ for $w \in V^{t-2}$ and $i, j \in V$. If $i > j+2$, then $j \notin S$ and so $wij \in N(wis)$ where $s \in \{j-1, j+1\}$. As a consequence, $i > s+2$ and $wis \in A_1$. If $i = j+2$ and $s = j+1$, then $wij = w(s+1)(s-1) \in C_2$, which is a contradiction, as $g(wij) = 0$. Hence, if $i = j+2$, then $s = j-1$ and $wis = w(s+3)s \in A_1$. Now, let $i < j+2$. If $i = j+1$, then $wij = wi(i-1) \in N(w(i-1)i)$ and $w(i-1)i \in A_2$. Also, if $i < j+1$, then wij is dominated by some vertex in $A_2 \cup A_3$. Hence, g is a Roman dominating function on $S(P_n,t)$. Thus,

$$\gamma_R(S(P_n,t)) \leq \omega(g) = 2\sum_{i=1}^{3}|A_i| + \sum_{i=1}^{3}|C_i|.$$

On one hand,

$$\sum_{i=1}^{3}|C_i| = n^{t-2}(2|S|+1) = n^{t-2}(2\gamma(P_n)-1) = 2k+1$$

and, on the other hand,

$$|A_1| = n^{t-2}\sum_{u\in S}u = n^{t-2}\left(\frac{3k^2+k}{2}\right),\ |A_2| = 2n^{t-2}$$

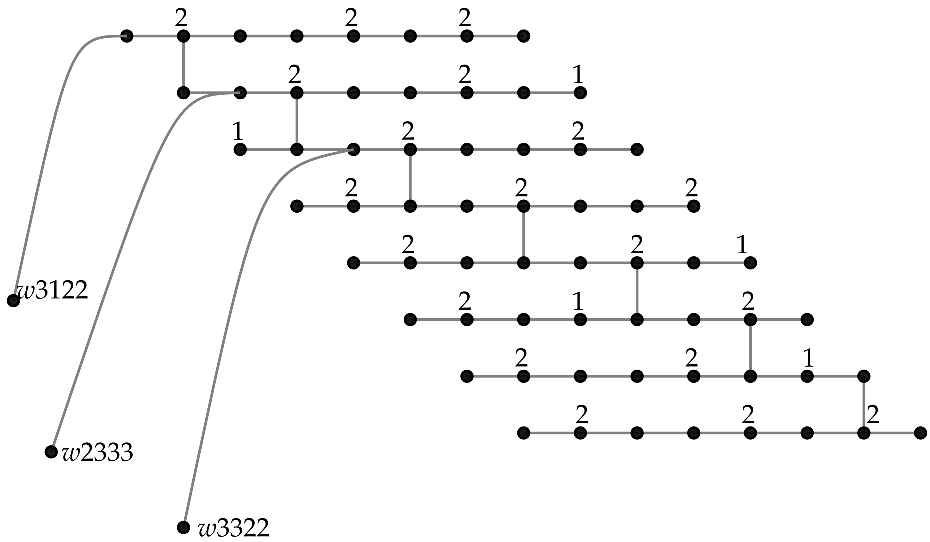

Figure 4: This figure shows the labelling of $\langle V_{w32}\rangle \cong S(P_8, 2)$ induced by g, where labels 0's are omitted.

and

$$|A_3| = n^{t-2}\left(k+2+\sum_{i=2}^{k} 3i\right) = n^{t-2}\left(\frac{3k^2+5k-2}{2}\right).$$

Thus,

$$\sum_{i=1}^{3}|A_i| = n^{t-2}\left(\frac{3k^2+k}{2}+2+\frac{3k^2+5k-2}{2}\right) = n^{t-2}(3k^2+3k+1).$$

Therefore, $\gamma_R(S(P_n,t)) \le n^{t-2}(6k^2+8k+3)$ and, since $n = 3k+2$,

$$\gamma_R(S(P_n,t)) \le n^{t-2}\left(n\left\lceil\frac{2n}{3}\right\rceil - 2\left\lceil\frac{n}{3}\right\rceil + 1\right),$$

as required. □

5. The Particular Case of Cycles

Theorem 5.1. *Let $n \ge 4$ and $t \ge 2$ be two integers. If $n \equiv 1,2 \pmod 3$, then $\gamma_R(S(C_n,t)) = n^{t-1}\lfloor\frac{2n}{3}\rfloor$, otherwise, $\frac{n^{t-1}(2n-3)}{3} \le \gamma_R(S(C_n,t)) \le \frac{n^{t-1}(2n-1)}{3}$.*

Proof. Let $V = \{1,\dots,n\}$ be the vertex set of C_n, where $i \in N_{C_n}(i+1)$, for any i, and the addition is taken modulo n. First, we proceed to deduce the upper bound for $\gamma_R(S(C_n,t))$. If $n \equiv 0 \pmod 3$, then Theorem 2.1 leads to

$$\gamma_R(S(C_n,t)) \le \frac{n^{t-1}}{3}(2n-1). \tag{3}$$

Suppose that $n = 3k+1$, for some integer k. Define $D = \{ij : i \in V,\ j = i+1+3k',\ 0 \le k' < k-1\}$ and $D_{t-2} = \{wx : w \in V^{t-2}, x \in D\}$. Notice that D is a 2-packing[1)] dominating set, and $D \cap \{ii : i \in V\} = \emptyset$, hence

[1)]A set S of vertices is called a 2-*packing* of G if for every pair of vertices $u, v \in S$, $N_G[u] \cap N_G[v] = \emptyset$.

D_{t-2} is also a 2-packing dominating set and therefore $\gamma(S(C_n,t)) = |D_{t-2}| = n^{t-2}|D| = n^{t-1}\left\lfloor \frac{n}{3} \right\rfloor$, which implies that

$$\gamma_R(S(C_n,t)) \le 2\gamma(S(C_n,t)) = n^{t-1}\left\lfloor \frac{2n}{3} \right\rfloor. \tag{4}$$

Now, let $n = 3k+2$ for any positive integer k. Set

$$A = \{wij : w \in V^{t-2}, i \in V, j = i+1+3k', 0 \le k' \le k-1\}$$

and

$$B = \{wij : w \in V^{t-2}, i \in V, j = i-2\}.$$

Define $f_2 : V^t \to \{0,1,2\}$ such that

$$f_2(x) = \begin{cases} 2, & x \in A; \\ 1, & x \in B; \\ 0, & \text{otherwise.} \end{cases}$$

Let $w \in V^{t-2}$ and $i, j \in V$ such that $g(wii) = 0$. If $j \equiv i-1 \pmod{n}$, then $wi(i-1) \in N(w(i-1)i) \subset N(A)$. Otherwise, $j \equiv i+3k'$ or $i+2+3k' \pmod{n}$, for $1 \le k' \le k-1$. Hence, $wij \in N(wi(j+1))$ or $wij \in N(wi(j-1))$ respectively. So $wij \in N(A)$. Therefore, g is a Roman dominating function on $S(C_n,t)$ and, as a consequence,

$$\gamma_R(S(C_n,t)) \le \omega(f_2) = 2|A| + |B| = n^{t-1}(2k+1) = n^{t-1}\left\lfloor \frac{2n}{3} \right\rfloor. \tag{5}$$

Now we will find the lower bound for $\gamma_R(S(C_n,t))$. Assume that $f = (B_0, B_1, B_2)$ is a γ_R-function on $S(C_n,t)$. Set

$$C_{wu} = \{wui \in V_{wu} : i \notin \{u-1, u, u+1\}\}$$

for $w \in V^{t-2}$ and $u \in V$. Hence, the subgraph induced by C_{wu} is isomorphic to P_{n-3} and $\omega(V_{wu}) = \omega(C_{wu}) + \sum_{i\in\{u-1,u,u+1\}} f(wui)$. Let

$$D_i = \left\{\langle V_{wu}\rangle : \omega(V_{wu}) = \left\lceil \frac{2n}{3} \right\rceil - 2 + i\right\} \text{ for } i \in \{0,1\}$$

and

$$D_2 = \left\{\langle V_{wu}\rangle : \omega(V_{wu}) = \left\lceil \frac{2n}{3} \right\rceil - 2 + j, \text{ for some } j \ge 2\right\}.$$

Notice that

$$\gamma_R(S(C_n,t)) = \sum_{\langle V_{wu}\rangle \in D_0} \omega(V_{wu}) + \sum_{\langle V_{wu}\rangle \in D_1} \omega(V_{wu}) + \sum_{\langle V_{wu}\rangle \in D_2} \omega(V_{wu}).$$

If $\langle V_{wu}\rangle \in D_0$, then $\{wu(u-1), wuu, wu(u+1)\} \subset B_0$ and so there exists $w' \in V^{t-2}$ and $v \in V$ such that $wuu \in N(w'vv)$ and $f(w'vv) = 2$. Thus, $\langle V_{w'v}\rangle \in D_2$. We can define an injective application $\phi : D_0 \longrightarrow D_2$, so that we emphasize that if $\langle V_{wu}\rangle \in D_0$, then the contribution of $\omega(V_{wu}) + \omega(\phi(\langle V_{wu}\rangle))$ to $\gamma_R(S(C_n,t))$ is greater than or equal to such contribution when both $\langle V_{wu}\rangle$ and $\phi(\langle V_{wu}\rangle)$ belong to D_1. The argument shows that,

$$\gamma_R(S(C_n,t)) = \sum_{w\in V^{t-2}} \sum_{u\in V} \omega(V_{wu}) \ge n^{t-1}\left(\left\lceil \frac{2n}{3} \right\rceil - 1\right).$$

Therefore, the result follows. □

6. The Particular Case of Complete Graphs

The domination number of $S(K_n, t)$ was previously studied by Klavžar, Milutinović and Petr in [13] where they obtained the following result.

Theorem 6.1. [13] *For any integers $n \geq 2$ and $t \geq 1$,*

$$\gamma(S(K_n, t)) = \begin{cases} \dfrac{n^t + n}{n+1}, & t \text{ even;} \\[2ex] \dfrac{n^t + 1}{n+1}, & t \text{ odd.} \end{cases}$$

The above result is an important tool to deduce an upper bound on the Roman domination number of $S(K_n, t)$.

Theorem 6.2. *For any integers $n \geq 2$ and $t \geq 1$,*

$$\gamma_R(S(K_n, t)) \leq \begin{cases} \dfrac{2n^t + n - 1}{n+1}, & t \text{ even;} \\[2ex] \dfrac{2(n^t + 1)}{n+1}, & t \text{ odd.} \end{cases}$$

Proof. Let $V = \{1, 2, \ldots, n\}$ be the vertex set of K_n. For t odd we deduce the bound from Theorem 6.1, as $\gamma_R(S(K_n, t)) \leq 2\gamma(S(K_n, t))$. We claim that for $t = 2k$ there exists a Roman dominating function such that $f(1\ldots1) = 1$ and $\omega(f) = \frac{2n^{2k}+n-1}{n+1}$. To show this we proceed by induction on k. For $k = 1$ we define the Roman dominating function f as follows. $f(11) = 1$, $f(i1) = 2$ for all $i \neq 1$ and $f(xy) = 0$ for others. Notice that $\omega(f) = 2(n-1) + 1 = \frac{2n^2+n-1}{n+1}$.

Now, suppose that f is a Roman dominating function on $S(K_n, 2k)$ such that $f(1\ldots1) = 1$ and $\omega(f) = \frac{2n^{2k}+n-1}{n+1}$. We shall construct a Roman dominating function f' on $S(K_n, 2k+2)$ in the following way:

- $f'(11w) = f(w)$ for all $w \in V^{2k}$.
- $f'(1i\ldots i) = 0$ for all $i \neq 1$ and $f(11w) = f(w')$ for all $w \in V^{2k-2} \setminus \{i\ldots i : i \in V\}$, where w' is obtained from w by exchanging i and 1.
- For any $i \in V \setminus \{1\}$ and $w \in V^{2k}$, we define $f(i1w)$ as follows. As shown in [13, Corollary 3.5], there exists a 1-perfect code C of $S(K_n, 2k)$ which contains all the extreme vertices. So, we set $f'(i1w) = 2$ for all $w \in C$ and $f'(i1w) = 0$ for others.
- $f'(ij1\ldots1) = 0$ and $f'(ijw) = f(w)$ for all $i, j \neq 1$ and $w \neq 1\ldots1$.

Notice that $f'(1\ldots1) = 1$. To conclude that f' is a Roman dominating function on $S(K_n, 2k+2)$ we only need to observe that all $x \in V^{2k+2}$ of the form $x = 1i\ldots i$, $i \neq 1$ are adjacent to $i1\ldots1$ and $f'(i1\ldots1) = 2$, and all $x \in V^{2k+2}$ of the form $x = ij1\ldots1$, $i, j \neq 1$ are adjacent to $i1j\ldots j$ and $f'(i1j\ldots j) = 2$. Finally, by Theorem 6.1, $|C| = \frac{n^{2k}+n}{n+1}$, and so

$$\omega(f') = \omega(f) + (n-1)(\omega(f) - 1) + 2|C|(n-1) + (n-1)^2(\omega(f) - 1) = \frac{2n^{2k+2} + n - 1}{n+1},$$

as required. □

By Remark 1.5 we deduce the following corollary.

Corollary 6.3. *For any graph G of order n and any integer $t \geq 1$,*

$$\gamma_R(S(G,t)) \geq \gamma_R(S(K_n,t)).$$

As the above corollary shows, a lower bound (or a closed formula) on the Roman domination number of $S(K_n,t)$ imposes a lower bound on $\gamma_R(S(G,t))$ for every graph G. Therefore, this issue definitely deserves further research.

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