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Some Additive Results for the Drazin Inverse and its Application

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Abstract. In this paper, we give some formulas for the Drazin inverses of the sum of two matrices under conditions weaker than those used in some current literature. Also, we obtain representation for the Drazin inverse of a complex block matrix under some conditions.

1. Introduction

Let $C^{n\times n}$ denote the set of all $n \times n$ complex matrices. For $A \in C^{n\times n}$, we call the smallest nonnegative integer k which satisfies $rank(A^{k+1}) = rank(A^k)$ the index of A, and denote k by $ind(A)$. Let $A \in C^{n\times n}$ with $ind(A) = k$. The matrix $X \in C^{n\times n}$ which satisfies

$$A^{k+1}X = A^k, XAX = X, XA = AX$$

is called the Drazin inverse of matrix A and is denoted by A^D(see [1]). The Drazin inverse of a square complex matrix always exists and is unique(see [1]). If $ind(A) = 1$, then A^D is called the group inverse of A and is denoted by $A^\sharp$. In this paper, we denote $A^\pi = I - AA^D$.

The Drazin inverse of square complex matrices has applications in several areas, such as singular differential or difference equations, iterative method and perturbation bounds for the relative eigenvalue problem(see [1,2]), respectively. For applications of the Drazin inverse of a 2×2 block matrix, we refer readers to [2,3,4].

Suppose $P, Q \in C^{n\times n}$. In 1958, Drazin offered the formula $(P + Q)^D = P^D + Q^D$, which is valid when $PQ = QP = 0$. In 2001, Hartwig et al. gave the representation of $(P+Q)^D$ when $PQ = 0$(see [5]). In the recent years, many authors have considered this problem and provided the representations of $(P+Q)^D$ with some specific conditions. Some of them are as follows:

(i) $P^2Q = 0, Q^2 = 0$ (see [6]);
(ii) $P^2Q = 0, Q^2P = 0$ (see [7]);
(iii) $QP^2 = 0, Q^2 = 0$ (see [6]);
(iv) $PQ^2 = 0, QP^2 = 0$ (see [7]).

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In this paper we derive the formulas for $(P+Q)^D$ under the conditions $P^3Q=0, P^2QP=0, Q^2PQ=0, Q^2P^2=0, PQ^2P=0, Q^3P=0$ and $PQ^3=0, QPQ^2=0, PQP^2=0, P^2Q^2=0, PQ^2P=0, QP^3=0$. These results are weaker than the formula (i-ii) and (iii-iv) from previous list, respectively.

Another aim of this paper is to derive a representation of the Drazin inverse of 2×2 complex block matrix

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

where A and D are square matrix. This problem was firstly posed in 1979 by Compbell and Meyer [8]. No formula for M^D has yet been offered without any restrictions upon the blocks. Special cases of this problem have been studied. In some papers the expression of M^D is given under conditions which concern the generalized Schur complement of matrix M defined by $S=D-CA^DB$. Here we list some of them:

(i) $CA^\pi=0$, $A^\pi B=0$, and $S=0$ (see [9]);
(ii) $CA^\pi B=0$, $AA^\pi B=0$, and $S=0$ (see [3]);
(iii) $CA^\pi B=0$, $CAA^\pi=0$, and $S=0$ (see [3]);
(iv) $ABCA^\pi=0$, BCA^π is nipotent, and $S=0$ (see [6]);
(v) $A^\pi BCA=0$, $A^\pi BC$ is nipotent, and $S=0$ (see [6]);
(vi) $ABCA^\pi=0$, $A^\pi ABC=0$, and $S=0$ (see [7]);
(vii) $ABCA^\pi=0$, $CBCA^\pi=0$, and $S=0$ (see [7]);
(viii) $ABCA^\pi A=0$, $ABCA^\pi B=0$, and $S=0$ (see [11]);
(ix) $AA^\pi BCA=0$, $CA^\pi BCA=0$, and $S=0$ (see [11]).

In this paper, we derive a new representation for M^D under the conditions $A^\pi ABC=0$, $A^2BCA^\pi A=0$, $A^2BCA^\pi B=0$ and $S=0$. This result generalizes the conditions in [7].

2. Some Lemmas

In order to give the main results, we first give some lemmas as follows.

Lemma 2.1. [6] *Let $P,Q\in C^{n\times n}$,*

(a) *If $PQ=QP=0$, then $(P+Q)^D=P^D+Q^D$.*

(b) *If $PQ=0$ and P is $r-$nilpotent, then $((P+Q)^D)^j=\sum\limits_{i=0}^{r-1}(Q^D)^{i+j}P^i,\ \forall j\geq 1$.*

If $PQ=0$ and Q is $s-$nilpotent, then $((P+Q)^D)^j=\sum\limits_{i=0}^{s-1}Q^i(P^D)^{i+j},\ \forall j\geq 1$.

Lemma 2.2. [5] *Let $P,Q\in C^{n\times n}$ be such that $ind(P)=r$ and $ind(Q)=s$. If $PQ=0$, then*

$$(P+Q)^D=\sum_{i=0}^{s-1}Q^\pi Q^i(P^D)^{i+1}+\sum_{i=0}^{r-1}(Q^D)^{i+1}P^iP^\pi.$$

Lemma 2.3. [9] *Let M be a matrix of the form* (1.1)*, such that $S=0$. If $A^\pi B=0, CA^\pi=0$, then*

$$M^D=\begin{bmatrix} I \\ CA^D \end{bmatrix}((AW)^D)^2A\begin{bmatrix} I & A^DB \end{bmatrix},$$

where $W=AA^D+A^DBCA^D$.

Lemma 2.4. [12] *Let $M=\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, where $A,C\in C^{n\times n}$, then*

$$M^D=\begin{pmatrix} A^D & 0 \\ C(A^D)^2 & 0 \end{pmatrix}.$$

Lemma 2.5. [12] *Let* $M = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}$*, where* $A, B \in C^{n\times n}$*, then*

$$M^D = \begin{pmatrix} 0 & 0 \\ (B^D)^2 A & B^D \end{pmatrix}.$$

Lemma 2.6. [12] *Let* $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$*, where* $A, B \in C^{n\times n}$*, then*

$$M^D = \begin{pmatrix} A^D & (A^D)^2 B \\ 0 & 0 \end{pmatrix}.$$

3. Additive Results

In [7], Bu, Feng and Bai gave the representation of $(P+Q)^D$ when $P^2Q = 0$, $Q^2P = 0$. In this paper, we give the representation of $(P+Q)^D$ when $P^3Q = 0$, $P^2QP = 0$, $Q^2PQ = 0$, $Q^3P = 0$, $Q^2P^2 = 0$,$PQ^2P = 0$, which generalizes the above result.

Theorem 3.1. *Let* $P, Q \in C^{n\times n}$*, such that* $ind(P^2+Q^2) = r$*,* $ind(PQ+QP) = s$*,* $ind(P^2) = r_1$*,* $ind(Q^2) = s_1$*,* $ind(PQ) = r_2$*,* $ind(QP) = s_2$*, if* $P^3Q = 0$*,* $P^2QP = 0$*,* $Q^2PQ = 0$*,* $Q^2P^2 = 0$*,* $PQ^2P = 0$*,* $Q^3P = 0$*, then*

$$(P+Q)^D = (P+Q)(\sum_{i=0}^{s-1}(PQ+QP)^\pi (PQ+QP)^i((Q^2+P^2)^D)^{i+1} + \sum_{i=0}^{r-1}((PQ+QP)^D)^{i+1}(Q^2+P^2)^i(Q^2+P^2)^\pi),$$

where for $n \in N$

$$((Q^2+P^2)^D)^n = (Q^D)^{2n} + (P^D)^{2n} + P^2(Q^D)^{2(n+1)},$$
$$((PQ+QP)^D)^n = \sum_{i=0}^{s_2-1}(QP)^\pi(QP)^i((PQ)^D)^{i+n} + \sum_{i=0}^{r_2-1}((QP)^D)^{i+n}(PQ)^i(PQ)^\pi - \sum_{i=1}^{n-1}((QP)^D)^i((PQ)^D)^{n-i},$$
$$(Q^2+P^2)^\pi = P^\pi - QQ^D - P^2(Q^D)^2,$$
$$(PQ+QP)^\pi = (QP)^\pi(PQ)^\pi - \sum_{i=0}^{s_2-2}(QP)^\pi(QP)^{i+1}((PQ)^D)^{i+1} - \sum_{i=0}^{r_2-2}((QP)^D)^{i+1}(PQ)^{i+1}(PQ)^\pi.$$

Proof. Using definition of the Drazin inverse, we have that

$$(P+Q)^D = (P+Q)((P+Q)^D)^2 = (P+Q)(P^2+Q^2+PQ+QP)^D. \tag{3.1}$$

Denote by $E = P^2 + Q^2$ and $F = PQ + QP$. Since $EF = 0$, matrices E and F satisfy the condition of Lemma 2.2 and therefore

$$(E+F)^D = \sum_{i=0}^{s-1} F^\pi F^i (E^D)^{i+1} + \sum_{i=0}^{r-1}(F^D)^{i+1}E^iE^\pi. \tag{3.2}$$

Furthermore, since $Q^2P^2 = 0$, $PQ^2P = 0$, $P^3Q = 0$, $Q^3P = 0$, by Lemma 2.2, we have

$$E^D = (Q^2+P^2)^D = (Q^D)^2 + (P^D)^2 + P^2(Q^D)^4. \tag{3.3}$$

$$F^D = (PQ+QP)^D = \sum_{i=0}^{s_2-1}(QP)^\pi(QP)^i((PQ)^D)^{i+1} + \sum_{i=0}^{r_2-1}((QP)^D)^{i+1}(PQ)^i(PQ)^\pi, \tag{3.4}$$

and for every $n \in N$, we have

$$(E^D)^n = (Q^D)^{2n} + (P^D)^{2n} + P^2(Q^D)^{2(n+1)}, \tag{3.5}$$

$$(F^D)^n = \sum_{i=0}^{s_2-1}(QP)^\pi(QP)^i((PQ)^D)^{i+n} + \sum_{i=0}^{r_2-1}((QP)^D)^{i+n}(PQ)^i(PQ)^\pi - \sum_{i=1}^{n-1}((QP)^D)^i((PQ)^D)^{n-i}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.2), combined with (3.1), we get that the statement of the theorem is valid. The proof is completed. □

The next theorem is a symmetrical formulation of Theorem 3.1.

Theorem 3.2. *Let $P, Q \in C^{n\times n}$, such that $ind(P^2+Q^2) = r$, $ind(PQ+QP) = s$, $ind(P^2) = r_1$, $ind(Q^2) = s_1$, $ind(PQ) = r_2$, $ind(QP) = s_2$, if $PQ^3 = 0$, $QPQ^2 = 0$, $PQP^2 = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$, $QP^3 = 0$, then*

$$(P+Q)^D = (\sum_{i=0}^{r-1}(P^2+Q^2)^\pi(P^2+Q^2)^i((PQ+QP)^D)^{i+1} + \sum_{i=0}^{s-1}((P^2+Q^2)^D)^{i+1}(PQ+QP)^i(PQ+QP)^\pi)(P+Q).$$

where for $n \in N$

$$\begin{aligned}
&((P^2+Q^2)^D)^n = (P^D)^{2n} + (Q^D)^{2n} + (Q^D)^{2(n+1)}P^2,\\
&((PQ+QP)^D)^n = \sum_{i=0}^{s_2-1}(QP)^\pi(QP)^i((PQ)^D)^{i+n} + \sum_{i=0}^{r_2-1}((QP)^D)^{i+n}(PQ)^i(PQ)^\pi - \sum_{i=1}^{n-1}((QP)^D)^{n-i}((PQ)^D)^i,\\
&(P^2+Q^2)^\pi = P^\pi - QQ^D - (Q^D)^2P^2,\\
&(PQ+QP)^\pi = (QP)^\pi(PQ)^\pi - \sum_{i=0}^{s_2-2}(QP)^\pi(QP)^{i+1}((PQ)^D)^{i+1} - \sum_{i=0}^{r_2-2}((QP)^D)^{i+1}(PQ)^{i+1}(PQ)^\pi.
\end{aligned}$$

Notice that one special case of Theorem 3.1 is when matrices P and Q satisfy the conditions $P^2QP = 0$, $P^3Q = 0$ and $Q^2P = 0$. Similarly, a special case of Theorem 3.2 is when $PQP^2 = 0$, $QP^3 = 0$ and $PQ^2 = 0$ is valid. The following we give the corollaries, which we will use in section 3 to obtain representations for the Drazin inverse of block matrix.

Corollary 3.3. *Let $P, Q \in C^{n\times n}$, such that $ind(P^2+Q^2) = r$, $ind(PQ+QP) = s$, $ind(P^2) = r_1$, $ind(Q^2) = s_1$, $ind(PQ) = r_2$, $ind(QP) = s_2$. If $P^2QP = 0$, $P^3Q = 0$, $Q^2P = 0$, then*

$$\begin{aligned}
(P+Q)^D = (P+Q)(&\sum_{i=2}^{s-1}(PQ+QP)^\pi((PQ)^i + (QP)^{i-1}PQ + (QP)^i)((Q^2+P^2)^D)^{i+1}\\
&+\sum_{i=2}^{r-1}((PQ+QP)^D)^{i+1}(P^{2i} + P^2Q^{2i-2} + Q^{2i})(Q^2+P^2)^\pi\\
&+(PQ+QP)^\pi(Q^2+P^2)^D + (PQ+QP)^D(Q^2+P^2)^\pi\\
&+((PQ+QP)^D)^2(Q^2+P^2)(Q^2+P^2)^\pi\\
&+(PQ+QP)^\pi(PQ+QP)((Q^2+P^2)^D)^2),
\end{aligned}$$

where for $n \in N$, $((Q^2+P^2)^D)^n$, $(Q^2+P^2)^\pi$, $((PQ+QP)^D)^n$ and $(PQ+QP)^\pi$ are defined as in Theorem 3.1.

Corollary 3.4. *Let $P, Q \in C^{n\times n}$, such that $ind(P^2+Q^2) = r$, $ind(PQ+QP) = s$, $ind(P^2) = r_1$, $ind(Q^2) = s_1$, $ind(PQ) = r_2$, $ind(QP) = s_2$. If $PQP^2 = 0$, $QP^3 = 0$, $PQ^2 = 0$, then*

$$(P+Q)^D = (\sum_{i=2}^{r-1}(P^2+Q^2)^\pi(Q^{2i} + Q^{2i-2}P^2 + P^{2i})((PQ+QP)^D)^{i+1}$$

$$+\sum_{i=2}^{s-1}((P^2+Q^2)^D)^{i+1}((PQ)^i+QP(PQ)^{i-1}+(QP)^i)(PQ+QP)^\pi$$
$$+(P^2+Q^2)^\pi(PQ+QP)^D+(P^2+Q^2)^\pi(P^2+Q^2)((PQ+QP)^D)^2$$
$$+(P^2+Q^2)^D(PQ+QP)^\pi+((P^2+Q^2)^D)^2(PQ+QP)(PQ+QP)^\pi)(P+Q),$$

where for $n\in N$, $((P^2+Q^2)^D)^n$, $(P^2+Q^2)^\pi$, $((PQ+QP)^D)^n$ *and* $(PQ+QP)^\pi$ *are defined as in Theorem 3.2.*

4. Representation for the Drazin Inverse of Block Matrix

In this section, we use the additive formula from section 3 to obtain representation for the Drazin inverse of a complex block matrix.

Let $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, where A and D are square matrices and let generalized Schur complement $S=D-CA^DB$ of matrix M be equal to zero. In [6], Martinez-Serrano and Castro-Gonzalez gave the representation for the Drazin inverse of M under the condition $ABC=0$. In [7], Bu et al. gave the formula for M^D under conditions $ABCA^\pi=0$ and $A^\pi ABC=0$. In the following Theorem 4.1, we give the representation for the Drazin inverse of M under the conditions $A^2BCA^\pi A=0$, $A^2BCA^\pi B=0$, $A^\pi ABC=0$. The result generalizes mentioned representations from [6] and [7].

Theorem 4.1. *Let M be a matrix of the form* (1.1) *such that* $S=0$. *If* $A^2BCA^\pi A=0$, $A^2BCA^\pi B=0$, $A^\pi ABC=0$, *then*

$$M^D=M(\sum_{i=2}^{s-1}G\begin{pmatrix}(BC)^{i-1}A^\pi BC & (BC)^{i-1}A^\pi BCA^DB\\ C(BC)^{i-1}A^\pi A & C(BC)^{i-1}A^\pi B\end{pmatrix}L^{i+1}$$
$$+\sum_{i=2}^{r-1}K^{i+1}\begin{pmatrix}(AA^\pi)^{2i} & (AA^\pi)^{2i-1}A^\pi B\\ 0 & 0\end{pmatrix}H+\sum_{i=2}^{r-1}K^{i+1}P^{2i}H+\sum_{i=2}^{r-1}K^{i+1}P^2$$
$$\times\begin{pmatrix}(AA^\pi)^{2i-2} & (AA^\pi)^{2i-3}A^\pi B\\ 0 & 0\end{pmatrix}H+GL+G\begin{pmatrix}A^\pi BC & A^\pi BCA^DB\\ CAA^\pi & CA^\pi B\end{pmatrix}L^2$$
$$+KH+K^2\begin{pmatrix}AA^DBC+A^2 & AA^DBCA^DB+AB\\ CA^2A^D+CA^DBC & CAA^DB+(CA^DB)^2\end{pmatrix}H).$$

where

$$G=\begin{pmatrix}(A^\pi BC)^\pi & -A^\pi BC(A^\pi BC)^DA^DB\\ -CAA^\pi(A^\pi BC)^D & I-CAA^\pi(A^\pi BC)^DA^DB\end{pmatrix},$$
$$H=P^\pi-\begin{pmatrix}AA^\pi(AA^\pi)^D+A^3A^D((AA^\pi)^D)^2 & (AA^\pi)^DA^\pi B+A^3A^D((AA^\pi)^D)^3A^\pi B\\ CA^2A^D((AA^\pi)^D)^2 & CA^2A^D((AA^\pi)^D)^3A^\pi B\end{pmatrix},$$
$$K=\begin{pmatrix}(A^\pi BC)^D & (A^\pi BC)^DA^DB\\ ((CA^\pi B)^D)^2CAA^\pi & (CA^\pi B)^D\end{pmatrix},$$
$$L=(P^D)^2+\begin{pmatrix}((AA^\pi)^D)^2+A^3A^D((AA^\pi)^D)^4 & ((AA^\pi)^D)^3A^\pi B+A^3A^D((AA^\pi)^D)^5A^\pi B\\ CA^2A^D((AA^\pi)^D)^4 & CA^2A^D((AA^\pi)^D)^5A^\pi B\end{pmatrix},$$

and where

$$P=\begin{pmatrix}A^2A^D & AA^DB\\ C & CA^DB\end{pmatrix},$$
$$(P^D)^j=\begin{pmatrix}I\\ CA^D\end{pmatrix}((AW)^D)^{j+1}\begin{pmatrix}A+(AW)^DAA^DBCA^\pi & AA^DB\end{pmatrix},$$
$$W=AA^D+A^DBCA^D,\ \textit{for}\ j\geq 1,$$

$$s = max\,\{ind(A^{\pi}BC), ind(CA^{\pi}B)\},\ r = ind\left(\begin{pmatrix} A^2 + AA^DBC & AB + AA^DBCA^DB \\ CA^2A^D + CA^DBC & CAA^DB + (CA^DB)^2 \end{pmatrix}\right).$$

Proof. Consider the splitting of matrix M,

$$M = \begin{pmatrix} A & B \\ C & CA^DB \end{pmatrix} = \begin{pmatrix} A^2A^D & AA^DB \\ C & CA^DB \end{pmatrix} + \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

If we denote by $P = \begin{pmatrix} A^2A^D & AA^DB \\ C & CA^DB \end{pmatrix}$ and $Q = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}$, we have that $P^3Q = 0, P^2QP = 0, Q^2P = 0$. Hence, the conditions of Corollary 3.3 are satisfied, and

$$\begin{aligned} M^D = (P+Q)^D &= (P+Q)(\sum_{i=2}^{s-1}(PQ+QP)^{\pi}((PQ)^i + (QP)^{i-1}PQ + (QP)^i) \\ &\times((Q^2+P^2)^D)^{i+1} + \sum_{i=2}^{r-1}((PQ+QP)^D)^{i+1}(P^{2i} + P^2Q^{2i-2} + Q^{2i})(Q^2+P^2)^{\pi} \\ &+(PQ+QP)^{\pi}(Q^2+P^2)^D + (PQ+QP)^{\pi}(PQ+QP)((Q^2+P^2)^D)^2 \\ &+(PQ+QP)^D(Q^2+P^2)^{\pi} + ((PQ+QP)^D)^2(Q^2+P^2)(Q^2+P^2)^{\pi}), \end{aligned} \tag{4.1}$$

where

$$ind(P^2+Q^2) = r, ind(PQ+QP) = s.$$

From

$$PQ = \begin{pmatrix} 0 & 0 \\ CAA^{\pi} & CA^{\pi}B \end{pmatrix},$$

$$QP = \begin{pmatrix} A^{\pi}BC & A^{\pi}BCA^DB \\ 0 & 0 \end{pmatrix},$$

$$(PQ)^i = \begin{pmatrix} 0 & 0 \\ C(BC)^{i-1}A^{\pi}A & C(BC)^{i-1}A^{\pi}B \end{pmatrix},\ for\ i \geq 1,$$

$$(QP)^i = \begin{pmatrix} (BC)^{i-1}A^{\pi}BC & (BC)^{i-1}A^{\pi}BCA^DB \\ 0 & 0 \end{pmatrix},\ for\ i \geq 1,$$

using Lemma 2.6, we get

$$Q^D = \begin{pmatrix} (AA^{\pi})^D & ((AA^{\pi})^D)^2A^{\pi}B \\ 0 & 0 \end{pmatrix},$$

$$(QP)^D = \begin{pmatrix} (A^{\pi}BC)^D & (A^{\pi}BC)^DA^DB \\ 0 & 0 \end{pmatrix},$$

applying Lemma 2.5, we know

$$(PQ)^D = \begin{pmatrix} 0 & 0 \\ ((CA^{\pi}B)^D)^2CAA^{\pi} & (CA^{\pi}B)^D \end{pmatrix},$$

then

$$(PQ)^{\pi} = \begin{pmatrix} I & 0 \\ -(CA^{\pi}B)^{D}CA^{\pi}A & (CA^{\pi}B)^{\pi} \end{pmatrix},$$

$$(QP)^{\pi} = \begin{pmatrix} (A^{\pi}BC)^{\pi} & -A^{\pi}BC(A^{\pi}BC)^{D}A^{D}B \\ 0 & I \end{pmatrix}.$$

Let $P_1 = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}$,
then

$$P = P_1 + P_2,\ P_2P_1 = 0,\ P_2^2 = 0,$$

applying Lemma 2.1,

$$(P^D)^j = ((P_1)^D)^j + ((P_1)^D)^{j+1}P_2, \quad \forall j \geq 1. \tag{4.2}$$

Let S_1 be the generalized Schur complement of P_1, then we have

$$S_1 = CA^DB - CAA^D(A^2A^D)^DAA^DB = 0,$$

and

$$(A^2A^D)^{\pi}AA^DB = 0, CAA^D(A^2A^D)^{\pi} = 0,$$

so from Lemma 2.3, we get

$$((P_1)^D)^i = \begin{pmatrix} I \\ CA^D \end{pmatrix}((AW)^D)^{i+1}A\begin{pmatrix} I & A^DB \end{pmatrix},$$

where

$$W = AA^D + A^DBCA^D, for\ i \geq 1.$$

After computation we get, $for\ n \in N$,

$$(PQ+QP)^{\pi} = \begin{pmatrix} (A^{\pi}BC)^{\pi} & -A^{\pi}BC(A^{\pi}BC)^{D}A^{D}B \\ -CAA^{\pi}(A^{\pi}BC)^{D} & I - CAA^{\pi}(A^{\pi}BC)^{D}A^{D}B \end{pmatrix},$$
$$(Q^2+P^2)^{\pi} = P^{\pi} - \begin{pmatrix} AA^{\pi}(AA^{\pi})^{D} + A^3A^D((AA^{\pi})^D)^2 & (AA^{\pi})^{D}A^{\pi}B + A^3A^D((AA^{\pi})^D)^3A^{\pi}B \\ CA^2A^D((AA^{\pi})^D)^2 & CA^2A^D((AA^{\pi})^D)^3A^{\pi}B \end{pmatrix},$$
$$((PQ+QP)^D)^n = \begin{pmatrix} ((A^{\pi}BC)^D)^n & ((A^{\pi}BC)^D)^nA^DB \\ ((CA^{\pi}B)^D)^{n+1}CAA^{\pi} & ((CA^{\pi}B)^D)^n \end{pmatrix},$$
$$((Q^2+P^2)^D)^n = (P^D)^{2n} + \begin{pmatrix} ((AA^{\pi})^D)^{2n} + A^3A^D((AA^{\pi})^D)^{2(n+1)} & ((AA^{\pi})^D)^{2n+1}A^{\pi}B + A^3A^D((AA^{\pi})^D)^{2n+3}A^{\pi}B \\ CA^2A^D((AA^{\pi})^D)^{2(n+1)} & CA^2A^D((AA^{\pi})^D)^{2n+3}A^{\pi}B \end{pmatrix},$$

where

$$(P^D)^j = ((P_1)^D)^j + ((P_1)^D)^{j+1}\begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \forall j \geq 1,$$

$$((P_1)^D)^i = \begin{pmatrix} I \\ CA^D \end{pmatrix}((AW)^D)^{i+1}A\begin{pmatrix} I & A^DB \end{pmatrix},$$

and where

$$W = AA^D + A^DBCA^D, for\ i \geq 1.$$

After substituting this expressions and (4.2) into (4.1), we complete the proof. □

5. Numerical Example

In this section, we give a numerical example to demonstrate the application of Theorem 4.1.

Example 5.1. *Consider the matrix* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

By computing we know the generalized Schur complement $S = D - CA^D B$ is zero and M does not satisfy the condition $A^\pi ABC = 0$, $ABCA^\pi = 0$ in Theorem 4.1 in Ref.[7], however it satisfies the conditions $A^\pi ABC = 0$, $A^2BCA^\pi A = 0$, $A^2BCA^\pi B = 0$ in Theorem 4.1 in this paper.

We have $ind(A) = 1$, and

$$A^D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix},$$

then according to the formula in Theorem 4.1, we have

$$M^D = \begin{pmatrix} \frac{1}{9} & -\frac{5}{36} & -\frac{1}{72} & \frac{1}{9} & -\frac{5}{72} \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{2}{9} & -\frac{1}{36} & \frac{7}{72} & \frac{2}{9} & -\frac{1}{72} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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