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Certain Derived WP-Bailey Pairs and Transformation Formulas for q-Hypergeometric Series

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Abstract. In this paper, several results involving the derived WP-Bailey pairs of sequences are established. Furthermore, by using these results, a number of transformation formulas for basic (or q-) hypergeometric series are derived.

1. Introduction, Notations and Definitions

For $q, \lambda, \mu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [2], [9], [11] and [12]; see also the recent works [3], [4], [5] and [10] dealing with the q-analysis)

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \qquad (|q| < 1;\ \lambda, \mu \in \mathbb{C}), \tag{1.1}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} \left(1 - \lambda q^j\right) \qquad (|q| < 1;\ \lambda \in \mathbb{C}), \tag{1.3}$$

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where, as usual, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}$ denotes the set of positive integers (*with* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). For convenience, we write

$$(a_1, \cdots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n \tag{1.4}$$

and

$$(a_1, \cdots, a_r; q)_\infty = (a_1; q)_\infty \cdots (a_r; q)_\infty. \tag{1.5}$$

In our investigation, we shall also make use of the basic (or q-) hypergeometric function ${}_r\Phi_s$ with r numerator and s denominator parameters, which is defined by (see, for example, [6] and [12, p. 347, Eq. 9.4 (272)])

$${}_r\Phi_s\left[\begin{array}{c} a_1, \cdots, a_r; \\ \\ b_1, \cdots, b_s; \end{array} q, z\right] := \sum_{k=0}^{\infty} (-1)^{(1-r+s)k}\, q^{(1-r+s)\binom{k}{2}} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k}, \tag{1.6}$$

provided that the *generalized basic* (or q-) *hypergeometric series* in (1.6) converges.

Following Andrews' work on the lemma and transform associated with the WP-Bailey pair (see, for details, [1]; see also [2], [8] and [13]), a WP-Bailey pair relative to the parameter a is a pair of sequences $\langle \alpha_n(a, q), \beta_n(a, q)\rangle$ constrained by

$$\begin{aligned} \beta_n(a, k) &= \sum_{r=0}^{n} \frac{(k/a; q)_{n-r}(k; q)_{n+r}}{(q; q)_{n-r}(aq; q)_{n+r}}\, \alpha_r(a, k) \\ &= \frac{(k, k/a; q)_n}{(q, aq; q)_n} \sum_{r=0}^{n} \frac{(q^{-n}, kq^n; q)_r}{(aq^{1-n}/k, aq^{1+n}; q)_r} \left(\frac{aq}{k}\right)^r \alpha_r(a, k). \end{aligned} \tag{1.7}$$

Bailey's definition of a conjugate Bailey pair can now be extended to define a conjugate WP-Bailey pair relative to the parameter a to be a pair of sequences $\langle \gamma_n(a, k), \delta_n(a, k)\rangle$ such that

$$\gamma_n(a, k) = \sum_{r=0}^{\infty} \frac{(k/a; q)_r (k; q)_{r+2n}}{(q; q)_r (aq; q)_{r+2n}}\, \delta_{r+n}. \tag{1.8}$$

Thus, analogous to the Bailey transform [2], we have the following result.

Theorem. *Let $\langle \alpha_n(a, k), \beta_n(a, k)\rangle$ be a WP-Bailey pair. Also let $\langle \gamma_n(a, k), \delta_n(a, k)\rangle$ be a conjugate WP-Bailey pair. Then, under suitable convergence conditions,*

$$\sum_{n=0}^{\infty} \alpha_n(a, k)\gamma_n(a, k) = \sum_{n=0}^{\infty} \beta_n(a, k)\delta_n(a, k). \tag{1.9}$$

For a WP-Bailey pair $\langle \alpha_n(a, q), \beta_n(a, q)\rangle$ and $n \in \mathbb{N}$, let us define

$$\alpha'_n(a) = \lim_{k\to 1} \alpha_n(a, k) \tag{1.10}$$

and

$$\beta'_n(a) = \lim_{k\to 1} \frac{\beta_n(a, k)}{1-k}, \tag{1.11}$$

assuming that each of the limits in (1.10) and (1.11) exists. Following the work presented in [7], such a pair of sequences $\langle \alpha'_n(a), \beta'_n(a)\rangle$ is called a derived WP-Bailey pair.

In our present investigation of (for example) several families of derived WP-Bailey pairs and related transformation formulas for basic (or q-) hypergeometric series defined by (1.6), we shall make use of following known summation formulas.

$$ {}_2\Phi_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; q, \frac{c}{ab}\right] = \frac{\left(\frac{c}{a}, \frac{c}{b}; q\right)_\infty}{\left(c, \frac{c}{ab}; q\right)_\infty} \qquad \left(\left|\frac{c}{ab}\right| < 1\right). \tag{1.12} $$

(see [6, Appendix II, p. 236, Entry (II.8)] and [12, p. 348, Eq. 9.4 (277)])

$$ {}_2\Phi_1\left[\begin{matrix} a, b; \\ cq; \end{matrix}\; q, \frac{c}{ab}\right] = \frac{\left(\frac{cq}{a}, \frac{cq}{b}; q\right)_\infty}{\left(cq, \frac{cq}{ab}; q\right)_\infty}\left(\frac{ab(1+c) - c(a+b)}{ab - c}\right) \qquad \left(\left|\frac{c}{ab}\right| < 1\right). \tag{1.13} $$

(see [14, p. 771, Eq. (1.4)])

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; \\ \frac{a^2q}{b}; \end{matrix}\; q, \frac{q^{3/2}}{b}\right] = \frac{1}{2a}\,\frac{(a^2, q^{1/2}; q)_\infty}{\left(\frac{a^2q}{b}, \frac{q^{1/2}}{b}; q\right)_\infty}\left(\frac{\left(\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(a; q^{1/2})_\infty} - \frac{\left(-\frac{aq^{1/2}}{b}; q^{1/2}\right)_\infty}{(-a; q^{1/2})_\infty}\right) \qquad \left(\left|\frac{q^{3/2}}{b}\right| < 1\right). \tag{1.14} $$

(see [15, p. 75, Eq. (3.6)])

$$ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}; \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{1+n}; \end{matrix}\; q, q\right] = \frac{\left(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{bcd}; q\right)_n} \qquad \left(a^2q = bcdeq^{-n}\right). \tag{1.15} $$

(see [6, Appendix II, p. 238, Entry (II.22)])

$$ \begin{aligned} {}_4\Phi_3\left[\begin{matrix} a, c, \frac{a}{c}\, q^{\frac{1}{2}+m}, q^{-m}; \\ \frac{aq}{c}, cq^{\frac{1}{2}-m}, aq^{1+m}; \end{matrix}\; q, q\right] &= \frac{(a; q)_{m+1}(q^{1/2}; q)_m\left(\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\left(\frac{aq}{c}; q\right)_m\left(\frac{q^{1/2}}{c}; q\right)_m(a^{1/2}; q^{1/2})_{2m+1}} \\ &\quad + \frac{(a; q)_{m+1}(q^{1/2}; q)_m\left(-\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\left(\frac{aq}{c}; q\right)_m(q^{1/2}/c; q)_m(-a^{1/2}; q^{1/2})_{2m+1}}. \end{aligned} \tag{1.16} $$

(see [15, p. 71, Eq. (1.3)])

$$ \begin{aligned} {}_4\Phi_3\left[\begin{matrix} a, c, \frac{a}{c}\, q^{\frac{1}{2}+m}, q^{-m}; \\ \frac{aq}{c}, cq^{\frac{1}{2}-m}, aq^{1+m}; \end{matrix}\; q, q^2\right] &= \frac{(a; q)_{m+1}(q^{1/2}; q)_m\left(\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\sqrt{a}\left(\frac{aq}{c}; q\right)_m\left(\frac{q^{1/2}}{c}; q\right)_m(a^{1/2}; q^{1/2})_{2m+1}} \\ &\quad - \frac{(a; q)_{m+1}(q^{1/2}; q)_m\left(-\frac{\sqrt{aq}}{c}; q^{1/2}\right)_{2m}}{2\sqrt{a}\left(\frac{aq}{c}; q\right)_m\left(\frac{q^{1/2}}{c}; q\right)_m(-a^{1/2}; q^{1/2})_{2m+1}}. \end{aligned} \tag{1.17} $$

(see [15, p. 77, Eq. (4.4)])

2. Transformation Formulas Involving Derived WP-Bailey Pairs

In this section, we consider the following cases.

(a) Equation (1.8) can be put in the following form:

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(k/a;q)_r (kq^{2n};q)_r}{(q;q)_r (aq^{1+2n};q)_r}\, \delta_{r+n}(a,k). \tag{2.1}$$

Upon setting

$$\delta_r(a,k) = \left(\frac{a^2 q}{k^2}\right)^r$$

in (2.1), and upon summing the series by using (1.12), we get

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \frac{(aq/k, a^2q/k;q)_\infty (a^2q/k^2)^n}{(aq, a^2q/k^2;q)_\infty}. \tag{2.2}$$

Substituting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ into (1.9), we have

$$\sum_{n=0}^{\infty} \beta_n(a,k) \left(\frac{a^2q}{k^2}\right)^n = \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a,k) \tag{2.3}$$

which can be rewritten as follows:

$$\sum_{n=1}^{\infty} \beta_n(a,k) \left(\frac{a^2q}{k^2}\right)^n - \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} \sum_{n=1}^{\infty} \frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a,k) = \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} - 1. \tag{2.4}$$

Dividing both sides of (2.4) by $1-k$ and then taking the limit $k \to 1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \beta_n'(a)(a^2q)^n - \sum_{n=1}^{\infty} \frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}} (a^2q)^n\, \alpha_n'(a) &= \lim_{k\to 1} \frac{\dfrac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} - 1}{1-k} \\ &= \lim_{k\to 1} -\frac{d}{dk}\left\{\frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}\right\}.\end{aligned} \tag{2.5}$$

We next assume that

$$y = \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty} = \frac{\prod_{r=0}^{\infty}\left(1 - \frac{aq^{r+1}}{k}\right)\prod_{r=0}^{\infty}\left(1 - \frac{a^2q^{r+1}}{k}\right)}{\prod_{r=0}^{\infty}(1 - aq^{r+1})\prod_{r=0}^{\infty}\left(1 - \frac{a^2q^{r+1}}{k^2}\right)}. \tag{2.6}$$

Now, taking the logarithm of both sides of (2.6), we have

$$\log y = \sum_{r=0}^{\infty} \log\left(1 - \frac{aq^{r+1}}{k}\right) + \sum_{r=0}^{\infty} \log\left(1 - \frac{a^2q^{r+1}}{k}\right) - \sum_{r=0}^{\infty} \log\left(1 - aq^{r+1}\right) - \sum_{r=0}^{\infty} \log\left(1 - \frac{a^2q^{r+1}}{k^2}\right). \tag{2.7}$$

Upon differentiating both sides of (2.7) with respect to k and then taking the limit as $k \to 1$, we have

$$\lim_{k\to 1}\frac{dy}{dk} = \lim_{k\to 1}\frac{d}{dk}\left\{\frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}\right\} = \sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r} - \sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r}. \tag{2.8}$$

Putting the value of $\lim_{k\to 1}\frac{dy}{dk}$ from this last equation (2.8) into (2.5), we finally obtain

$$\sum_{n=1}^{\infty}\beta_n'(a)(a^2q)^n - \sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}(a^2q)^n}{(a^2q;q)_{2n}}\,\alpha_n'(a) = \sum_{r=1}^{\infty}\frac{a^2q^r}{1-a^2q^r} - \sum_{r=1}^{\infty}\frac{aq^r}{1-aq^r}, \tag{2.9}$$

where $\langle \alpha_n'(a), \beta_n'(a)\rangle$ is a *derived* WP-Bailey pair.

(b) Upon setting

$$\delta_r(a,k) = \left(\frac{a^2}{k^2}\right)^r$$

in (1.8) and using the summation formula (1.13), we find that

$$\gamma_n(a,k) = \frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}\left(\frac{k}{k+a}\right)\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}\left(\frac{a^2}{k^2}\right)^n (1+aq^{2n}). \tag{2.10}$$

Substituting these values of $\delta_n(a,k)$ and $\gamma_n(a,k)$ given by (2.10) into (1.9), we get

$$\sum_{n=0}^{\infty}\beta_n(a,k)\left(\frac{a^2}{k^2}\right)^n = \left(\frac{k}{k+a}\right)\frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}\left(\frac{a^2}{k^2}\right)^n(1+aq^{2n})\alpha_n(a,k), \tag{2.11}$$

which can be rewritten as follows:

$$\sum_{n=1}^{\infty}\beta_n(a,k)\left(\frac{a^2}{k^2}\right)^n - \left(\frac{k}{k+a}\right)\frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}\sum_{n=1}^{\infty}\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}\left(\frac{a^2}{k^2}\right)^n(1+aq^{2n})\alpha_n(a,k) = \left(\frac{k}{k+a}\right)\frac{(aq/k, a^2q/k;q)_\infty}{(aq, a^2q/k^2;q)_\infty}(1+a) - 1. \tag{2.12}$$

Dividing both sides of (2.12) by $1-k$ and then taking the limit as $k \to 1$, we get the following result after some simplifications:

$$\sum_{n=1}^{\infty}\beta_n'(a)a^{2n} - \frac{1}{1+a}\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}}\,a^{2n}(1+aq^{2n})\alpha_n'(a) = \sum_{n=1}^{\infty}\frac{a^2q^n}{1-a^2q^n} - \sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n} - \frac{a}{1+a}, \tag{2.13}$$

where $\langle \alpha_n'(a), \beta_n'(a)\rangle$ is a derived WP-Bailey pair.

(c) By setting

$$\delta_r(a,k) = \left(\frac{a}{k}\,q^{3/2}\right)^r$$

in (1.8) and making use of the summation formula (1.14), we have

$$\gamma_n(a,k) = \frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty}{(aq, aq^{1/2}/k; q)_\infty} \left(\frac{a}{k} q^{1/2}\right)^n \cdot \left(\frac{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty (k^{1/2}; q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n (k^{1/2}; q^{1/2})_\infty} - \frac{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty (-k^{1/2}; q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n (-k^{1/2}; q^{1/2})_\infty} \right). \tag{2.14}$$

Putting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ in (1.9), we obtain

$$\sum_{n=0}^{\infty} \beta_n(a,k) \left(\frac{a}{k} q^{3/2}\right)^n = \frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty \left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (k^{1/2}; q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(k^{1/2}; q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(\frac{a}{k} q^{1/2}\right)^n \alpha_n(a,k)$$
$$- \frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty \left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (-k^{1/2}; q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(-k^{1/2}; q^{1/2})_n \left(\frac{a}{k} q^{1/2}\right)^n}{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \alpha_n(a,k), \tag{2.15}$$

which can be rewritten as follows:

$$\sum_{n=1}^{\infty} \beta_n(a,k) \left(\frac{a}{k} q^{3/2}\right)^n - \left(\frac{1+k^{1/2}}{2k^{1/2}}\right) \frac{(kq, q^{1/2}; q)_\infty \left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (k^{1/2} q^{1/2}; q^{1/2})_\infty} \sum_{n=1}^{\infty} \frac{(k^{1/2}; q^{1/2})_n}{\left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(\frac{a}{k} q^{1/2}\right)^n \alpha_n(a,k)$$
$$+ \left(\frac{1-k^{1/2}}{2k^{1/2}}\right) \frac{(kq, q^{1/2}; q)_\infty \left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (-k^{1/2} q^{1/2}; q^{1/2})_\infty} \sum_{n=1}^{\infty} \frac{(-k^{1/2}; q^{1/2})_n}{\left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_n} \left(\frac{a}{k} q^{1/2}\right)^n \alpha_n(a,k)$$
$$= \left(\frac{1+k^{1/2}}{2k^{1/2}}\right) \frac{(kq, q^{1/2}; q)_\infty \left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (k^{1/2} q^{1/2}; q^{1/2})_\infty} - \left(\frac{1-k^{1/2}}{2k^{1/2}}\right) \frac{(kq, q^{1/2}; q)_\infty \left(-\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (-k^{1/2} q^{1/2}; q^{1/2})_\infty} - 1. \tag{2.16}$$

Now, dividing both sides of (2.16) by $1-k$ and then taking the limit as $k \to 1$. we find that

$$\sum_{n=1}^{\infty} \beta_n'(a) (aq^{3/2})^n - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(q^{1/2}; q^{1/2})_{n-1}}{(aq^{1/2}; q^{1/2})_n} (aq^{1/2})^n \alpha_n'(a)$$
$$+ \frac{1}{2} \frac{(q^{1/2}; q^{1/2})_\infty (-aq^{1/2}; q^{1/2})_\infty}{(aq^{1/2}; q^{1/2})_\infty (-q^{1/2}; q^{1/2})_\infty} \sum_{n=1}^{\infty} \frac{(-q^{1/2}; q^{1/2})_{n-1}}{(-aq^{1/2}; q^{1/2})_n} (aq^{1/2})^n \alpha_n'(a)$$
$$= \lim_{k \to 1} \frac{\left(\frac{1+k^{1/2}}{2k^{1/2}}\right) \frac{(kq, q^{1/2}; q)_\infty \left(\frac{aq^{1/2}}{k^{1/2}}; q^{1/2}\right)_\infty}{(aq, aq^{1/2}/k; q)_\infty (k^{1/2} q^{1/2}; q^{1/2})_\infty} - 1}{1-k} - \frac{1}{4} \frac{(q^{1/2}; q^{1/2})_\infty (-aq^{1/2}; q^{1/2})_\infty}{(-q^{1/2}; q^{1/2})_\infty (aq^{1/2}; q^{1/2})_\infty}, \tag{2.17}$$

which yields following result after some simplifications:

$$\sum_{n=1}^{\infty}\beta'_n(a)(aq^{3/2})^n-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(q^{1/2};q^{1/2})_{n-1}}{(aq^{1/2};q^{1/2})_n}(aq^{1/2})^n\alpha'_n(a)$$
$$+\frac{1}{2}\frac{(q^{1/2};q^{1/2})_\infty(-aq^{1/2};q^{1/2})_\infty}{(aq^{1/2};q^{1/2})_\infty(-q^{1/2};q^{1/2})_\infty}\sum_{n=1}^{\infty}\frac{(-q^{1/2};q^{1/2})_{n-1}}{(-aq^{1/2};q^{1/2})_n}(aq^{1/2})^n\alpha'_n(a)$$
$$=-\frac{1}{4}-\frac{1}{4}\frac{(q^{1/2};q^{1/2})_\infty(-aq^{1/2};q^{1/2})_\infty}{(-q^{1/2};q^{1/2})_\infty(aq^{1/2};q^{1/2})_\infty}-\sum_{r=1}^{\infty}\frac{q^r}{1-q^r}$$
$$+\frac{1}{2}\sum_{r=1}^{\infty}\frac{q^{r/2}}{1-q^{r/2}}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{aq^{r/2}}{1-aq^{r/2}}-\sum_{r=1}^{\infty}\frac{aq^{r-\frac{1}{2}}}{1-aq^{r-\frac{1}{2}}}. \tag{2.18}$$

3. Families of Derived WP-Bailey Pairs

In this section, we investigate the following derived WP-Bailey pairs.

(i) In the summation formula (1.15), if we put

$$d=kq^n \qquad \text{and} \qquad e=\frac{a^2q}{bck},$$

then it takes the following form:

$${}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, kq^n, \frac{a^2q}{bck}, q^{-n}; \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}, \frac{a}{k}q^{1-n}, aq^{1+n}; \end{matrix} q, q\right]=\frac{\left(aq, \frac{aq}{bc}, \frac{bk}{a}, \frac{ck}{a}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{k}{a}, \frac{bck}{a}; q\right)_n}. \tag{3.1}$$

Now, by setting

$$\alpha_n(a,k)=\frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bck}; q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bck}{a}; q\right)_n}\left(\frac{k}{a}\right)^n \tag{3.2}$$

in (1.7), and then using (3.1), we have

$$\beta_n(a,k)=\frac{(k, aq/bc, bk/a, ck/a; q)_n}{(q, aq/b, aq/c, bck/a; q)_n}. \tag{3.3}$$

From the WP-Bailey pair $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ given by (3.2) and (3.3), we find that

$$\alpha'_n(a)=\lim_{k\to 1}\alpha_n(a,k)=\frac{\left(a, q\sqrt{a}, -q\sqrt{a}, b, c, \frac{a^2q}{bc}; q\right)_n}{\left(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{bc}{a}; q\right)_n}\left(\frac{1}{a}\right)^n \tag{3.4}$$

and

$$\beta'_n(a)=\lim_{k\to 1}\frac{\beta_n(a,k)}{1-k}=\frac{(q;q)_{n-1}(aq/bc, b/a, c/a; q)_n}{(q;q)_n(aq/b, aq/c, bc/a; q)_n}$$
$$=\frac{(aq/bc, b/a, c/a; q)_n}{(aq/b, aq/c, bc/a; q)_n(1-q^n)}. \tag{3.5}$$

The pair $\langle \alpha'_n(a), \beta'_n(a)\rangle$ given by (3.4) and (3.5) forms a derived WP-Bailey pair.

(ii) If we set

$$c = \frac{a}{k}\, q^{1/2}$$

in (1.16), we get

$$
{}_4\Phi_3\left[\begin{array}{c} a, \frac{a}{k}\, q^{1/2}, kq^n, q^{-n}; \\ kq^{1/2}, \frac{a}{k} q^{1-n}, aq^{1+n}; \end{array} q, q\right] = \left(\frac{1+\sqrt{a}}{2}\right) \frac{(aq, \sqrt{q}; q)_n}{(kq^{1/2}, k/a; q)_n} \frac{\left(\frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(\sqrt{aq}, q\sqrt{a}; q)_n}
$$

$$
+ \left(\frac{1-\sqrt{a}}{2}\right) \frac{(aq, \sqrt{q}; q)_n}{(kq^{1/2}, k/a; q)_n} \frac{\left(-\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(-\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.6}
$$

Now, if we choose

$$\alpha_n(a,k) = \frac{(a, aq^{1/2}/k; q)_n}{(q, kq^{1/2}; q)_n} \left(\frac{k}{a}\right)^n \tag{3.7}$$

in (1.7) and make use of (3.6), we find that

$$
\beta_n(a,k) = \left(\frac{1+\sqrt{a}}{2}\right) \frac{\left(k, \sqrt{q}, \frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, \sqrt{aq}, q\sqrt{a}; q)_n} + \left(\frac{1-\sqrt{a}}{2}\right) \frac{\left(k, \sqrt{q}, -\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{(q, kq^{1/2}, -\sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{3.8}
$$

Now, by making use of (3.7) and (3.8), we obtain the following derived WP-Bailey pair:

$$\alpha'_n(a) = \lim_{k\to 1} \alpha_n(a,k) = \frac{(a, aq^{1/2}; q)_n}{(q, q^{1/2}; q)_n} \left(\frac{1}{a}\right)^n \tag{3.9}$$

and

$$
\beta'_n(a) = \lim_{k\to 1} \frac{\beta_n(a,k)}{1-k} = \left(\frac{1+\sqrt{a}}{2}\right) \frac{\left(\frac{1}{\sqrt{a}}, \sqrt{\frac{q}{a}}; q\right)_n}{(\sqrt{aq}, q\sqrt{a}; q)_n (1-q^n)} + \left(\frac{1-\sqrt{a}}{2}\right) \frac{\left(-\frac{1}{\sqrt{a}}, -\sqrt{\frac{q}{a}}; q\right)_n}{(-\sqrt{aq}, -q\sqrt{a}; q)_n (1-q^n)}. \tag{3.10}
$$

The pair $\langle \alpha'_n(a), \beta'_n(a)\rangle$ given by (3.7) and (3.10) forms a derived WP-Bailey pair.

(iii) Putting

$$c = \frac{a}{k}\, q^{1/2}$$

in (1.17), we obtain

$$
{}_4\Phi_3\left[\begin{array}{c} a, \frac{a}{k}\, q^{1/2}, kq^n, q^{-n}; \\ kq^{1/2}, \frac{a}{k}\, q^{1-n}, aq^{1+n}; \end{array} q, q^2\right] = \left(\frac{1+\sqrt{a}}{2\sqrt{a}}\right) \frac{\left(aq, \sqrt{q}, \frac{k}{\sqrt{a}}, k\sqrt{\frac{q}{a}}; q\right)_n}{\left(kq^{1/2}, \frac{kq^{1/2}}{a}, \sqrt{aq}, q\sqrt{a}; q\right)_n}
$$

$$
- \left(\frac{1-\sqrt{a}}{2\sqrt{a}}\right) \frac{\left(aq, \sqrt{q}, -\frac{k}{\sqrt{a}}, -k\sqrt{\frac{q}{a}}; q\right)_n}{\left(kq^{1/2}, \frac{kq^{1/2}}{a}, -\sqrt{aq}, -q\sqrt{a}; q\right)_n}. \tag{3.11}
$$

Thus, if we choose

$$\alpha_n(a,k)=\frac{\left(a,\frac{aq^{1/2}}{k};q\right)_n}{(q,kq^{1/2};q)_n}\left(\frac{kq}{a}\right)^n \tag{3.12}$$

in (1.7) and make use of (3.11), we find that

$$\beta_n(a,k)=\left(\frac{1+\sqrt{a}}{2\sqrt{a}}\right)\frac{\left(k,\sqrt{q},\frac{k}{\sqrt{a}},k\sqrt{\frac{q}{a}};q\right)_n}{(q,kq^{1/2},\sqrt{aq},q\sqrt{a};q)_n}-\left(\frac{1-\sqrt{a}}{2\sqrt{a}}\right)\frac{\left(k,\sqrt{q},-\frac{k}{\sqrt{a}},-k\sqrt{\frac{q}{a}};q\right)_n}{(q,kq^{1/2},-\sqrt{aq},-q\sqrt{a};q)_n}. \tag{3.13}$$

Now, by applying (3.12) and (3.13), we have

$$\alpha'_n(a)=\lim_{k\to 1}\alpha_n(a,k)=\frac{(a,aq^{1/2};q)_n}{(q,q^{1/2};q)_n}\left(\frac{q}{a}\right)^n \tag{3.14}$$

and

$$\beta'_n(a)=\lim_{k\to 1}\frac{\beta_n(a,k)}{1-k}=\left(\frac{1+\sqrt{a}}{2\sqrt{a}}\right)\frac{\left(\frac{1}{\sqrt{a}},\sqrt{\frac{q}{a}};q\right)_n}{(1-q^n)(\sqrt{aq},q\sqrt{a};q)_n}-\left(\frac{1-\sqrt{a}}{2\sqrt{a}}\right)\frac{\left(-\frac{1}{\sqrt{a}},-\sqrt{\frac{q}{a}};q\right)_n}{(1-q^n)(-\sqrt{aq},-q\sqrt{a};q)_n}. \tag{3.15}$$

The pair $\langle\alpha'_n(a),\beta'_n(a)\rangle$ given by (3.14) and (3.15) forms another derived WP-Bailey pair.

4. Transformation Formulas for q-Series

In this section, we establish the following transformations of q-series into Lambert series.

(1) Putting the derived WP-Bailey pair $\langle\alpha'_n(a),\beta'_n(a)\rangle$ given by (3.4) and (3.5) in the equation (2.9), we get

$$\sum_{n=1}^{\infty}\frac{(aq/bc,b/a,c/a;q)_n(a^2q)^n}{(1-q^n)(aq/b,aq/c,bc/a;q)_n}-\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}}{(a^2q;q)_{2n}}(aq^n)\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2q/bc;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bc/a;q)_n}$$
$$=\sum_{n=1}^{\infty}\frac{a^2q^n}{1-a^2q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}. \tag{4.1}$$

(2) Putting the derived WP-Bailey pair $\langle\alpha'_n(a),\beta'_n(a)\rangle$ given by (3.4) and (3.5) in the equation (2.13) instead of (2.9), we find that

$$\sum_{n=1}^{\infty}\frac{(aq/bc,b/a,c/a;q)_na^{2n}}{(1-q^n)(aq/b,aq/c,bc/a;q)_n}-\frac{1}{(1+a)}\sum_{n=1}^{\infty}\frac{(q;q)_{2n-1}a^{2n}(1+aq^{2n})}{(a^2q;q)_{2n}}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2q/bc;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bc/a;q)_n}$$
$$=\sum_{n=1}^{\infty}\frac{a^2q^n}{1-a^2q^n}-\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n}-\frac{a}{1+a}. \tag{4.2}$$

(3) We put the derived WP-Bailey pair $\langle\alpha'_n(a),\beta'_n(a)\rangle$ given by (3.4) and (3.5) in the equation (2.18). We thus obtain

$$\sum_{n=1}^{\infty}\frac{(aq/bc,b/a,c/a;q)_n(aq^{3/2})^n}{(1-q^n)(aq/b,aq/c,bc/a;q)_n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(q^{1/2};q^{1/2})_{n-1}}{(aq^{1/2};q^{1/2})_n}q^{n/2}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2q/bc;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bc/a;q)_n}$$
$$+\frac{1}{2}\frac{(q^{1/2};q^{1/2})_\infty(-aq^{1/2};q^{1/2})_\infty}{(-q^{1/2};q^{1/2})_\infty(aq^{1/2};q^{1/2})_\infty}\sum_{n=1}^{\infty}\frac{(-q^{1/2};q^{1/2})_{n-1}}{(-aq^{1/2};q^{1/2})_n}q^{n/2}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,a^2q/bc;q)_n}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,bc/a;q)_n}$$
$$=-\frac{1}{4}-\frac{1}{4}\frac{(q^{1/2};q^{1/2})_\infty(-aq^{1/2};q^{1/2})_\infty}{(-q^{1/2};q^{1/2})_\infty(aq^{1/2};q^{1/2})_\infty}-\sum_{r=1}^{\infty}\frac{q^r}{(1-q^r)}-\sum_{r=1}^{\infty}\frac{aq^{r-\frac{1}{2}}}{1-aq^{r-\frac{1}{2}}}$$
$$+\frac{1}{2}\sum_{r=1}^{\infty}\frac{aq^{r/2}}{(1-aq^{r/2})}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{q^{r/2}}{1-q^{r/2}}. \tag{4.3}$$

Proceeding as in the above three cases, if we substitute the derived WP-Bailey pair $\langle\alpha_n'(a),\beta_n'(a)\rangle$ given by (3.9) and (3.10), and the derived Bailey pair $\langle\alpha_n'(a),\beta_n'(a)\rangle$ given by (3.14) and (3.15), in the equations (2.9), (2.13) and (2.18), we shall get three more transformation formulas of the above type. The details involved are being left as an exercise for the interested reader.

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