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Entire Functions Sharing a Linear Polynomial with Linear Differential Polynomials

Goutam Kumar Ghosh[a]

[a] *Assistant Professor, Department of Mathematics, Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya, Hatgobindapur, Burdwan, W.B, India*

Abstract. In the paper we study the uniqueness of entire functions sharing a linear polynomial with linear differential polynomials generated by them. The results of the paper improves the corresponding results of P. Li (Kodai Math J. 22: 446–457, 1999), Lahiri-Present author(G. K. Ghosh) (Analysis (Munich)31: 331–340,2011) and Lahiri-Mukherjee(Bull. Aust. Math. Soc. 85: 295–306, 2012).

1. Introduction, Definitions and Results

In the paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [1]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$, as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let a be a small function of f. We denote by $E(a; f)$ the set of a-pionts of f, where each point is counted according its multiplicity. We denote by $\overline{E}(a; f)$ the reduced form of $E(a; f)$. We say that f, g share a CM, provided that $E(a; f) = E(a; g)$, and we say that f and g share a IM, provided that $\overline{E}(a; f) = \overline{E}(a; g)$. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM.

We require the following definitions.

Definition 1.1. *A meromorphic function $a = a(z)$ is called a small function of f if $T(r, a) = S(r, f)$.*

Definition 1.2. *Let f and g be two non-constant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g \neq b)$($\overline{N}(r, a; f \mid g \neq b)$ the counting function (reduced counting function) of those a-points of f which are not the b-points of g.*

Definition 1.3. *Let f and g be two non-constant meromorphic functions defined in $\mathbb{C}$. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid g = b)$($\overline{N}(r, a; f \mid g = b)$ the counting function (reduced counting function) of those a-points of f which are the b-points of g.*

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Email address: g80g@rediffmail.com (Goutam Kumar Ghosh)

In 1986 G. Jank, E. Mues and L. Volkman [2] considered the case when an entire function shared a single value with its first two derivatives and proved the following result.

Theorem 1.4. *[2] Let f be a nonconstant entire function and $a(\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

In fact, in Theorem 1.4 f and $f^{(1)}$ share the value a CM(counting multiplicities). Again considering $f = e^{wz} + w - 1$, where $w^{m-1} = 1, w \neq 1$ and $m(\geq 3)$ is an integer and $a = w$, we can verify that the second derivative in Theorem 1.4 can not be simply replaced by the m^{th} derivative for $m \geq 3$(see [9]) .

In 1995 H. Zhong[9] generalised Theorem 1.4 and proved the following theorem.

Theorem 1.5. *[9] Let f be a non-constant entire function and $a(\neq 0)$ be a finite complex number. If f and $f^{(1)}$ share the value a CM and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n \geq 1$, then $f \equiv f^{(n)}$.*

For $A \subset \mathbb{C} \cup \{\infty\}$, we denote by $N_A(r, a; f)(\overline{N}_A(r, a; f))$ the counting function (reduced counting function) of those a-points of f which belong to A.

In 2011 I. Lahiri and Present author(G. K. Ghosh) [3] improved Theorem 1.5 in the following manner.

Theorem 1.6. *[3] Let f be a nonconstant entire function and a, b be two nonzero finite constants. Suppose further that $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(b; f^{(n+1)})\}$ for $n(\geq 1)$. If each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity and $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, then $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ or $f = \lambda e^{\frac{bz}{a}} + a$, where $\lambda(\neq 0)$ is a constant.*

In 1999 P. Li [6] improved Theorem 1.5 by considering a linear differential polynomial instead of the derivative. The result of P. Li may be stated as follows:

Theorem 1.7. *[6] Let f be nonconstant entire function and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \cdots, a_n(\neq 0)$ are constants. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.*

In the same paper P. Li [6] also proved the following result.

Theorem 1.8. *[6] Let f be a non-constant entire function and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \ldots, a_n(\neq 0)$ are constants. If $\overline{E}(a; f) = \overline{E}(a; L)$, $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})$ and $\sum_{k=1}^{n} 2^k a_k \neq 0$ or $\sum_{k=1}^{n} a_k \neq -1$, then $f \equiv f^{(1)} \equiv L$.*

In 2011 I. Lahiri and G. K. Ghosh [4] improved Theorem 1.8 by replacing the nature of sharing in the following manner.

Theorem 1.9. *[4] Let f be a non-constant entire function in $\mathbb{C}$, a be a finite nonzero complex number and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$,where $a_1, a_2, \ldots, a_n(\neq 0)$ are constants.*

Further suppose that $E_{1)}(a; f) \subset E(a; f^{(1)})$ and $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$, where $A = E(a; f) \setminus E(a; L)$ and $B = E(a; L) \setminus \{E(a; f^{(1)}) \cap E(a; L^{(1)})\}$. Then one of the following cases holds:

(i) *$f = a + \alpha e^z$ and $L = \alpha e^z$, where α is a nonzero constant;*
(ii) *$f = L = \alpha e^z$, where α is a nonzero constant;*
(iii) *$f = a + \frac{\alpha^2}{a} e^{2z} - \alpha e^z$ and $L = \alpha e^z$, where $\sum_{k=1}^{n} 2^k a_k = 0$, $\sum_{k=1}^{n} a_k = -1$ and α is a nonzero constant.*

In the same paper I. Lahiri and G. K. Ghosh also proved the following result.

Theorem 1.10. *[4] Let f be a nonconstant entire function in $\mathbb{C}$, a be a finite nonzero complex number and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$,where $a_1, a_2, \ldots, a_n(\neq 0)$are constants. Further let $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$, where $A = E(a; f) \setminus E(a; L)$ and $B = E(a; L) \setminus \{E(a; f^{(1)}) \cap E(a; L^{(1)})\}$. If $f \not\equiv L$ then one of the following holds:*

(i) *$f = a + \alpha e^z$ and $L = \alpha e^z$, where α is a nonzero constant;*

(ii) $f = a + \frac{\alpha^2}{a}e^{2z} - \alpha e^z$ *and* $L = \alpha e^z$, *where* $\sum_{k=1}^{n} 2^k a_k = 0, \sum_{k=1}^{n} a_k = -1$ *and* α *is a nonzero constant.*

In 2012 I. Lahiri and R. Mukherjee [5] improved Theorem 1.7 in the following manner.

Theorem 1.11. *[5] Let* f *be a non-constant entire function in* $\mathbb{C}$, a *be a finite nonzero complex number and* $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$ *,where* $a_1, a_2, \ldots, a_n (\neq 0)$ *are constants.*
Suppose further that:

(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, *where* $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)})$ *and* $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$;
(ii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; L^{(1)})$; *and*
(iii) $\overline{E}_{(2}(a; f) \cap \overline{E}(0; L^{(1)}) = \emptyset$.
Then $L = \alpha e^z$ *and* $f = \alpha e^z$ *or* $f = a + \alpha e^z$, *where* $\alpha (\neq 0)$ *is a constant.*

In the paper we consider the situation when a nonconstant entire function f share a linear polynomial $a(z) = \alpha z + \beta$, $\alpha (\neq 0)$ and β are constants, with their linear differential polynomial L, $L^{(1)}$.
We now state the main result of the paper.

Theorem 1.12. *Let* f *be a nonconstant entire function in* $\mathbb{C}$, $a = \alpha z + \beta (\not\equiv f)$, *where* $\alpha (\neq 0)$ *and* β *are constants, and* $L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)}$ *,where* $a_2, a_3, \ldots, a_n (\neq 0)$ *are constants.*
Further suppose that

(i) $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$, *where* $A = \overline{E}(a; f) \setminus \overline{E}(a; L)$ *and* $B = \overline{E}(a; L) \setminus \{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(2)}) \cap \overline{E}(a; L^{(1)})\}$;
(ii) $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$; *and*
(iii) $\overline{N}_{(2}(r, a; f) = S(r, f)$.
Then $f = L = ce^z$, *or* $f = a + ce^z$ *and* $L = L^{(1)} = ce^z$ *and* $\sum_{k=2}^{n} a_k = 1$, *where* $c (\neq 0)$ *is a constant.*

In the next theorem we see the possible form of an entire function if we drop the hypothesis $\overline{E}_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$. In fact the Case 2. of the proof of Theorem 1.12 suggests the following theorem.

Theorem 1.13. *Let* f *be a nonconstant entire function in* $\mathbb{C}$, $a = \alpha z + \beta (\not\equiv f)$, *where* $\alpha (\neq 0)$ *and* β *are constants, and* $L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)}$ *,where* $a_2, a_3, \ldots, a_n (\neq 0)$ *are constants. Further let* $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$ *and* $\overline{N}_{(2}(r, a; f) = S(r, f)$, *where* $A = \overline{E}(a; f) \setminus \overline{E}(a; L)$ *and* $B = \overline{E}(a; L) \setminus \{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(2)}) \cap \overline{E}(a; L^{(1)})\}$. *If* $f \not\equiv L$ *then* $f = a + ce^z$ *and* $L = L^{(1)} = ce^z$ *and* $\sum_{k=2}^{n} a_k = 1$, *where* $c (\neq 0)$ *is a constant.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. { *p.47[1]* } *Let* f *be a nonconstant meromorphic function and* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\mu) = S(r, f)$ *for* $\mu = 1, 2, 3$. *Then*

$$T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

Lemma 2.2. { *p.57 [1]* } *Suppose that* g *is a nonconstant meromorphic function and* $\Psi = \sum_{\mu=0}^{l} a_\mu g^{(\mu)}$ *where* $a'_\mu s$ *are meromorphic functions satisfying* $T(r, a_\mu) = S(r, g)$ *for* $\mu = 0, 1, 2, \cdots, l$. *If* Ψ *is nonconstant, then*

$$T(r, g) \leq \overline{N}(r, \infty; g) + N(r, 0; g) + \overline{N}(r, 1; \Psi) + S(r, g).$$

Lemma 2.3. *Let f be a transcendental meromorphic function and $a = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. Then for a positive integer n*

$$T(r, f) \leq \overline{N}(r, \infty; f) + N(r, a; f) + \overline{N}(r, a; L) + S(r, f).$$

Proof: The lemma follows from Lemma 2.2 for $g = f - a, a_0 = a_1 = 0$ and $\Psi = \frac{L}{a}$. This proves the lemma.

Lemma 2.4. { *p.68 [1]* } *Let f be meromorphic and transcendental function in $\mathbb{C}$ and $f^n P = Q$, where P, Q are differential polynomials in f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

Proof. [Proof of Theorem 1.12] First we verify that f can not be a polynomial. If f is a polynomial, then $T(r, f) = O(\log r)$. Since f is a polynomial so $f - a$ and $L - a$ have only finite number of zeros. If $A \neq \emptyset$ then A contains finite number of zeros of $f - a$. Then $N_A(r, a; f) = O(\log r)$, similarly $N_B(r, a; L) = O(\log r)$ so $N_A(r, a; f) + N_B(r, a; L) = O(\log r)$. But by the hypothesis $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$. Therefore $T(r, f) = O(\log r) = S(r, f)$, a contradiction. Hence $A = B = \emptyset$. Therefore $\overline{E}(a; f) \subset \overline{E}(a; L) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(2)}) \cap \overline{E}(a; L^{(1)})$.

Let $f = A_1 z + B_1$, where $A_1(\neq 0), B_1$ are constants. Then $f^{(1)} = A_1, f^{(2)} = 0, L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)} = 0 = L^{(1)}$. Now $f - a = A_1 z + B_1 - \alpha z - \beta = 0$, implies $z = \frac{\beta - B_1}{A_1 - \alpha}$ is the only zero of $f - a$, $\frac{A_1 - \beta}{\alpha}$ is the only zero of $f^{(1)} - a$ and $-\frac{\beta}{\alpha}$ is the only zero of $L - a$ and also since $\overline{E}(a; L) \subset \overline{E}(a; f^{(1)})$ so, $\frac{A_1 - \beta}{\alpha} = -\frac{\beta}{\alpha}$ implies $A_1 = 0$, which is a contradiction.

We denote by $N_{(2}(r, a; f \mid L = a)$ the counting function (counted with multiplicities) of those multiple a-points of f which are a-points of L. We first note that

$$\begin{aligned} N_{(2}(r, a; f) &\leq N_A(r, a; f) + N_{(2}(r, a; f \mid L = a) \\ &\leq n\overline{N}_{(2}(r, a; f) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Now let f be a polynomial of degree greater than 1. Since $N_{(2}(r, a; f) = S(r, f)$, we see that $f - a$ has no multiple zero and so all the zeros of $f - a$ are distinct. Since $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)})$ and $deg(f - a) = deg(f^{(1)} - a) + 1$, we arrive at a contradiction.

Therefore f is a transcendental entire function. Now we consider the following cases.

Case 1. Let $f \equiv L$. Then $f^{(1)} \equiv L^{(1)}$. Now

$$\begin{aligned} m(r, a; f) &= m(r, \frac{1}{f - a}) \\ &= m(r, \frac{f^{(1)} - a^{(1)}}{f - a} \cdot \frac{1}{f^{(1)} - a^{(1)}}) \\ &\leq m(r, \frac{1}{f^{(1)} - a^{(1)}}) + S(r, f) \\ &\leq m(r, \frac{a^{(1)}}{f^{(1)} - a^{(1)}} + 1) + S(r, f) \\ &= m(r, \frac{L^{(1)}}{f^{(1)} - a^{(1)}}) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{1}$$

We now define λ to be

$$\lambda = \frac{f^{(1)} - a}{f - a}. \tag{2}$$

From the hypotheses we see that λ has no simple pole and

$$\begin{aligned} N(r,\lambda) &\leq N_A(r,a;f)+N_B(r,a;L)+S(r,f) \\ &= S(r,f) \end{aligned}$$

and from (1) we get

$$\begin{aligned} m(r,\lambda) &= m(r,\frac{f^{(1)}-a}{f-a}) \\ &= m(r,\frac{f^{(1)}-a^{(1)}}{f-a}+\frac{a^{(1)}-a}{f-a})+S(r,f) \\ &\leq m(r,\frac{a^{(1)}-a}{f-a})+S(r,f) \\ &= S(r,f). \end{aligned}$$

Hence $T(r,\lambda)=S(r,f)$. From (2) we get

$$f^{(1)}=\lambda_1 f+\mu_1, \tag{3}$$

where $\lambda_1=\lambda$ and $\mu_1=a(1-\lambda)$.

We repeat the above argument $(k-1)$-times by differentiating (3) we get

$$f^{(k)}=\lambda_k f+\mu_k (k=1,2,\ldots), \tag{4}$$

where λ_k and μ_k are meromorphic functions satisfying $\lambda_k=\lambda_{k-1}^{(1)}+\lambda_1\lambda_{k-1}$ and $\mu_k=\mu_{k-1}^{(1)}+\mu_1\lambda_{k-1}$ for $k=2,3,\ldots$. Also we note that $T(r,\lambda_k)+T(r,\mu_k)=S(r,f)$ for $k=1,2,\ldots$.

Now

$$L=\sum_{k=2}^{n}a_k f^{(k)}=(\sum_{k=2}^{n}a_k\lambda_k)f+\sum_{k=2}^{n}a_k\mu_k=\xi f+\eta, \text{say}. \tag{5}$$

Clearly $T(r,\xi)+T(r,\eta)=S(r,f)$. Differentiating (5) we get

$$L^{(1)}=\xi f^{(1)}+\xi^{(1)}f+\eta^{(1)}. \tag{6}$$

Let z_0 be a simple zero of $f-a$ such that $z_0\notin A\cup B\cup C$ where $C=\{z: a(z)-a^{(1)}(z)=0\}$. Then from (5) and (6) we get $a(z_0)\xi(z_0)+\eta(z_0)=a(z_0)$ and $a(z_0)\xi(z_0)+a(z_0)\xi^{(1)}(z_0)+\eta^{(1)}(z_0)=a(z_0)$. First suppose that $a\xi+\eta\not\equiv a$. Since every multiple zero of $f-a$ must belong to $A\cup B\cup C$ then we get

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f)+N_B(r,a;L)+N(r,a;a\xi+\eta) \\ &= S(r,f), \end{aligned}$$

which is impossible because we have from (1) $m(r,a;f)=S(r,f)$. Hence

$$a\xi+\eta\equiv a. \tag{7}$$

Similarly

$$a\xi+a\xi^{(1)}+\eta^{(1)}\equiv a. \tag{8}$$

Differentiating (7) and then subtract (8) we get $a-a^{(1)}=\xi(a-a^{(1)})$. Since $a\not\equiv a^{(1)}$ we get $\xi\equiv 1$ and $\eta\equiv 0$. Then from (5) we get $f\equiv L$.

By actual calculation we see that $\lambda_2=\lambda^2+\lambda^{(1)}$ and $\lambda_3=\lambda^3+3\lambda\lambda^{(1)}+\lambda^{(2)}$. In general, we now verify that

$$\lambda_k=\lambda^k+P_{k-1}[\lambda], \tag{9}$$

where $P_{k-1}[\lambda]$ is a differential polynomial in λ with constant coefficients having degree at most $k-1$ and weight at most k. Also we note that each term of $P_{k-1}[\lambda]$ contains some derivative of λ.

Let (9) be true. Then

$$\begin{aligned}\lambda_{k+1} &= \lambda_k^{(1)} + \lambda_1\lambda_k \\ &= (\lambda^k + P_{k-1}[\lambda])^{(1)} + \lambda(\lambda^k + P_{k-1}[\lambda]) \\ &= \lambda^{k+1} + k\lambda^{k-1}\lambda^{(1)} + (P_{k-1}[\lambda])^{(1)} + \lambda P_{k-1}[\lambda] \\ &= \lambda^{k+1} + P_k[\lambda],\end{aligned}$$

noting that differentiation does not increase the degree of a differential polynomial but increase its weight by 1. So (9) is verified by mathematical induction.

Since $\sum\limits_{k=2}^{n} a_k\lambda_k = \xi = 1$, we get from (9)

$$\sum_{k=2}^{n} a_k\lambda^k + \sum_{k=2}^{n} a_k P_{k-1}[\lambda] \equiv 1. \tag{10}$$

If z_0 is a pole of λ with multiplicity $p(\geq 2)$, then z_0 is a pole of $\sum\limits_{k=2}^{n} a_k\lambda^k$ with multiplicity np and it is a pole of $\sum\limits_{k=2}^{n} a_k P_{k-1}[\lambda]$ with multiplicity not exceeding $(n-1)p+1$. Since $np > (n-1)p+1$, it follows that z_0 is a pole of the left hand side of (10) with multiplicity np, which is impossible. So λ is an entire function. If λ is transcendental, from (10)we get by Lemma 2.4 that $m(r,\lambda) = S(r,\lambda)$ and if λ is a polynomial then following the proof of Lemma 2.4 we get $m(r,\lambda) = O(1)$. Therefore λ is a constant. Hence from (9) we obtain $\lambda_k = \lambda^k$ for $k = 1, 2, \ldots$.

Since $\xi \equiv 1$, we see that $\sum\limits_{k=2}^{n} a_k\lambda^k \equiv 1$. Also from (3) we obtain $f^{(1)} = \lambda f + a(1-\lambda)$ then $f^{(2)} = \lambda f^{(1)} + \alpha(1-\lambda)$ and $f^{(3)} = \lambda f^{(2)}$ and so $f^{(2)} = ce^{\lambda z}$, where $c(\neq 0)$ is a constant. Then $f^{(1)} = \frac{ce^{\lambda z}}{\lambda} + d$. Since $L \equiv f$ then also $L^{(1)} \equiv f^{(1)}$ implies
$L^{(1)} = a_2 f^{(3)} + a_3 f^{(4)} + \cdots + a_n f^{(n+1)} = ce^{\lambda z}(a_2\lambda + a_3\lambda^2 + \cdots + a_n\lambda^{n-1}) = f^{(1)} = \frac{ce^{\lambda z}}{\lambda} + d$ then $d = 0$ and $\sum\limits_{k=2}^{n} a_k\lambda^k = 1$. So $f^{(1)} = \frac{ce^{\lambda z}}{\lambda}$ then $f = \frac{ce^{\lambda z}}{\lambda^2} + d_1$. Since $m(r,a;f) = S(r,f)$ then obviously $N(r,a;f) \neq S(r,f)$. By hypothesis $N_A(r,a;f) + N_B(r,a;L) = S(r,f)$ so $E(a;f) \cap E(a;f^{(1)}) \neq \emptyset$. Hence from $f^{(1)} = \frac{ce^{\lambda z}}{\lambda}$ and $f = \frac{ce^{\lambda z}}{\lambda^2} + d_1$ we get $d_1 = 0$ and $\lambda = 1$. Hence $L \equiv f \equiv ce^z$ and $\sum\limits_{k=2}^{n} a_k = 1$.

Case 2. Let $f \not\equiv L$.

Subcase 2.1. Let $L \equiv L^{(1)} \equiv f^{(1)}$. Then $L \equiv L^{(1)}$ implies $L = ce^z$. Hence $L \equiv L^{(1)} \equiv f^{(1)} = ce^z$ then $f = ce^z + d$, which implies f does not assume the values d and ∞, by Lemma 2.1 we get

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f-a) + \overline{N}(r,0;f-\infty) + \overline{N}(r,0;f-d) \\ &\leq \overline{N}(r,a;f).\end{aligned}$$

This implies $\overline{N}(r,a;f) \neq S(r,f)$. Also since $N_A(r,a;f) + N_B(r,a;L) = S(r,f)$ and $f = ce^z + d = L + d$ we see that $\overline{E}(r,a;f) \cap \overline{E}(r,a;L) \neq \emptyset$ this implies $d = 0$ and so $f \equiv L$, we arrive at a contradiction.

Subcase 2.2. Suppose that $L^{(1)} \not\equiv f^{(1)}$. Here we have to consider following subcases.

Subcase 2.2.1. Suppose $L \equiv L^{(1)}$ and $L \not\equiv f^{(1)}$. Then we have two possibilities either $L \equiv L^{(1)}$ and $L^{(1)} \equiv f^{(2)}$ or $L \equiv L^{(1)}$ and $L^{(1)} \not\equiv f^{(2)}$.

If we consider the possibility $L \equiv L^{(1)}$ and $L^{(1)} \equiv f^{(2)}$. Then $L \equiv L^{(1)}$ implies $L = ce^z$(c is a non zero constant) and so $L^{(1)} = f^{(2)} = ce^z$ then $f^{(1)} = ce^z + \gamma$, and $f = ce^z + \gamma z + \delta$. Since $L \neq f^{(1)}$ obviously $\gamma \neq 0$.

If we consider $\gamma z+\delta \neq a$. Then by Lemma 2.1 we get

$$\begin{aligned} T(r, ce^z) &\leq \overline{N}(r,0;ce^z)+\overline{N}(r,\infty;ce^z)+\overline{N}(r,a-\gamma z-\delta;ce^z)\\ &= \overline{N}(r,a;f)+S(r,ce^z). \end{aligned} \tag{11}$$

Since $f = L^{(1)}+\gamma z+\delta$, we see that if z_1 is a zero of $f-a$ such that $z_1 \notin A\cup B$ then $\gamma z+\delta = 0$. Therefore

$$\begin{aligned} \overline{N}(r,a;f) &\leq N_A(r,a;f)+N_B(r,a;L)+N(r,0;\gamma z+\delta)\\ &= S(r,f). \end{aligned}$$

Which contradicts (11).
Next we consider $\gamma z+\delta \equiv a$, then $f = ce^z+a$ and so $f^{(1)} = ce^z+\alpha$ and $f^{(2)} = ce^z$.
Hence $L = (a_2+a_3+\cdots+a_n)ce^z = f^{(2)} = ce^z$ implies $\sum\limits_{k=2}^{n} a_k = 1$. Hence we get $L = L^{(1)} = ce^z$ and $f = a+ce^z$ where $c(\neq 0)$ is a constant and $\sum\limits_{k=2}^{n} a_k = 1$.

Next we consider the possibility $L \equiv L^{(1)}$ and $L^{(1)} \not\equiv f^{(2)}$. Hence $L \not\equiv f^{(2)}$. Then by the hypothesis we get

$$\begin{aligned} \overline{N}(r,a;L) &\leq N_B(r,a;L)+N(r,1;\frac{L}{f^{(2)}})\\ &\leq T(r,\frac{L}{f^{(2)}})+S(r,f)\\ &= N(r,\frac{L}{f^{(2)}})+S(r,f)\\ &\leq N(r,0;f^{(2)})+S(r,f). \end{aligned} \tag{12}$$

Again

$$\begin{aligned} m(r,a;f) &= m(r,\frac{f^{(2)}}{f-a}\cdot\frac{1}{f(2)})\\ &\leq m(r,0;f^{(2)})+S(r,f)\\ &= T(r,f^{(2)})-N(r,0;f^{(2)})+S(r,f)\\ &= m(r,f^{(2)})-N(r,0;f^{(2)})+S(r,f)\\ &\leq m(r,f)-N(r,0;f^{(2)})+S(r,f)\\ &= T(r,f)-N(r,0;f^{(2)})+S(r,f) \end{aligned}$$

and so

$$N(r,0;f^{(2)}) \leq N(r,a;f)+S(r,f). \tag{13}$$

Hence from (12) and (13) we get

$$\overline{N}(r,a;L) \leq N(r,a;f)+S(r,f), \tag{14}$$

which implies by Lemma 2.3 that

$$T(r,f) \leq 2N(r,a;f)+S(r,f). \tag{15}$$

We put $\Phi = \frac{f^{(2)}-L}{f-a}$ and $\Psi = \frac{(a-a^{(1)})f^{(2)}-a(f^{(1)}-a^{(1)})}{f-a}$.
Then

$$\begin{aligned} N(r,\Phi) &\leq N_A(r,a;f)+N_B(r,a;L)+N_{(2}(r,a;f)+S(r,f)\\ &= S(r,f), \end{aligned}$$

also $N(r,\Psi) = S(r,f)$, and $m(r,\Phi) = S(r,f)$, $m(r,\Psi) = S(r,f)$. Therefore $T(r,\Phi) = S(r,f)$ and $T(r,\Psi) = S(r,f)$.
Since $L \neq f^{(2)}$ so $\Phi \not\equiv 0$.
Let z_2 be a simple zero of $f - a$ such that $z_2 \notin A \cup B \cup C$ where $C = \{z : a(z) - a^{(1)}(z) = 0\}$.
Then by Taylor's expansion in some neighbourhood of z_2 we get

$$\begin{aligned} f - a &= (f-a)(z_2) + (f-a)^{(1)}(z_2)(z-z_2) + (f-a)^{(2)}(z_2)\frac{(z-z_2)^2}{2} + (f-a)^{(3)}(z_2)\frac{(z-z_2)^3}{6} + \dots \\ &= (a(z_2) - a^{(1)}(z_2))(z-z_2) + a(z_2)\frac{(z-z_2)^2}{2} + f^{(3)}(z_2)\frac{(z-z_2)^3}{6} + \dots \end{aligned}$$

Now differentiating we obtain

$$f^{(1)} - \alpha = a(z_2) - a^{(1)}(z_2) + a(z_2)(z-z_2) + f^{(3)}(z_2)\frac{(z-z_2)^2}{2} + \dots$$

and

$$f^{(2)} = a(z_2) + f^{(3)}(z_2)(z-z_2) + \dots$$

Also,

$$\begin{aligned} L &= L(z_2) + L^{(1)}(z_2)(z-z_2) + L^{(2)}(z_2)\frac{(z-z_2)^2}{2} + \dots \\ &= a(z_2) + a(z_2)(z-z_2) + L^{(2)}(z_2)\frac{(z-z_2)^2}{2} + \dots \end{aligned}$$

Therefore in some neighbourhood of z_2 we get

$$\begin{aligned} \Phi(z) &= \frac{a(z_2) + f^{(3)}(z_2)(z-z_2) - a(z_2) - a(z_2)(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{(f^{(3)}(z_2) - a(z_2))(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{f^{(3)}(z_2) - a(z_2) + O(z-z_2)}{a(z_2)-\alpha + O(z-z_2)} \end{aligned}$$

Noting that $a(z_2) - \alpha \neq 0$, then

$$\Phi(z_2) = \frac{f^{(3)}(z_2) - a(z_2)}{a(z_2) - \alpha}. \tag{16}$$

Also in some neighbourhood of z_2 we get

$$\begin{aligned} \Psi(z) &= \frac{\{a(z) - a^{(1)}(z)\}\{a(z_2) + f^{(3)}(z_2)(z-z_2)\} - a(z)\{a(z_2) - a^{(1)}(z_2) + a(z_2)(z-z_2)\} + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{\alpha^2(z-z_2)\{(a(z)-\alpha)f^{(3)}(z_2) - a(z)a(z_2)\}(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{\alpha^2 + (a(z)-\alpha)f^{(3)}(z_2) - a(z)a(z_2) + O(z-z_2)}{a(z_2)-\alpha + O(z-z_2)}. \end{aligned}$$

Hence

$$\begin{aligned}\Psi(z_2) &= \frac{(f^{(3)}(z_2) - a(z_2) - \alpha)(a(z_2) - \alpha)}{a(z_2) - \alpha} \\ &= f^{(3)}(z_2) - a(z_2) - \alpha. \end{aligned} \tag{17}$$

From (16) and (17) we get

$$(a(z_2) - \alpha)\Phi(z_2) = \Psi(z_2) + a(z_2) + \alpha - a(z_2)$$

implies

$$(a(z_2) - \alpha)\Phi(z_2) - \Psi(z_2) - \alpha = 0.$$

If

$$(a - \alpha)\Phi - \Psi - \alpha \not\equiv 0,$$

then we get

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N_B(r, a; L) + N_{(2}(r, a; f) + N(r, 0; (a - \alpha)\Phi - \Psi - \alpha) \\ &= S(r, f), \end{aligned}$$

which contradicts (15).
Therefore

$$(a - \alpha)\Phi - \Psi - \alpha \equiv 0. \tag{18}$$

First we suppose that $\Psi \equiv 0$. Then from (18) and the definitions of Φ and Ψ we get $(a-\alpha)\frac{f^{(2)}-L}{f-a} = \alpha$ and $(a - \alpha)f^{(2)} - a(f^{(1)} - \alpha) = 0$ implies

$$(a - \alpha)f^{(2)} - (a - \alpha)L = \alpha(f - a) \tag{19}$$

and

$$(a - \alpha)f^{(2)} = a(f^{(1)} - \alpha). \tag{20}$$

From (19) and (20) we get

$$a(f^{(1)} - \alpha) - (a - \alpha)L = \alpha(f - a). \tag{21}$$

Differentiating (21) we get

$$af^{(2)} + \alpha(f^{(1)} - \alpha) - \alpha L - (a - \alpha)L^{(1)} = \alpha(f^{(1)} - \alpha). \tag{22}$$

Since $L \equiv L^{(1)}$ then from (22) we get $af^{(2)} = aL$ implies $a(f^{(2)} - L) = 0$, since $a \neq 0$ so $f^{(2)} - L \equiv 0$ and so $\Phi \equiv 0$, which is a contradiction.

Next we suppose that $\Psi \not\equiv 0$. Then from (18) and the definitions of Φ and Ψ we get

$$(a - \alpha)\frac{f^{(2)} - L}{f - a} - \frac{(a - \alpha)f^{(2)} - a(f^{(1)} - \alpha)}{f - a} = \alpha$$

this implies

$$-(a - \alpha)L + a(f^{(1)} - \alpha) = \alpha(f - a). \tag{23}$$

Differentiating both sides of (23) and put $L \equiv L^{(1)}$ we get $a(L - f^{(2)}) = 0$, since $a \neq 0$ so $f^{(2)} - L \equiv 0$ and so $\Phi \equiv 0$, which is a contradiction.

Subcase 2.2.2. Let $L \not\equiv L^{(1)}$ and $L^{(1)} \equiv f^{(1)}$.
Since $L \not\equiv L^{(1)}$. Then by hypothesis we get

$$\begin{aligned}\overline{N}(r,a;L) &\leq N_B(r,a;L) + N(r,1;\frac{L^{(1)}}{L})\\ &\leq T(r,\frac{L^{(1)}}{L}) + S(r,f)\\ &= N(r,\frac{L^{(1)}}{L}) + S(r,f)\\ &\leq N(r,0;L) + S(r,f).\end{aligned} \tag{24}$$

Again

$$\begin{aligned}m(r,a;f) &= m(r,\frac{L}{f-a}.\frac{1}{L})\\ &\leq m(r,0;L) + S(r,f)\\ &= T(r,L) - N(r,0;L) + S(r,f)\\ &= m(r,L) - N(r,0;L) + S(r,f)\\ &\leq m(r,f) - N(r,0;L) + S(r,f)\\ &= T(r,f) - N(r,0;L) + S(r,f)\end{aligned}$$

and so

$$N(r,0;L) \leq N(r,a;f) + S(r,f). \tag{25}$$

Hence from (24) and (25) we get

$$\overline{N}(r,a;L) \leq N(r,a;f) + S(r,f),$$

which implies by Lemma 2.3 that

$$T(r,f) \leq 2N(r,a;f) + S(r,f). \tag{26}$$

Therefore $N(r,a;f) \neq S(r,f)$. Also since $L^{(1)} \equiv f^{(1)}$. Then $L \equiv f + c$, where c is a constant. Also since $N(r,a;f) \neq S(r,f)$ and by hypothesis we get $c = 0$. Hence $L \equiv f$, which contradicts the initial supposition of Case 2.
Subcase 2.2.3. Let $L \not\equiv L^{(1)}$ and $L \equiv f^{(1)}$.
We put

$$\tau = \frac{(a - a^{(1)})L - a(f^{(1)} - a^{(1)})}{f - a}.$$

Then

$$\begin{aligned}N(r,\tau) &\leq N_A(r,a;f) + N_B(r,a;L) + N_{(2}(r,a;f) + S(r,f)\\ &= S(r,f),\end{aligned}$$

also $m(r,\tau) = S(r,f)$. Therefore $T(r,\tau) = S(r,f)$.
Let z_4 be a simple zero of $f - a$ such that $z_4 \notin A \cup B \cup C$ where $C = \{z : a(z) - a^{(1)}(z) = 0\}$.
Then by Taylor's expansion in some neighbourhood of z_4 we get

$$\begin{aligned}f - a &= (f-a)(z_4) + (f-a)^{(1)}(z_4)(z-z_4) + (f-a)^{(2)}(z_4)\frac{(z-z_4)^2}{2} + O(z-z_4)^3\\ &= (a(z_4) - \alpha)(z-z_4) + a(z_4)\frac{(z-z_4)^2}{2} + O(z-z_4)^3\end{aligned}$$

Now differentiating we obtain

$$f^{(1)} - \alpha = (a(z_4) - \alpha) + a(z_4)(z - z_4) + (z - z_4)^2$$

and

$$\begin{aligned} L &= L(z_4) + L^{(1)}(z_4)(z - z_4) + O(z - z_4)^2 \\ &= a(z_4) + a(z_4)(z - z_4) + O(z - z_4)^2 \end{aligned}$$

Therefore in some neighbourhood of z_4 we get

$$\begin{aligned} \tau(z) &= \frac{\{a(z) - a^{(1)}(z)\}\{a(z_4) + a(z_4)(z - z_4)\} - a(z)\{a(z_4) - \alpha + a(z_4)(z - z_4)\} + O(z - z_4)^2}{(a(z_4) - \alpha)(z - z_4) + O(z - z_4)^2} \\ &= \frac{\alpha^2(z - z_4) - \alpha a(z_4)(z - z_4) + O(z - z_4)^2}{(z - z_4)(a(z_4) - \alpha + O(z - z_4))} \\ &= \frac{-\alpha(a(z_4) - \alpha) + O(z - z_4)}{a(z_4) - \alpha + O(z - z_4)} \\ &= -\alpha + O(z - z_4). \end{aligned}$$

Let $P = \tau + \alpha$. Then in some neighbourhood of z_4 we get $P(z) = O(z - z_4)$.
First we suppose that $P(z) \not\equiv 0$. Since every multiple zero of $f - a$ must belongs to $A \cup B \cup C$, then we get

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N_B(r, a; L) + N(r, 0; P) \\ &= S(r, f). \end{aligned}$$

Then from (26) we get $T(r, f) = S(r, f)$, a contradiction. Hence $P \equiv 0$ and so

$$(a - \alpha)L - a(f^{(1)} - \alpha) + \alpha(f - a) = 0.$$

Since $L \equiv f^{(1)}$ then we get

$$(a - \alpha)f^{(1)} - a(f^{(1)} - \alpha) + \alpha(f - a) = 0$$

which implies $\alpha(f - f^{(1)}) = 0$, since $\alpha \neq 0$ then $f \equiv f^{(1)}$. So $f = ce^z$ where $c(\neq 0)$ is a constant. Then $L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)} = (a_2 + a_3 + \cdots + a_n)ce^z$ and $L^{(1)} = a_2 f^{(3)} + a_3 f^{(4)} + \cdots + a_n f^{(n+1)} = (a_2 + a_3 + \cdots + a_n)ce^z$. So $L \equiv L^{(1)}$ which is a contradiction. This completes the proof of the theorem. □

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