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On Endomorphism Rings of Leavitt Path Algebras

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Abstract. Let E be an arbitrary graph, K be any field and A be the endomorphism ring of $L := L_K(E)$ considered as a right L-module. Among the other results, we prove that: (1) if A is a von Neumann regular ring, then A is dependent if and only if for any two paths in L satisfying some conditions are initial of each other, (2) if A is dependent then $L_K(E)$ is morphic, (3) L is morphic and von Neumann regular if and only if L is semisimple and every homogeneous component is artinian.

1. Introduction

Leavitt algebras $L_K(1, n)$ for $2 \leq n$ and any field K were introduced and studied by W. G. Leavitt [10] in 1962 as universal examples of algebras not satisfying the IBN (invariant bases number) property. A ring R is said to have the IBN property in case for any pair of positive integers $m \neq n$ we have that the free left R-modules R^m and R^n which are not isomorphic. If $R = L_K(1, n)$, then ${}_RR^1 \cong_R R^n$ which shows Leavitt algebras fail to have the IBN property. A generalization of Leavitt algebras, the Leavitt path algebras $L_K(E)$ for row-finite graphs E were independently introduced by P. Ara, M. A. Moreno-Frías and E. Pardo in [4], and by G. Abrams and G. Aranda Pino in [1]. These $L_K(E)$ are algebras associated to directed graphs and are the algebraic analogs of the Cuntz-Krieger graph C^*-algebras [15].

Let E be a graph and K a field. G. Aranda Pino, K. M. Rangaswamy and M. Siles Molina [5] studied conditions on a graph E which are necessary and sufficient for the endomorphism ring A of the Leavitt path algebra $L : L_K(E)$ considered as a right L-module to be von Neumann regular (recall that a ring R is von Neumann regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$). The algebra L embeds in A and $A = L$ if the graph E has finitely many vertices. The authors of [5] state that their focus is on the case when the graph E has infinitely many vertices since some earlier works in the literature (for instance, [3]) contain necessary and sufficient conditions on E for L to be von Neumann regular, and they show in [5, Theorem 3.5] that, if E is a row-finite graph, A is von Neumann regular if and only if E is cyclic and every infinite path ends in a sink (equivalently, L is left and right self-injective and von Neumann regular if and only if L is semisimple right L-module).

In the literature on von Neumann regular rings, various conditions have been shown to characterize the subclass of unit regular rings (recall that a ring R is unit regular if for every $a \in R$ there exists a unit $u \in R$ such that $a = aua$). We remark that the Leavitt path algebras that we look at will not necessary have a unit. If E is a graph and K is a field, the Leavitt path algebra $L_K(E)$ is unital if and only if the vertex set E^0 is

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finite, in which case $\sum_{v\in E^0} v = 1_{L_K(E)}$. However, every Leavitt path algebra does have a set of local units (A set of local units for a ring R is a set $E \subseteq R$ of commuting idempotents with the property that for any $x \in R$ there exists $t \in E$ such that $tx = xt = x$. If R is a ring with a set of local units E, then for any finite number of elements $x_1, ..., x_n \in R$, there exists $t \in E$ such that $tx_i = x_i t = x_i$ for all $1 \leq i \leq n$.).

According to M. Henriksen [8], R is called a dependent ring if, for every $a, b \in R$, there are $s, t \in R$, not both zero, such that $sa+tb = 0$. In [6, Theorem 6], Ehrlich showed that every unit regular ring R is dependent. In [8, Corollary 10], Henriksen shows that not all dependent regular rings are unit regular. In view of this useful fact, our aim is to understand and study dependent rings for the ring A of endomorphisms of $L_K(E)$ (viewed as a right $L_K(E)$-module). We prove that: (1) assume that A is a von Neumann regular ring. Then A is dependent if and only if for any two paths in L satisfying some conditions are initial of each other, (2) if A is dependent then $L_K(E)$ is morphic, (3) L is morphic and von Neumann regular if and only if L is semisimple and every homogeneous component is an artinian ring, (4) if L is morphic and A is von Neumann regular ring, then L is a morphic and a Rickart module, and if L is a morphic and a d-Rickart module, then A is dependent.

2. Notations and key observations

We begin this section by recalling the basic definitions and examples of Leavitt path algebras. Also, we will include some of the graph-theoretic definitions that will be needed later in the paper.

A (directed) graph $E = (E^0, E^1, r, s)$ consist of a set E^0 of vertices, a set E^1 of edges, and maps $r, s : E^1 \to E^0$. For each edge v, the vertex $s(v)$ is the source of v, and $r(v)$ is the range of v.

We say that a vertex $v \in E^0$ is a sink if $s^{-1}(v) = \emptyset$, and we say that a vertex $v \in E^0$ is an infinite emitter if $|s^{-1}(v)| = \infty$. A singular vertex is a vertex that is either a sink or an infinite emitter, and we denote the set of singular vertices by E^0_{sing}. We also let $E^0_{reg} = E^0 \setminus E^0_{sing}$, and refer to the element of E^0_{reg} as regular vertices; i.e., a vertex $v \in E^0$ is a regular vertex if and only if $0 < |s^{-1}(v)| < \infty$. A graph is row-finite if it has no infinite emitters. A graph is finite if both sets E^0 and E^1 are finite (or equivalently, when E^0 is finite and E is row-finite).

A path in a graph is a sequence $p = e_1...e_n$ $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. We say the path p has length $|p| = n$, and we let E^n denote set of paths of length n. We consider the vertices in E^0 to be paths of length zero. We also let $E^* = \bigcup_{n=0}^{\infty} E^n$ denote the paths of finite length in E, and we extend the maps r and s to E^* as follows: For $p = e_1...e_n \in E^n$ with $n \geq 1$, we set $s(p) = s(e_1)$ and $r(p) = r(e_n)$; for $p = v \in E^0$, we set $s(v) = v = r(v)$. In this case, $s(p) = s(e_1)$ is the source of p, $r(p) = r(e_n)$ is the range of p. If $p = e_1...e_n$ is a path then we denote by p^0 the set of its vertices, that is, $p^0 = \{s(e_1), r(e_i) : 1 \leq i \leq n\}$.

A path $p = e_1...e_n$ is closed if $r(e_n) = s(e_1)$, in which case p is said to be based at the vertex $s(e_1)$. A closed path $p = e_1...e_n$ based v is a closed simple path if $r(e_i) \neq v$ for every $i < n$, i.e., if p visits the vertex v only once. A cycle is a path $p = e_1...e_n$ with length $|p| \geq 1$ and $r(p) = s(p)$. In other word, a cycle is a path that begins and ends on the same vertex and does not pass through any vertex more than once. If p is a cycle with $s(p) = r(p) = v$, then we say that p is based at v. A graph E is called acyclic if it does not have any cycles. If $p = e_1...e_n$ is a cycle, an exit for p is an edge $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$ for some i.

The elements of E^1 are called (real) edges, while for $e \in E^1$ we call e^* a ghost edge. The set $\{e^* : e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. Let E be a graph and K be a field. The Leavitt path $K-$algebra $L_K(E)$ is defined to be the $K-$algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$, which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
(3) $e^*f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.
(4) $v = \sum_{\{e\in E^1, s(e)=v\}} ee^*$ whenever E^0_{reg}.

The conditions (3) and (4) are called Cuntz-Krieger relations. If $p = e_1...e_n$ is a path, we define $p^* = e_n^*...e_1^*$ of $L_K(E)$. One can show that

$$L_K(E) = span_K\{pq^* : p \text{ and } q \text{ are paths in } E \text{ and } r(p) = r(q)\}$$

The Leavitt path algebras that we look at will not necessary have a unit. If E is a graph and K is a field, the Leavitt path algebra $L_K(E)$ is unital if and only if the vertex set E^0 is finite, in which case $\sum_{v\in E^0} v = 1_{L_K(E)}$. However, every Leavitt path algebra does have a set of local units.

In [7], Fuller proved a ring R has enough idempotents if there exists a collection of mutually orthogonal idempotents $\{e_\alpha\}_{\alpha\in\Lambda}$ such that $R = \bigoplus e_\alpha R = \bigoplus Re_\alpha$. Note that if we let $S = \{e_\alpha\}_{\alpha\in\Lambda}$ be the mutually orthogonal idempotents of above definition, then $E = \{\sum_{k=1}^{n} e_k : e_1, ..., e_n \in S\}$ is set of local unit for R. Thus rings with enough idempotents are rings with local units. If E is a graph and $L_K(E)$ is the associated Leavitt path algebra, then

$$L_K(E) = \bigoplus_{v\in E^0} vL_K(E) = \bigoplus_{v\in E^0} L_K(E)v$$

so $L_K(E)$ is a ring with enough idempotents. Furthermore, if we list the vertices of E as $E^0 = \{v_1, v_2, ...\}$, let

$$\Lambda = \begin{cases} \{1, 2, ..., |E^0|\} & \text{if } E^0 \text{ is finite} \\ \{1, 2, ...\} & \text{if } E^0 \text{ is infinite} \end{cases}$$

and set $t_n = \sum_{k=1}^{n} v_k$, then $\{t_n\}_{n\in\Lambda}$ is a set of local units for $L_K(E)$.

We will now outline some easily derivable basic facts about the endomorphism ring A of $L := L_K(E)$. Let E be any graph and K be any field. Denote by A the unital ring $End(L_L)$. Then we may identify L with subring of A, concretely, the following is a monomorphism of rings:

$$\begin{aligned} \phi : L &\to End(L_L) \\ x &\mapsto \lambda_x \end{aligned}$$

where $\lambda_x : L \to L$ is the left multiplication by x, i.e., for every $y \in L$, $\lambda_x(y) = xy$ which is a homomorphism of right $L-$module. The map ϕ also a monomorphism because given a nonzero $x \in L$ there exists an idempotent $u \in L$ such that $xu = x$, hence $0 \neq x = \lambda_x(u)$.

Fact 2.1. For any $f \in A$ and $x \in L$, $f\lambda_x = \lambda_{f(x)} \in L$. Moreover, L is a left ideal of A. (see [5, Lemma 2.3] and [5, Corollary 2.4], respectively).

Fact 2.2. If E is a finite graph, then $L_K(E)$ is unital with $\sum_{v\in E^\circ} v = 1_{L_K(E)}$. Furthermore, we assume that E is a finite graph, u is a unit element in A and e is an idempotent in $L_K(E)$. Then $\lambda_{u(e)}$ is a unit element in $L_K(E)$.

Proof. Since u is an unit element in A there exist an element λ_b in A such that $\lambda_b u = u\lambda_b = 1_{L_K(E)}$ and so $eb = b = be$. Then we get

$$1_{L_K(E)} = u\lambda_b \quad = u\lambda_{eb} \quad = \lambda_{u(e)b} = \lambda_{u(e)}\lambda_b$$

which implies

$$\lambda_b\lambda_{u(e)} = \lambda_b u\lambda_e = u\lambda_b\lambda_e = u\lambda_{be} = u\lambda_{eb} = u\lambda_e\lambda_b = \lambda_{u(e)}\lambda_b.$$

□

Fact 2.3. If E is a infinite graph, then $L_K(E)$ is a ring with a set of local units. Furthermore, we assume that E is a infinite graph, u is a local unit element in A and e is an idempotent element in $L_K(E)$. Then $\lambda_{u(e)}$ is a local unit element in A.

Proof. Since u is an local unit element in A, there exist an element λ_b in A such that $\lambda_b u = \lambda_b = u\lambda_b$ and so $eb = b = be$. Then we get

$$\lambda_b = u\lambda_b = u\lambda_{eb} = \lambda_{u(e)b} = \lambda_{u(e)}\lambda_b$$

which implies

$$\lambda_b\lambda_{u(e)} = \lambda_b u\lambda_e = u\lambda_b\lambda_e = u\lambda_{be} = u\lambda_{eb} = u\lambda_e\lambda_b = \lambda_{u(e)}\lambda_b.$$

□

Fact 2.4. If E is an infinite graph, then $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices of the graph. On the other hand, $L_K(E)$ has plenty of idempotents (in fact, it is an algebra with local units), and this is true also for A, Now we assume that E is an infinite graph, A is a unit regular ring and $a \in L$. Since idempotents play a significant role in the theory of Leavitt path algebra, we remark that λ_a is an idempotent in L.

Proof. For any $a \in L$, by the hypothesis, there is a local unit $u \in A$ satisfying $\lambda_a u = \lambda_a = u\lambda_a$ such that $\lambda_a = \lambda_a u \lambda_a$. Then $\lambda_a = \lambda_a u \lambda_a = \lambda_a \lambda_a$ which implies that λ_a is an idempotent in L. □

3. The Results

Let E be any graph and K be any field. In [5, Proposition 3.1], it is shown that if A is von Neumann regular then $L_K(E)$ is von Neumann regular.

Lemma 3.1. *Let E be an arbitrary graph, K be any field and A be the endomorphism ring of $L := L_K(E)$ considered as a right L-module. If A is dependent so is L.*

Proof. Suppose A is dependent. To show that L is dependent, let $a, b \in L$. By hypothesis, there are elements $f, g \in A$, not both zero, such that $f\lambda_a + g\lambda_b = 0$. If u_1 and u_2 are local units in L satisfying $u_1 a = a = au_1$ and $u_2 b = b = bu_2$, then

$$f\lambda_a = f\lambda_{u_1 a} = f\lambda_{u_1}\lambda_a = \lambda_{f(u_1)}\lambda_a$$

and

$$g\lambda_b = g\lambda_{u_2 b} = g\lambda_{u_2}\lambda_b = \lambda_{g(u_2)}\lambda_b.$$

Now

$$\begin{aligned} 0 &= f\lambda_a + g\lambda_b \\ &= \lambda_{f(u_1)}\lambda_a + \lambda_{g(u_2)}\lambda_b, \end{aligned}$$

and hence L is dependent. □

In the literature on von Neumann regular rings, various conditions have been shown to characterize the subclass of unit regular rings. In [6, Theorem 6], Ehrlich showed that every unit regular ring R is dependent. In [8, Corollary 10], Henriksen shows that not all dependent regular rings are unit regular. The following observation gives one more such condition for dependent rings.

Given paths $p, q \in E$, we say that q is an initial segment of p if $p = qm$ for some path $m \in E$. It is well known that, given nonzero paths pq^* and mn^* in $L_K(E)$, q is an initial segment of m if and only if $(pq^*)(mn^*) \neq 0$.

Theorem 3.2. *Let E be a graph, K be any field and A be the endomorphism ring of $L := L_K(E)$ considered as a right L-module. Assume that A is a von Neumann regular ring. Then the following conditions are equivalent.*

(1) A is dependent.

(2) If, for all paths nq^* and pm^* in $L_K(E)$, $An = Aq$ and $Ap = Am$ imply q is an initial segment of p.

Proof. (1) ⇒ (2) Let A be dependent. Then, for all paths $nq^*, pm^* \in L_K(E)$, there exists both non zero $u, v \in A$ such that $u(nq^*) + v(pm^*) = 0$. By assumption, let $n = fq$ and $p = gm$ for some $f, g \in A$. Assume that $(nq^*)(pm^*) = 0$. Then

$$\begin{aligned} 0 &= u(nq^*) + v(pm^*) \\ &= u(nq^*)(pm^*) + v(pm^*)(pm^*) \\ &= v(pm^*) \end{aligned}$$

which implies $v = 0$. Similarly, we also get $u = 0$, which is a contradiction. Hence $(nq^*)(pm^*) \neq 0$ and so q is an initial segment of p.

(2) ⇒ (1) Let $p, q \in A$. Since A is a von Neumann regular ring, for $p, q \in A$, choose $f, g \in A$ such that $p = pfp$ and $q = qgq$. Let $fp = m$ and $gq = n$ for some $m, n \in L_K(E)$. Then, by (2), $Ap = Afp = Am$ and $Aq = Agq = An$ imply q is an initial segment of p. So there exists a path r such that $p = qr$, hence A is dependent. □

Theorem 3.3. *Let E be any graph, K be any field and e be an idempotent in a Leavitt path algebra $L = L_K(E)$. If L is dependent, so is eLe.*

Proof. Let L dependent. Then for each $a, b \in L$ there are $s, t \in L$, not both zero, such that $sa + tb = 0$. Now, let e be an idempotent in L. Then

$$\begin{aligned} 0 &= esa + etb \\ &= esae + etbe \\ &= seae + tebe \\ &= eseae + etebe \\ &= \underbrace{ese}_{s'}\underbrace{eae}_{a'} + \underbrace{ete}_{t'}\underbrace{ebe}_{b'} \end{aligned}$$

for some both nonzero $a', b' \in eLe$ and $s', t' \in eLe$. Hence eLe is dependent. □

Let R be a ring. For every element $a, b \in R$, if $Ra = ann(b)$ and $Rb = ann(a)$ then we say $a \sim b$.

Proposition 3.4. *Let E be a (finite) graph, K be any field and A be the endomorphism ring of $L := L_K(E)$ considered as a right L-module.*

1. *If $x \sim y$ for all x, y in $L_K(E)$, then $\lambda_x \sim \lambda_y$ in A.*
2. *The following conditions are equivalent for all $\alpha, \beta \in A$.*
   (a) $\alpha \sim \beta$
   (b) $u\alpha \sim \beta u^{-1}$
   (c) $\alpha u \sim u^{-1}\beta$

Proof. (1) Let E be a any graph and $x \sim y$ for all x, y in $L_K(E)$. We must show that $A\lambda_x = ann(\lambda_y)$ and $A\lambda_y = ann(\lambda_x)$. Let $f \in A$. For some idempotent e in L, we can write $x = \lambda_x(e)$ and $y = \lambda_y(e)$. By hypothesis, since $x \sim y$, $Lx = ann(y)$ and $Ly = ann(x)$, then $Lxy = 0$ and $Lyx = 0$ so $L\lambda_x(e)\lambda_y(e) = 0$ and $L\lambda_y(e)\lambda_x(e) = 0$. Then, by Fact 2.1, $f\lambda_x\lambda_y = \lambda_{f(x)}\lambda_y = \lambda_{f(e)}\lambda_x\lambda_y = 0$ and we get $A\lambda_x \subseteq ann(\lambda_y)$. Conversely, $ann(y) \subseteq Lx \Rightarrow ann(\lambda_y) \subseteq L\lambda_x \subseteq A\lambda_x$. So $A\lambda_x = ann(\lambda_y)$.

By Fact 2.1, $f\lambda_y\lambda_x = \lambda_{f(y)}\lambda_x = \lambda_{f(e)}\lambda_y\lambda_x = 0$ and we get $A\lambda_y \subseteq ann(\lambda_x)$. Conversely, $ann(x) \subseteq Ly \Rightarrow ann(\lambda_x) \subseteq L\lambda_y \subseteq A\lambda_y$. So $A\lambda_y = ann(\lambda_x)$.

(2)(*a*) ⇒ (*b*) Let $\alpha \sim \beta$. Then we can write $A\alpha = ann(\beta)$ and $A\beta = ann(\alpha)$. Take a local unit u in A. Clearly, u^{-1} is a local unit element in A. Hence

$$A(u\alpha) = A\alpha = ann(\beta) = ann(\beta u^{-1})$$

and

$$A(\beta u^{-1}) = A\beta = ann(\alpha) = ann(u\alpha).$$

(*b*) ⇒ (*c*) This is obvious.

(*c*) ⇒ (*a*) Let $\alpha u \sim u^{-1}\beta$. Then we can write $A(\alpha u) = ann(u^{-1}\beta)$ and $A(u^{-1}\beta) = ann(\alpha u)$. Take a local unit u in A. We get

$$A\alpha = A(\alpha u) = ann(u^{-1}\beta) = ann(\beta)$$

and

$$A\beta = AA(u^{-1}\beta) = ann(\alpha u) = ann(\beta).$$

□

According to [14], an endomorphism α of a module M is called morphic if $M/M\alpha \cong \mathrm{Ker}(\alpha)$, equivalently there exists $\beta \in \mathrm{End}(M)$ such that $M\beta = \mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta) = M\alpha$ by [14, Lemma 1]. The module M is called a morphic module if every endomorphism is morphic. If R is a ring, an element a in R is called left morphic if right multiplication $\cdot a : {}_R R \to {}_R R$ is a morphic endomorphism, that is if $R/Ra \cong l(a)$. The ring itself is called a left morphic ring if every element is left morphic, that is if ${}_R R$ is a morphic module.

Corollary 3.5. *Let E be any graph and K be any field. If A is dependent then $L_K(E)$ is morphic.*

Proof. This follows from Proposition 3.4 and Lemma 3.1. □

We continue to obtain some characterizations which are similar to Theorem 3.2.

Theorem 3.6. *Let E be an arbitrary graph and A be the endomorphism ring of $L = L_K(E)$ as a right $L_K(E)$-module. Then*

1. *L is morphic and von Neumann regular if and only if L is semisimple and every homogeneous component is an artinian ring, concretely, $L \cong \bigoplus_{i\in\Lambda} \mathbb{M}_{n_i}(K)$, where every n_i is an integer (the set of n_i's might not be bounded)..*
2. *If L is morphic and von Neumann regular, then A is dependent.*

Proof. (1) See [2, Theorem 2.4].

(2) We show that A is dependent. Let $\alpha, \beta \in A$. Since L has local units there are idempotents $u, v \in L$ such that

$$\alpha\lambda_u = \lambda_{\alpha(u)} = \lambda_\alpha\lambda_u \in L$$

and

$$\beta\lambda_v = \lambda_{\beta(v)} = \lambda_\beta\lambda_v \in L$$

Since L is morphic, if $(\alpha\lambda_u)\alpha \in ann(\beta)$ then $\beta(\alpha\lambda_u)\alpha = 0$ and $(\beta\lambda_v)\beta \in ann(\alpha)$ which implies $\alpha(\beta\lambda_v)\beta = 0$. So, $\beta(\alpha\lambda_u)\alpha + \alpha(\beta\lambda_v)\beta = 0$. Hence A is a dependent ring. □

A module M is called kernel-direct if $Ker(\alpha)$ is a direct summand of M for every $\alpha \in \mathrm{End}(M)$; and M is called image-direct if $Im(\alpha)$ is a direct summand of M for each $\alpha \in \mathrm{End}(M)$ (see [14]). Modules with regular endomorphism ring (and hence all semisimple modules) have both properties. As pointed out of the authors [14],a morphic module is kernel direct if and only if it is image direct by [14, Lemma 1].

Theorem 3.7. *Let E be an arbitrary graph and A be the endomorphism ring of $L = L_K(E)$ as a right $L_K(E)$-module. Assume*

1. *L is morphic and kernel-direct,*
2. *L is morphic and image-direct,*
3. *A is dependent.*

Then we have (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) This follows from Proposition 3.4.

(2) ⇒ (3) Let $\alpha \in A$. Then $L\alpha$ is a direct summand of L as $Ker(\alpha)$ is a direct summand of L. By [16, Corollary 3.2], A is von Neumann regular and so A is dependent. □

A module M is called Rickart if the kernel of every endomorphism of M is a direct summand of M. M is called a d-Rickart module if the image of every endomorphism of M is a direct summand of M (see [11, 12] for details).

Theorem 3.8. *Let E be an arbitrary graph and A be the endomorphism ring of $L = L_K(E)$ as a right $L_K(E)$-module. Assume*

1. *L is morphic and A is von Neumann regular ring.*
2. *L is a morphic and a Rickart module.*
3. *L is a morphic and a d-Rickart module.*
4. *A is dependent.*

Then we have (1) ⇒ (2) ⇒ (3) ⇒ (4).

Proof. (1) ⇒ (2) Let L be morphic and A is von Neumann regular. By [13, Theorem 1.1], for all $\alpha \in A$, $Ker(\alpha)$ is direct summand of L. So L is a Rickart Module.

(2) ⇒ (3) By [17, Proposition 7], L is a d-Rickart module.

(3) ⇒ (4) Let L be a morphic and a d-Rickart module. Then the image of every endomorphism of L is a direct summand of L. So, by Theorem 3.7, A is dependent. □

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