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On Certain Subclasses of Multivalent Analytic Functions

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Dedicated to Professor Hari M. Srivastava

Abstract. In the present paper the authors introduce two new subclasses of multivalent analytic functions. Distortion inequalities and partial sums of functions in these classes are given.

1. Introduction

Throughout this paper, we assume that

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \le B < 0, B < A \le 1 \text{ and } 0 \le \lambda \le 1. \tag{1.1}$$

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \quad (p \in N), \tag{1.2}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

For functions f and g analytic in U, we say that f is subordinate to g in U and write $f \prec g$, if there exists a Schwarz function $w(z)$ in U such that

$$|w(z)| \le |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Let

$$f_j(z) = z^p + \sum_{n=2p}^{\infty} a_{n,j} z^n \in A(p) \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=2p}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

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The following lemma will be required in our investigation.

Lemma 1. Let $f \in A(p)$ defined by (1.2) satisfy

$$\sum_{n=2p}^{\infty}[\lambda n + p(1-\lambda)\delta_{n,p,k}]|a_n| \le \frac{p(A-B)}{1-B} \quad (z \in U). \tag{1.3}$$

Then

$$(1-\lambda)z^{-p}f_{p,k}(z) + \frac{\lambda}{p}z^{-p+1}f'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.4}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-jp}f(\varepsilon_k^j z), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \tag{1.5}$$

and

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right). \end{cases} \tag{1.6}$$

Proof. For $f \in A(p)$ defined by (1.2), the function $f_{p,k}(z)$ in (1.5) can be expressed as

$$f_{p,k}(z) = z^{-p} + \sum_{n=2p}^{\infty}\delta_{n,p,k}a_n z^n \tag{1.7}$$

with

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{j(n-p)} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right). \end{cases}$$

Let the inequality (1.3) hold. Then from (1.1) we deduce that

$$\begin{aligned}\left|\frac{\left[(1-\lambda)z^{-p}f_{p,k}(z) + \frac{\lambda}{p}z^{-p+1}f'(z)\right]-1}{A-B\left[(1-\lambda)z^{-p}f_{p,k}(z) + \frac{\lambda}{p}z^{-p+1}f'(z)\right]}\right| &= \left|\frac{\sum_{n=2p}^{\infty}[\lambda n + p(1-\lambda)\delta_{n,p,k}]a_n z^{n-p}}{p(A-B) - B\sum_{n=2p}^{\infty}[\lambda n + p(1-\lambda)\delta_{n,p,k}]a_n z^{n-p}}\right| \\ &\le \frac{\sum_{n=2p}^{\infty}[\lambda n + p(1-\lambda)\delta_{n,p,k}]|a_n|}{p(A-B) + B\sum_{n=2p}^{\infty}[\lambda n + p(1-\lambda)\delta_{n,p,k}]|a_n|} \\ &\le 1 \quad (|z|=1).\end{aligned}$$

Hence, by the maximum modulus theorem, we arrive at (1.4).

We now consider the following two subclasses of $A(p)$.

Definition 1. A function $f \in A(p)$ is said to be in the class $R_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

Definition 2. A function $f \in A(p)$ is said to be in the class $T_{p,k}(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=2p}^{\infty}n[\lambda n + p(1-\lambda)\delta_{n,p,k}]|a_n| \le \frac{p^2(A-B)}{1-B}. \tag{1.8}$$

From the Definitions 1 and 2 one can see that $T_{p,k}(\lambda, A, B) \subset R_{p,k}(\lambda, A, B)$. Also, it is obvious that

$$f(z) \in T_{p,k}(\lambda, A, B) \text{ if and only if } \frac{zf'(z)}{p} \in R_{p,k}(\lambda, A, B). \tag{1.9}$$

Many interesting classes of multivalent analytic functions were considered by earlier authors (see, e.g., [1-11] and the references therein). Inspired by some recent works of Srivastava et al. [8], the main object of the paper is to derive some distortion inequalities of functions in the classes $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$. In particular some results of partial sums and convolution of functions in these classes are also given.

2. Main Results

Our first theorem is given by the following.

Theorem 1. Let

$$\frac{p}{k} \notin N \text{ and } \frac{A-B}{2(1-B)} \leq \lambda \leq 1.$$

(i) If $f \in R_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|z|^p - \frac{A-B}{2\lambda(1-B)}|z|^{2p} \leq |f(z)| \leq |z|^p + \frac{A-B}{2\lambda(1-B)}|z|^{2p}. \tag{2.1}$$

(ii) If $f \in T_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$p\left(|z|^{p-1} - \frac{A-B}{2\lambda(1-B)}|z|^{2p-1}\right) \leq |f'(z)| \leq p\left(|z|^{p-1} + \frac{A-B}{2\lambda(1-B)}|z|^{2p-1}\right). \tag{2.2}$$

The bounds in (2.1) and (2.2) are sharp.

Proof. Let $\frac{p}{k} \notin N$. For $n \geq 2p$ and $\frac{n-p}{k} \notin N$, we have $\delta_{n,p,k} = \delta_{2p,p,k} = 0$, and so

$$\frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)} \geq \frac{2\lambda(1-B)}{A-B} \geq 1. \tag{2.3}$$

For $n \geq 2p$ and $\frac{n-p}{k} \in N$, we have $\delta_{n,p,k} = 1$ and

$$\frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)} \geq \frac{(1-B)[p + \lambda(p+1)]}{p(A-B)} \geq \frac{2\lambda(1-B)}{A-B}. \tag{2.4}$$

(i) If $f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in R_{p,k}(\lambda, A, B)$, then it follows from (2.3) and (2.4) that

$$\frac{2\lambda(1-B)}{A-B} \sum_{n=2p}^{\infty} |a_n| \leq 1.$$

Hence we have

$$|f(z)| \leq |z|^p + |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \leq |z|^p + \frac{A-B}{2\lambda(1-B)}|z|^{2p}$$

and

$$|f(z)| \geq |z|^p - |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \geq |z|^p - \frac{A-B}{2\lambda(1-B)}|z|^{2p} > 0$$

for $z \in U$.

The bounds in (2.1) are best possible which can be seen from the function f defined by

$$f(z) = z^p + \frac{A-B}{2\lambda(1-B)} z^{2p} \in R_{p,k}(\lambda, A, B). \tag{2.5}$$

(ii) If $f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in T_{p,k}(\lambda, A, B)$, then (2.3) and (2.4) yield

$$\frac{2\lambda(1-B)}{p(A-B)} \sum_{n=2p}^{\infty} n|a_n| \leq 1.$$

This leads to (2.2).

The bounds in (2.2) are best possible which can be seen from the function f defined by

$$f(z) = z^p + \frac{A-B}{4\lambda(1-B)} z^{2p} \in T_{p,k}(\lambda, A, B). \tag{2.6}$$

Theorem 2. Let $\frac{p}{k} \in N$ and $\frac{p}{p+1} \le \lambda \le 1$.
(i) If $f \in R_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|z|^p - \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p} \le |f(z)| \le |z|^p + \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p}. \tag{2.7}$$

(ii) If $f \in T_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$p\left(|z|^{p-1} - \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p-1}\right) \le |f'(z)| \le p\left(|z|^{p-1} + \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p-1}\right). \tag{2.8}$$

The bounds in (2.7) and (2.8) are sharp.

Proof. Let $\frac{p}{k} \in N$. For $n \ge 2p$ and $\frac{n-p}{k} \in N$, we have $\delta_{n,p,k} = \delta_{2p,p,k} = 1$, and so

$$\frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)} \ge \frac{(1+\lambda)(1-B)}{A-B} \ge 1. \tag{2.9}$$

For $n \ge 2p$ and $\frac{n-p}{k} \notin N$, we have $\delta_{n,p,k} = 0$ and

$$\frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)} \ge \frac{(1-B)(2p+1)\lambda}{p(A-B)} \ge \frac{(1+\lambda)(1-B)}{A-B} \tag{2.10}$$

for $\frac{p}{p+1} \le \lambda \le 1$.

(i) If $f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in R_{p,k}(\lambda, A, B)$, then it follows from (2.9) and (2.10) that

$$\frac{(1+\lambda)(1-B)}{A-B} \sum_{n=2p}^{\infty} |a_n| \le 1.$$

Hence we have

$$|f(z)| \le |z|^p + |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \le |z|^p + \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p}$$

and

$$|f(z)| \ge |z|^p - |z|^{2p} \sum_{n=2p}^{\infty} |a_n| \ge |z|^p - \frac{A-B}{(1+\lambda)(1-B)}|z|^{2p} > 0.$$

The bounds in (2.7) are sharp for the function f defined by

$$f(z) = z^p + \frac{A-B}{(1+\lambda)(1-B)} z^{2p} \in R_{p,k}(\lambda, A, B).$$

(ii) If $f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in T_{p,k}(\lambda, A, B)$, then (2.9) and (2.10) yield

$$\frac{(1+\lambda)(1-B)}{p(A-B)} \sum_{n=2p}^{\infty} n|a_n| \le 1.$$

This leads to (2.8).

The bounds in (2.8) are sharp for the function f defined by

$$f(z) = z^p + \frac{A-B}{2(1+\lambda)(1-B)} z^{2p} \in T_{p,k}(\lambda, A, B).$$

Next, we derive certain convolution properties of functions in the classes $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$.
Theorem 3. Let $f \in R_{p,k}(\lambda, A, B)$. Then

$$(f * h_\sigma)(z) \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1), \tag{2.11}$$

where

$$h_\sigma(z) = z^p - \frac{2\lambda(1+B\sigma)}{\sigma(A-B)} \cdot \frac{z^{2p}}{1-z} - \frac{\lambda(1+B\sigma)}{p\sigma(A-B)} \cdot \frac{z^{2p+1}}{(1-z)^2} - \frac{(1-\lambda)(1+B\sigma)}{\sigma(A-B)} g_{p,k}(z)$$

and

$$g_{p,k}(z) = \begin{cases} \frac{z^{2p}}{1-z^k} & \left(\frac{p}{k} \in N\right), \\ \frac{z^{k([\frac{p}{k}]+1)+p}}{1-z^k} & \left(\frac{p}{k} \notin N\right). \end{cases}$$

Proof. For $f \in R_{p,k}(\lambda, A, B)$, from Lemma 1 we have (1.4), which is equivalent to

$$(1-\lambda)z^{-p} f_{p,k}(z) + \frac{\lambda}{p} z^{-p+1} f'(z) \neq \frac{1+A\sigma}{1+B\sigma} \quad (z \in U;\ \sigma \in C,\ |\sigma| = 1,\ 1+B\sigma \neq 0),$$

or to

$$(1+B\sigma)[(1-\lambda)f_{p,k}(z) + \frac{\lambda}{p} z f'(z)] - (1+A\sigma)z^p \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1). \tag{2.12}$$

Obviously

$$z^p = f(z) * z^p$$

and

$$\begin{aligned} \frac{zf'(z)}{p} &= f(z) * \left(z^p + \frac{1}{p} \sum_{n=2p}^{\infty} n z^n \right) \\ &= f(z) * \left(z^p + \frac{2z^{2p}}{1-z} + \frac{z^{2p+1}}{p(1-z)^2} \right). \end{aligned} \tag{2.13}$$

If we put

$$f_{p,k}(z) = f(z) * (z^p + g_{p,k}(z)), \tag{2.14}$$

then for $\frac{p}{k} \in N$,

$$g_{p,k}(z) = \sum_{n=2p}^{\infty} \delta_{n,p,k} z^n = \sum_{l=0}^{\infty} z^{2p+lk} = \frac{z^{2p}}{1-z^k}, \tag{2.15}$$

and for $\frac{p}{k} \notin N$,

$$g_{p,k}(z) = \sum_{l=1}^{\infty} z^{k([\frac{p}{k}]+l)+p} = \frac{z^{k([\frac{p}{k}]+1)+p}}{1-z^k}. \tag{2.16}$$

Now, making use of (2.12) to (2.16), we arrive at

$$f(z) * \left\{(1+B\sigma)\left[(1-\lambda)(z^p + g_{p,k}(z)) + \lambda\left(z^p + \frac{2z^{2p}}{1-z} + \frac{z^{2p+1}}{p(1-z)^2}\right)\right] - (1+A\sigma)z^p\right\} \neq 0$$

for $z \in U \setminus \{0\}$, $\sigma \in C$ and $|\sigma| = 1$. This gives the desired result (2.11). The proof of the theorem is complete.

Corollary 1. Let $f \in T_{p,k}(\lambda, A, B)$. Then

$$f(z) * zh'_\sigma(z) \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1),$$

where $h_\sigma(z)$ is the same as in Theorem 3.

Proof. Since $f \in T_{p,k}(\lambda, A, B)$ if and only if

$$\frac{zf'(z)}{p} \in R_{p,k}(\lambda, A, B),$$

it follows from Theorem 3 that

$$f(z) * \frac{zh'_\sigma(z)}{p} = \frac{zf'(z)}{p} * h_\sigma(z) \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1).$$

Thus we complete the proof.

Finally, we derive some results of the partial sums of functions in the classes $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$.

Let $f \in A(p)$ be given by (1.2) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$s_1(z) = z^p \text{ and } s_m(z) = z^p + \sum_{n=2p}^{2p+m-2} a_n z^n \quad (m \in N \setminus \{1\}). \tag{2.17}$$

For simplicity we use the notation α_n $(n \geq 2p)$ as following:

$$\alpha_n = \frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)}. \tag{2.18}$$

Theorem 4. Let $f \in R_{p,k}(\lambda, A, B)$ and let

$$\max\left\{\frac{A-B}{2(1-B)}, \frac{p}{p+1}\right\} \leq \lambda \leq 1. \tag{2.19}$$

Then for $m \in N$, we have

$$\mathrm{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{1}{\alpha_{2p+m-1}} \quad (z \in U) \tag{2.20}$$

and

$$\mathrm{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{\alpha_{2p+m-1}}{1+\alpha_{2p+m-1}}. \tag{2.21}$$

The bounds in (2.20) and (2.21) are best possible for each m.

Proof. For $n \geq 2p$, we have from (2.18) and (2.19) that

$$\alpha_n = \frac{(1-B)[\lambda n + p(1-\lambda)\delta_{n,p,k}]}{p(A-B)} \geq \frac{2\lambda(1-B)}{A-B} \geq 1 \tag{2.22}$$

and

$$\begin{aligned}\alpha_{n+1} &= \frac{(1-B)[\lambda(n+1)+p(1-\lambda)\delta_{n+1,p,k}]}{p(A-B)} \\ &= \alpha_n + \frac{(1-B)[\lambda+p(1-\lambda)(\delta_{n+1,p,k}-\delta_{n,p,k})]}{p(A-B)} \\ &\geq \alpha_n + \frac{(1-B)[\lambda-p(1-\lambda)]}{p(A-B)} \\ &\geq \alpha_n. \end{aligned} \tag{2.23}$$

Let $f \in R_{p,k}(\lambda, A, B)$. Then it follows from (2.22) and (2.23) that

$$\sum_{n=2p}^{2p+m-2} |a_n| + \alpha_{2p+m-1} \cdot \sum_{n=2p+m-1}^{\infty} |a_n| \leq \sum_{n=2p}^{\infty} \alpha_n |a_n| \leq 1 \quad (m \in N \setminus \{1\}). \tag{2.24}$$

If we put

$$p_1(z) = 1 + \alpha_{2p+m-1}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

for $z \in U$ and $m \in N \setminus \{1\}$, then $p_1(0) = 1$ and we deduce from (2.24) that

$$\begin{aligned}\left|\frac{p_1(z)-1}{p_1(z)+1}\right| &= \left|\frac{\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}{2\left(1+\sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right)+\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}\right| \\ &\leq \frac{\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty}|a_n|}{2-2\sum_{n=2p}^{2p+m-2}|a_n|-\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty}|a_n|} \\ &\leq 1 \quad (z \in U;\ m \in N \setminus \{1\}).\end{aligned}$$

This implies that $\mathrm{Re}(p_1(z)) > 0$ for $z \in U$, and so (2.20) holds true for $m \in N \setminus \{1\}$.

Similarly, by setting

$$p_2(z) = (1+\alpha_{2p+m-1})\frac{s_m(z)}{f(z)} - \alpha_{2p+m-1},$$

it follows from (2.24) that

$$\begin{aligned}\left|\frac{p_2(z)-1}{p_2(z)+1}\right| &= \left|\frac{-(1+\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}{2\left(1+\sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right)+(1-\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}\right| \\ &\leq \frac{(1+\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty}|a_n|}{2-2\sum_{n=2p}^{2p+m-2}|a_n|-(\alpha_{2p+m-1}-1)\sum_{n=2p+m-1}^{\infty}|a_n|} \\ &\leq 1 \quad (z \in U;\ m \in N \setminus \{1\}).\end{aligned}$$

Hence we obtain (2.21) for $m \in N \setminus \{1\}$.

For $m = 1$, replacing (2.24) by

$$\alpha_{2p}\sum_{n=2p}^{\infty}|a_n| \leq \sum_{n=2p}^{\infty}\alpha_n|a_n| \leq 1$$

and proceeding as the above, we see that (2.20) and (2.21) are also true.

Furthermore, taking the function f defined by

$$f(z) = z^p + \frac{z^{2p+m-1}}{\alpha_{2p+m-1}} \in R_{p,k}(\lambda, A, B),$$

we have $s_m(z) = z^p$,

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) \to 1 - \frac{1}{\alpha_{2p+m-1}} \quad \text{as} \quad z \to \exp\left(\frac{\pi i}{p+m-1}\right)$$

and

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) \to \frac{\alpha_{2p+m-1}}{1+\alpha_{2p+m-1}} \quad \text{as} \quad z \to 1.$$

Thus the proof of the theorem is completed.

Corollary 2. Let the assumptions of Theorem 4 hold. Then, for $z \in U$, we have

$$\operatorname{Re}\left(\frac{f(z)}{z^p}\right) > \begin{cases} \frac{1-A+\lambda(1-B)}{(1+\lambda)(1-B)} & \left(\frac{p}{k} \in N\right), \\ \frac{2\lambda(1-B)-(A-B)}{2\lambda(1-B)} & \left(\frac{p}{k} \notin N\right). \end{cases}$$

and

$$\operatorname{Re}\left(\frac{z^p}{f(z)}\right) > \begin{cases} \frac{(1+\lambda)(1-B)}{A-B+(1+\lambda)(1-B)} & \left(\frac{p}{k} \in N\right), \\ \frac{2\lambda(1-B)}{A-B+2\lambda(1-B)} & \left(\frac{p}{k} \notin N\right). \end{cases}$$

The results are sharp.

Theorem 5. Let $f \in T_{p,k}(\lambda, A, B)$ and let the condition (2.19) be satisfied. Then for $m \in N$, we have

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{p}{(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U) \tag{2.25}$$

and

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{(2p+m-1)\alpha_{2p+m-1}}{p+(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U). \tag{2.20}$$

The bounds in (2.25) and (2.26) are sharp for the function f defined by

$$f(z) = z^p + \frac{pz^{2p+m-1}}{(2p+m-1)\alpha_{2p+m-1}} \in T_{p,k}(\lambda, A, B).$$

Proof. In view of the assumptions of Theorem 4, it follows from (2.22) and (2.23) that

$$\sum_{n=2p}^{2p+m-2} |a_n| + \frac{(2p+m-1)\alpha_{2p+m-1}}{p} \sum_{n=2p+m-1}^{\infty} |a_n| \le \sum_{n=2p}^{\infty} \frac{n}{p}\alpha_n |a_n| \le 1 \ \ (m \in N \setminus \{1\}) \tag{2.28}$$

and

$$\alpha_{2p} \sum_{n=2p}^{\infty} |a_n| \le \sum_{n=2p}^{\infty} \frac{n}{p}\alpha_n |a_n| \le 1. \tag{2.29}$$

If we put

$$p_1(z) = 1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

and

$$p_2(z) = \left(1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\right)\frac{s_m(z)}{f(z)} - \frac{(2p+m-1)\alpha_{2p+m-1}}{p},$$

then (2.28) and (2.29) lead to $\operatorname{Re}(p_j(z)) > 0$ ($z \in U$; $m \in N$; $j = 1, 2$). Hence we have (2.25) and (2.26). Sharpness can be verified easily.

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