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Strong Convergence of Halpern Iteration for Products of Finitely Many Resolvents of Maximal Monotone Operators in Banach Spaces

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Abstract. In this paper, using Bregman functions, we introduce a new Halpern-type iterative algorithm for finding common zeros of finitely many maximal monotone operators and obtain a strongly convergent iterative sequence to the common zeros of these operators in a reflexive Banach space. Furthermore, we study Halpern-type iterative schemes for finding common solutions of a finite system of equilibrium problems and null spaces of a γ-inverse strongly monotone mapping in a 2-uniformly convex Banach space. Some applications of our results to the solution of equations of Hammerstein-type are presented. Our scheme has an advantage that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. So the simple construction of Halpern iteration provides more flexibility in defining the algorithm parameters which is important from the numerical implementation perspective. Presented results improve and generalize many known results in the current literature.

1. Introduction

In this paper, we investigate the problem of finding zeros of mappings $A : E \to 2^{E^*}$; that is, find $x \in \mathrm{dom}A$ such that

$$0^* \in Ax. \tag{1.1}$$

The *domain* of a mapping A is defined by the set $\{x \in E : Ax \neq \emptyset\}$, where E is a Banach space.

Constructing iterative algorithms to approximate zeros of maximal monotone operators is a very active topic in pure and applied mathematics. The proximal point method (see [54]) is among the main tools for finding zeros of maximal monotone operators in Hilbert spaces. However, it was shown in [54] that the iterative sequence converges weakly but not strongly (see also [27]). To get the result of strong convergence, Kamimura and Takahashi [32] proposed a modified proximal point algorithm and obtained a strongly convergent iterative sequence to the zeros of a maximal monotone operator in a Hilbert space (see also [9, 21, 59]). On the other hand, the equilibrium problem, introduced by Blum and Oettli [10] in 1994, has been attracting a growing attention of researchers; see, e.g., [43, 44] and the references therein. Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium

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problem. In order to approximate the solution to this problem, various types of iterative schemes have been proposed (see [8, 30, 31, 52, 58, 62]). Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^* \rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.2}$$

exists. In this case, E is called *smooth*. If the limit (1.2) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [60, 61].

Let C be a nonempty subset of a Banach space E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. A point $p \in C$ is said to be an *asymptotic fixed point* [48] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

In recent years, several types of iterative schemes have been constructed and proved in order to get strong convergence results for nonexpansive mappings in various settings. The concept of nonexpansivity plays an important role in the study of Halpern-type iteration for finding fixed points of a mapping $T : C \to C$. Recall that the Halpern iteration is given by

$$\begin{cases} u \in C,\ x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. The construction of fixed points of nonexpansive mappings via Halpern's algorithm [28] has been extensively investigated recently in the current literature (see, for example, [47] and the references therein). Numerous results have been proved on Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces. Because of a simple construction, Halpern's iterations are widely used to approximate a solution of fixed points for nonexpansive mappings and other classes of nonlinear mappings by many authors in different styles (see, e.g., [1, 35, 40, 63, 64]).

1.1. Some facts about gradients

For any convex function $g : E \to (-\infty, +\infty]$ we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int} \operatorname{dom} g$ and any $y \in E$, we denote by $g^o(x, y)$ the *right-hand derivative* of g at x in the direction y, that is,

$$g^o(x, y) = \lim_{t\downarrow 0} \frac{g(x+ty) - g(x)}{t}. \tag{1.3}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case $g^o(x, y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [16](Proposition 1.1.10)). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [34](p. 508)). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty} \frac{g(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded subsets of* E if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (1.3) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by dom $A = \{x \in E : Ax \neq \emptyset\}$ and ran $A = \cup_{x\in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E\times E^*$ is said to be *monotone* [54] if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [55] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. A mapping A : dom $A \subset E \to E^*$ is called γ-inverse strongly monotone if there exists a positive real number γ such that for all $x, y \in$ dom A, $\langle x - y, Ax - Ay\rangle \geq \gamma\|Ax - Ay\|^2$.

1.2. Some facts about Legendre functions

Let E be a reflexive Banach space. For any proper, lower semicontinuous and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - g(x)\}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^*\rangle. \tag{1.4}$$

Here, ∂g is the subdifferential of g [56, 57]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function; see [61] for more details on convex analysis.

Let $g : E \to (-\infty, +\infty]$ be a mapping. The function g is said to be:
(i) *essentially smooth*, if ∂g is both locally bounded and single-valued on its domain.
(ii) *essentially strictly convex*, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of dom ∂g.
(iii) *Legendre*, if it is both essentially smooth and essentially strictly convex (for more details, we refer to [5](Definition 5.2)).

If E is a reflexive Banach space and $g : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [11](p. 83)

$$\nabla g^* = (\nabla g)^{-1}, \ \text{ran}\, \nabla g = \text{dom}\, g^* = \text{int dom}\, g^*, \ \text{and} \ \text{ran}\, \nabla g = \text{int dom}\, g.$$

Examples of Legendre functions are given in [4, 5]. One important and interesting Legendre function is $\frac{1}{s}\|.\|^s$ $(1 < s < \infty)$, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space.

1.3. Some facts about Bregman distances

Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [13, 19] corresponding to g is the function $D_g : E\times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \ \ \forall x, y \in E. \tag{1.5}$$

It is clear that $D_g(x, y) \geq 0$ for all $x, y \in E$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x, y) = \phi(x, y)$ for all $x, y \in E$.

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then, we know from [38] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y \in C} D_g(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \le 0, \quad \forall y \in C. \tag{1.6}$$

Furthermore, if C is a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

The *Bregman projection* proj_C^g from E onto C is defined by $\text{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property:

$$D_g\Big(y, \text{proj}_C^g x\Big) + D_g\Big(\text{proj}_C^g x, x\Big) \le D_g(y, x) \tag{1.7}$$

for all $y \in C$ and $x \in E$ (see [16] for more details).

1.4. Some facts about uniformly convex and totally convex functions

Let E be a Banach space and let $B_s := \{z \in E : \|z\| \le s\}$ for all $s > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of E* ([66](pp. 203, 221)) if $\rho_s(t) > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)} \tag{1.8}$$

for all $t \ge 0$. The function ρ_s is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of E* ([66](pp. 207, 221)) if $\lim_{t \downarrow 0} \frac{\sigma_s(t)}{t} = 0$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B_s, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \ge 0$. The function g is said to be *uniformly convex* if the function $\delta_g : [0, +\infty) \to [0, +\infty]$, defined by

$$\delta_g(t) := \sup \left\{ \frac{1}{2} g(x) + \frac{1}{2} g(y) - g\left(\frac{x+y}{2}\right) : \|y - x\| = t \right\},$$

satisfies that $\lim_{t \downarrow 0} \frac{\delta_g(t)}{t} = 0$.

Let $g : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that, in view of [16](Section 1.2, p. 17) (see also [15]), the function g is called *totally convex* at a point $x \in \text{int dom } g$ if its *modulus of total convexity* at x, that is, the function $v_g : \text{int dom } g \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_g(x, t) := \inf \Big\{ D_g(y, x) : y \in \text{int dom } g, \ \|y - x\| = t \Big\},$$

is positive whenever $t > 0$. The function g is called *totally convex* when it is *totally convex* at every point $x \in \text{int dom } g$. Moreover, the function f is called *totally convex on bounded subsets of E* if $v_g(x, t) > 0$ for any bounded subset X of E and for any $t > 0$, where the *modulus of total convexity of the function g* on the set X is the function $v_g : \text{int dom } g \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_g(X, t) := \inf \Big\{ v_g(x, t) : x \in X \cap \text{int dom } g \Big\}.$$

It is well known that any uniformly convex function is totally convex, but the converse is not true in general (see [16](Section 1.3, p. 30)).
It is also well known that g is totally convex on bounded subsets if and only if g is uniformly convex on bounded subsets (see [18](Theorem 2.10, p. 9)).
Examples of totally convex functions can be found, for instance, in [16–18].

1.5. Some facts about resolvents

Let E be a reflexive Banach space with the dual space E^* and let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $\mathrm{Res}^g_{rA} : E \to \mathrm{dom}\, A$ be defined by

$$\mathrm{Res}^g_{rA} = (\nabla g + rA)^{-1}\nabla g.$$

The mapping Res^g_{rA} is called the *g-resolvent* of A (see [6]). It is well known that $A^{-1}(0) = F\left(\mathrm{Res}^g_{rA}\right)$ for each $r > 0$ (for more details, see, for example [60]).
Examples and some important properties of such operators are discussed in [12].

1.6. Some facts about Bregman quasi-nonexpansive mappings

Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman quasi-nonexpansive* [51], if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \quad \forall x \in C,\ p \in F(T).$$

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* [51] if the following conditions are satisfied:
(1) $F(T)$ is nonempty;
(2) $D_g(p, Tv) \leq D_g(p, v),\ \forall p \in F(T),\ v \in C;$
(3) $\hat{F}(T) = F(T)$.

Recently, Sabach [58] proved the following two strong convergence theorems for the products of finitely many resolvents of maximal monotone operators in a reflexive Banach space.

Theorem 1.1. *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$, $i = 1, 2, ..., N$, be N maximal monotone operators such that $Z := \cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ y_n = \mathrm{Res}^g_{\lambda_n^N A_N} ... \mathrm{Res}^g_{\lambda_n^1 A_1}(x_n + e_n), \\ C_n = \{z \in E : D_g(z, y_n) \leq D_g(z, x_n + e_n)\}, \\ Q_n = \{z \in E : \langle z - x_n, \nabla g(x_0) - \nabla g(x_n)\rangle \leq 0\}, \\ x_{n+1} = \mathrm{proj}^g_{C_n \cap Q_n} x_0 \ \textit{and}\ n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.9}$$

If, for each $i = 1, 2, ..., N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, and the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^g_Z(x_0)$ as $n \to \infty$.

Theorem 1.2. *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$, $i = 1, 2, ..., N$, be N maximal monotone operators such that $Z := \cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative* *algorithm*

$$
\begin{cases}
x_0 \in E \quad \textit{chosen arbitrarily}, \\
H_0 = E, \\
y_n = \mathrm{Res}^g_{\lambda_n^N A_N} ... \mathrm{Res}^g_{\lambda_n^1 A_1}(x_n + e_n), \\
H_{n+1} = \{z \in H_n : D_g(z, y_n) \le D_g(z, x_n + e_n)\}, \\
x_{n+1} = \mathrm{proj}^g_{H_{n+1}} x_0 \textit{ and } n \in \mathbb{N} \cup \{0\},
\end{cases}
\tag{1.10}
$$

If, for each $i = 1, 2, ..., N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, and the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{proj}^g_Z(x_0)$ as $n \to \infty$.

The theory of fixed points with respect to Bregman distances has been studied in the last ten years and much intensively in the last four years. In [7], Bauschke and Combettes introduced an iterative method to construct the Bregman projection of a point onto a countable intersection of closed and convex sets in reflexive Banach spaces. They proved strong convergence theorem of the sequence produced by their method; for more detail see [42, Theorem 4.7]. In [49], Reich and Sabach introduced a proximal method for finding common zeros of finitely many maximal monotone operators in a reflexive Banach space. Then they proved that the sequence produced by their method converges strongly to a common zeros of finitely many maximal monotone operators. In [50], Reich and Sabach introduced a Mann type process to approximate fixed points of quasi Bregman firmly nonexpansive mappings defined on a nonempty, closed and convex subset C of a reflexive Banach space E. Then they proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ produced by their method converges strongly to a common fixed point of finitely many quasi Bregman firmly nonexpansive mappings. In [52], Reich and Sabach introduced iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in a reflexive Banach space. For some recent articles on the existence of fixed points for Bregman nonexpansive type mappings, we refer the readers to [4–7, 11, 12, 14, 15, 20, 30, 37, 46, 49–53, 58].

Remark 1.1. Though the iteration processes (1.9)-(1.10) and the algorithms in [50–52], as introduced by the authors mentioned above worked, it is easy to see that these processes seem cumbersome and complicated in the sense that at each stage of iteration, two different sets C_n and Q_n are computed and the next iterate taken as the Bregman projection of x_0 on the intersection of C_n and Q_n. This seems difficult in real world application.

But it is worth mentioning that, in all the above results for Bregman nonexpansive type mappings, the computation of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are required. So, the following question arises naturally in a Banach space setting.

Question 1.1. *Is it possible to obtain strong convergence of modified Halpern's type schemes to the common zeros of finitely many maximal monotone operators without using the Bregman projection of a point on the intersection of closed and convex sets?*

In this paper, using Bregman functions, we introduce a new Halpern-type iterative algorithm for finding common zeros of finitely many maximal monotone operators and obtain a strongly convergent iterative sequence to the common zeros of these mappings in a reflexive Banach space. First, we consider disadvantages of the iterative sequences in known results. Namely, Bregman projections are not always available in a practical calculation. We attempt to improve these schemes and, by combining them with iterative method of the Halpern type, we obtain a new type of strong convergence theorem, which overcomes the drawbacks of the previous results. Next, we study Halpern-type iterative schemes for finding common solutions of an equilibrium problem and null spaces of a γ-inverse strongly monotone mapping in a 2-uniformly convex Banach space. Some application of our results to the solution of equations of Hammerstein-type is presented. The computations of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are not required. Consequently, the above question is answered in the affirmative in a reflexive Banach space setting. Our results improve and generalize many known results in the current literature; see, for example, [7, 8, 12, 49–53].

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel. The following definition is slightly different from that in Butnariu and Iusem [16].

Definition 2.1 ([34])**.** *Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:*
(1) *g is continuous, strictly convex and Gâteaux differentiable;*
(2) *the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.*

The following lemma follows from Butnariu and Iusem [16] and Zălinscu [66].

Lemma 2.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;*
(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*
(3) *$\{x \in E : D_g(x, y) \le r\}$ is bounded for all $y \in E$ and $r > 0$;*
(4) *$dom\ g^* = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.*

We know the following two results; see [66].

Theorem 2.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*
(1) g is strongly coercive and uniformly convex on bounded subsets of E;
(2) $dom\ g^* = E^*$, g^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;
(3) $dom\ g^* = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.

Theorem 2.2. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*
(1) g is bounded on bounded subsets and uniformly smooth on bounded subsets of E;
(2) g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*;
(3) $dom\ g^* = E^*$, g^* is strongly coercive and uniformly convex on bounded subsets of E^*.

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [22] (see also [13, 19]) satisfies the *three point identity* that is

$$D_g(x, z) = D_g(x, y) + D_g(y, z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x, y, z \in E. \tag{2.1}$$

In particular, it can be easily seen that

$$D_g(x, y) = -D_g(y, x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x, y \in E. \tag{2.2}$$

Indeed, by letting $z = x$ in (2.1) and taking into account that $D_g(x, x) = 0$, we get the desired result.
The following result has been proved in [16] (see also [34]).

Lemma 2.2. *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent:*
(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$.
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following result was first proved in [18] (see also [34]).

Lemma 2.3. *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function and V the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad x \in E,\ x^* \in E^*.$$

Then the following assertions hold:
(1) $D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ *for all* $x \in E$ *and* $x^* \in E^*$.
(2) $V(x, x^*) + \langle \nabla g^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$ *for all* $x \in E$ *and* $x^*, y^* \in E^*$.

Corollary 2.1 *([66]). Let* E *be a Banach space,* $g : E \to (-\infty, \infty]$ *be a proper, lower semicontinuous and convex function and* $p, q \in \mathbb{R}$, *with* $1 \leq p \leq 2 \leq q$ *and* $p^{-1} + q^{-1} = 1$. *Then the following statements are equivalent:*
(1) There exists $c_1 > 0$ *such that* g *is* ρ*-convex with* $\rho(t) := \frac{c_1}{q} t^q$ *for all* $t \geq 0$.
(2) There exists $c_2 > 0$ *such that for all* $(x, x^*), (y, y^*) \in G(\partial g)$; $\|x^* - y^*\| \geq \frac{2c_2}{q} \|x - y\|^{q-1}$.

Lemma 2.4. *Let* E *be a Banach space,* $s > 0$ *be a constant,* ρ_s *be the gauge of uniform convexity of* g *and* $g : E \to \mathbb{R}$ *be a convex function which is uniformly convex on bounded subsets of* E. *Then*
(i) For any $x, y \in B_s$ *and* $\alpha \in (0, 1)$

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(\|x - y\|).$$

(ii) For any $x, y \in B_s$

$$\rho_s(\|x - y\|) \leq D_g(x, y).$$

Here, $B_s := \{z \in E : \|z\| \leq s\}$.

Proof. Let ρ_s be the gauge of uniform convexity of g. In view of (1.10), we get (i). Let us prove (ii). If $x, y \in B_s$ and $\alpha \in (0, 1)$, then we obtain

$$\frac{g(\alpha x + (1-\alpha)y) - g(y)}{\alpha} \leq g(x) - g(y) - (1-\alpha)\rho_s(\|x - y\|).$$

Letting $\alpha \to 0$ in the above inequality, we arrive at

$$\langle x - y, \nabla g(y) \rangle \leq g(x) - g(y) - \rho_s(\|x - y\|).$$

This implies that

$$\rho_s(\|x - y\|) \leq D_g(x, y),$$

which completes the proof.

Lemma 2.5 [36]. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}_{i\in\mathbb{N}}$ *of* $\{n\}_{n\in\mathbb{N}}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a subsequence* $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.6 [45, 65]. *Let* $\{s_n\}_{n\in\mathbb{N}}$ *be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\delta_n, \qquad \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ *and* $\{\delta_n\}_{n\in\mathbb{N}}$ *satisfy the conditions:*
(i) $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$, *or equivalently,* $\Pi_{n=1}^{\infty}(1-\gamma_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or*
(ii)' $\sum_{n=1}^{\infty} \gamma_n\delta_n < \infty$.
Then, $\lim_{n\to\infty} s_n = 0$.

3. Strong Convergence Theorems for Products of Resolvents

In this section, we propose a new Halpern-type iterative scheme for finding common zeros of finitely many maximal monotone operators in a Banach space and prove the following strong convergence theorem.

Using ideas in [23], we can prove the following important result.

Theorem 3.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Let $A_i : E \to 2^{E^*}$, $i = 1, 2, ..., N$, be N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0, 1]$ satisfying the following control conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E, \ x_1 \in E \quad \text{chosen arbitrarily,} \\ y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(\mathrm{Res}^g_{r_N A_N} ... \mathrm{Res}^g_{r_1 A_1}(x_n))], \\ x_{n+1} = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] \qquad \text{and} \ \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where ∇g is the gradient of g. If $r_i > 0$, for each $i = 1, 2, ..., N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.1) converges strongly to $\mathrm{proj}^g_Z u$ as $n \to \infty$.

Proof. We divide the proof into several steps.
Let $z = \mathrm{proj}^g_Z u$. We denote by T_i the resolvent $\mathrm{Res}^g_{r_i A_i}$ and by S_i the composition $T_i...T_1$ for any $i = 1, 2, ..., N$. Therefore,

$$y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(T_N...T_1 x_n)] = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(S_N x_n)].$$

Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{S_i x_n : n \in \mathbb{N}, \ i = 1, 2, ..., N\}$ are bounded sequences in E.
We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in Z$ be fixed. In view of Lemma 2.3 and (3.1), we have

$$\begin{aligned} D_g(p, y_n) &= D_g(p, \nabla g^*[(1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(S_N x_n)]) \\ &= V(p, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(S_N x_n)) \\ &\leq (1-\beta_n)V(p, \nabla g(x_n)) + \beta_n V(p, \nabla g(S_N x_n)) \\ &= (1-\beta_n)D_g(p, x_n) + \beta_n D_g(p, S_N x_n) \\ &\leq (1-\beta_n)D_g(p, x_n) + \beta_n D_g(p, x_n) \\ &= D_g(p, x_n). \end{aligned} \tag{3.2}$$

This implies that

$$\begin{aligned} D_g(p, x_{n+1}) &= D_g(p, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\ &= V(p, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\ &\leq \alpha_n V(p, \nabla g(u)) + (1-\alpha_n)V(p, \nabla g(y_n)) \\ &= \alpha_n D_g(p, u) + (1-\alpha_n)D_g(p, y_n) \\ &\leq \alpha_n D_g(p, u) + (1-\alpha_n)D_g(p, y_n) \\ &\leq \alpha_n D_g(p, u) + (1-\alpha_n)D_g(p, x_n) \\ &\leq \max\{D_g(p, u), D_g(p, x_n)\}. \end{aligned} \tag{3.3}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \leq \max\{D_g(p, u), D_g(p, x_1)\} \tag{3.4}$$

for all $n \in \mathbb{N}$. It follows from (3.4) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_1 > 0$ such that

$$D_g(x_n, x) \leq M_1, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

In view of Lemma 2.2 (3), we have that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{S_i\}_{i=1}^N$ is a finite family of Bregman relatively nonexpansive mappings from E into itself, we conclude that

$$D_g(p, S_i x_n) \le D_g(p, x_n), \quad \forall n \in \mathbb{N} \quad \text{and } i = 1, 2, ..., N. \tag{3.6}$$

This together with Definition 2.1 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ implies that $\{S_i x_n : n \in \mathbb{N},\ i = 1, 2, ..., N\}$ is bounded. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [16](Proposition 1.1.11) for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(S_i x_n) : n \in \mathbb{N},\ i = 1, 2, ..., N\}$ are bounded in E^*. In view of Theorem 2.2 (3), we obtain that dom $g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_1 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(S_N x_n)\| : n \in \mathbb{N}\}$ and $\rho^*_{s_1} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 2. We prove that for any $n \in \mathbb{N}$

$$D_g(z, y_n) \le D_g(z, x_n) - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|). \tag{3.7}$$

Let us show (3.7). For each $n \in \mathbb{N}$, in view of the definition of Bregman distance (see (1.5)), Lemma 2.4 and (3.2), we obtain

$$\begin{aligned}
D_g(z, y_n) &= g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*((1 - \beta_n)\nabla g(x_n) + \beta_n \nabla g(S_N x_n)) \\
&\quad - \langle z, (1 - \beta_n)\nabla g(x_n) + \beta_n \nabla g(S_N x_n))\rangle \\
&\le (1 - \beta_n)g(z) + \beta_n g(z) + (1 - \beta_n)g^*(\nabla g(x_n)) + \beta_n g^*(\nabla g(S_N x_n)) \\
&\quad - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \\
&\quad - (1 - \beta_n)\langle z, \nabla g(x_n)\rangle - \beta_n\langle z, \nabla g(S_N x_n)\rangle \\
&= (1 - \beta_n)[g(z) + g^*(\nabla g(x_n)) - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(z) + g^*(\nabla g(S_N x_n)) - \langle z, \nabla g(S_N x_n)\rangle] \\
&\quad - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \\
&= (1 - \beta_n)[g(z) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(z) - g(S_N x_n) + \langle S_N x_n, \nabla g(S_N x_n)\rangle - \langle z, \nabla g(S_N x_n)\rangle] \\
&\quad - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \\
&= (1 - \beta_n)D_g(z, x_n) + \beta_n D_g(z, S_N x_n) \\
&\quad - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \\
&\le (1 - \beta_n)D_g(z, x_n) + \beta_n D_g(z, x_n) \\
&\quad - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \\
&= D_g(z, x_n) - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|).
\end{aligned}$$

In view of Lemma 2.3 and (3.7), we obtain

$$\begin{aligned}
D_g(z, x_{n+1}) &= D_g(z, \nabla g^*[\alpha_n \nabla g(z) + (1 - \alpha_n)\nabla g(y_n)]) \\
&= D_g(z, \nabla g^*[\alpha_n \nabla g(z) + (1 - \alpha_n)\nabla g(y_n)]) \\
&= V(z, \alpha_n \nabla g(u) + (1 - \alpha_n)\nabla g(y_n)) \\
&\le \alpha_n V(z, \nabla g(u)) + (1 - \alpha_n)V(z, \nabla g(y_n)) \\
&= \alpha_n D_g(z, u) + (1 - \alpha_n)D_g(z, y_n) \\
&\le \alpha_n D_g(z, u) + (1 - \alpha_n)D_g(z, y_n) \\
&\le \alpha_n D_g(z, u) \\
&\quad + (1 - \alpha_n)[D_g(z, x_n) - \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|)].
\end{aligned} \tag{3.8}$$

Let $M_2 := \sup\{|D_g(z, u) - D_g(z, x_n)| + \beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) : n \in \mathbb{N}\}$. It follows from (3.8) that

$$\beta_n(1 - \beta_n)\rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) \le D_g(z, x_n) - D_g(z, x_{n+1}) + \alpha_n M_2. \tag{3.9}$$

In view of Lemma 2.3 and (3.7) we obtain

$$\begin{aligned} D_g(z, x_{n+1}) &= D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\ &= V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\ &\leq V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n) - \alpha_n(\nabla g(u) - \nabla g(z))) \\ &\quad - \langle \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] - z, -\alpha_n(\nabla g(u) - \nabla g(z))\rangle \\ &= V(z, \alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)) + \alpha_n\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle \\ &= D_g(z, \nabla g^*[\alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)]) \\ &\quad + \alpha_n\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq \alpha_n D_g(z,z) + (1-\alpha_n)D_g(z, y_n) + \alpha_n\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle \\ &= (1-\alpha_n)D_g(z, x_n) + \alpha_n\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned} \tag{3.10}$$

The rest of the proof will be divided into two parts:
Case 1. If there exists $n_0 \in \mathbb{N}$ such that $\{D_g(z, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing, then $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have $D_g(z, x_n) - D_g(z, x_{n+1}) \to 0$ as $n \to \infty$. This, together with condition (c) and (3.9), implies that

$$\lim_{n\to\infty} \rho^*_{s_1}(\|\nabla g(x_n) - \nabla g(S_N x_n)\|) = 0.$$

Therefore, from the property of $\rho^*_{s_1}$ we deduce that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(S_N x_n)\| = 0.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} \|x_n - S_N x_n\| = 0. \tag{3.11}$$

Since S_N is a Bregman relatively nonexpansive mapping, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup y \in F(S_N)$ and

$$\limsup_{n\to\infty}\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle = \lim_{i\to\infty}\langle x_{n_i+1} - z, \nabla g(u) - \nabla g(z)\rangle. \tag{3.12}$$

This, together with (1.6), implies that

$$\limsup_{n\to\infty}\langle x_n - z, \nabla g(u) - \nabla g(z)\rangle = \langle y - z, \nabla g(u) - \nabla g(z)\rangle \leq 0. \tag{3.13}$$

In view of Lemma 2.2 and (3.11) we obtain that

$$\lim_{n\to\infty} D_g(S_N x_n, x_n) = 0.$$

This implies that

$$D_g(S_N x_n, y_n) \leq (1-\beta_n)D_g(S_N x_n, x_n) + \beta_n D_g(S_N x_n, S_N x_n) = (1-\beta_n)D_g(S_N x_n, x_n) \to 0 \tag{3.14}$$

as $n \to \infty$. Also, we have

$$D_g(y_n, x_{n+1}) \leq \alpha_n D_g(y_n, u) + (1-\alpha_n)D_g(y_n, y_n) = \alpha_n D_g(y_n, u) \to 0 \tag{3.15}$$

as $n \to \infty$. In view of Lemma 2.2 and (3.10), (3.14) and (3.15), we conclude that

$$\lim_{n\to\infty}\|y_n - S_N x_n\| = \lim_{n\to\infty}\|y_n - x_{n+1}\| = 0 \text{ and } \lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.16}$$

This, together with Lemma 2.2. implies that

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0.$$

For any $w \in Z$, it follows from the three point identity (see (2.2)) that

$$\begin{aligned} |D_g(w, x_n) - D_g(w, y_n)| &= |D_g(w, u_n) + D_g(y_n, x_n) \\ &\quad + \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle - D_g(w, y_n)| \\ &= |D_g(y_n, x_n) - \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle| \\ &\leq D_g(y_n, x_n) + \|w - y_n\|\|\nabla g(y_n) - \nabla g(x_n)\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. On the other hand, we have from (3.2) that

$$\begin{aligned} D_g(S_i(x_n), S_{i-1}(x_n)) &= D_g(T_i(S_{i-1}(x_n)), S_{i-1}(x_n)) \\ &\leq D_g(w, S_{i-1}(x_n)) - D_g(w, S_i(x_n)) \\ &\leq D_g(w, x_n) - D_g(w, y_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} D_g(S_i x_n, x_n) = 0, \quad i = 1, 2, ..., N.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|S_i x_n - x_n\| = 0, \quad i = 1, 2, ..., N.$$

Hence $y \in Z$. From (3.13) and (3.16), we deduce that

$$\limsup_{n\to\infty}\langle x_n - z, \nabla g(u) - \nabla g(z)\rangle = \limsup_{n\to\infty}\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\rangle \leq 0.$$

Thus we have the desired result by Lemma 2.6.
Case 2. If there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_g(z, x_{n_i}) < D_g(z, x_{n_i+1})$$

for all $i \in \mathbb{N}$, then by Lemma 2.5, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_g(z, x_{m_k}) < D_g(z, x_{m_k+1}) \quad \text{and} \quad D_g(z, x_k) \leq D_g(z, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This, together with (3.9), implies that

$$\beta_{m_k}(1 - \beta_{m_k})\rho^*_{s_1}(\|\nabla g(x_{m_k}) - \nabla g(S_N x_{m_k})\|) \leq D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) + \alpha_{m_k} M_2 \leq \alpha_{m_k} M_2$$

for all $k \in \mathbb{N}$. Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} \rho^*_{s_1}(\|\nabla g(x_{m_k}) - \nabla g(S_N x_{m_k})\|) = 0.$$

By the same argument as Case 1, we arrive at

$$\limsup_{k\to\infty}\langle x_{m_k+1} - z, \nabla g(u) - \nabla g(z)\rangle = \limsup_{k\to\infty}\langle x_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \leq 0.$$

It follows from (3.10) that

$$D_g(z, x_{m_k+1}) \leq (1 - \alpha_{m_k}) D_g(z, x_{m_k}) + \alpha_{m_k}\langle x_{m_k+1} - z, \nabla g(u) - \nabla g(z)\rangle. \tag{3.17}$$

Since $D_g(z, x_{m_k}) \leq D_g(z, x_{m_k+1})$, we have that

$$\begin{aligned} \alpha_{m_k} D_g(z, x_{m_k}) &\leq D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) + \alpha_{m_k}\langle x_{m_k+1} - z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq \alpha_{m_k}\langle x_{m_k+1} - z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned} \tag{3.18}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$D_g(z, x_{m_k}) \le \langle x_{m_k+1} - z, \nabla g(u) - \nabla g(z) \rangle.$$

In view of (3.17), we deduce that

$$\lim_{k\to\infty} D_g(z, x_{m_k}) = 0.$$

This, together with (3.18), implies that

$$\lim_{k\to\infty} D_g(z, x_{m_k+1}) = 0.$$

On the other hand, we have $D_g(z, x_k) \le D_g(z, x_{m_k+1})$ for all $k \in \mathbb{N}$ which implies that $x_k \to z$ as $k \to \infty$. Thus, we have $x_n \to z$ as $n \to \infty$.

Remark 3.1. We propose a new type of Halpern iterative scheme for finding common zeros of finitely many maximal monotone operators in a reflexive Banach space E. This scheme has an advantage that we do not use any projection which creates some difficulties in a practical calculation of the iterative sequence.

4. Equilibrium Problems and Inverse Strongly Monotone Mappings

Let C be a nonempty, closed and convex of a reflexive Banach space E. Let $f : C\times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: Find $p \in C$ such that

$$f(p, y) \ge 0, \qquad \forall y \in C. \tag{4.1}$$

For solving the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for each $y \in C$, the function $x \longmapsto f(x, y)$ is upper semicontinuous;
(A4) for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.
The set of solutions of problem (4.1) is denoted by $EP(f)$.

In this section, we propose Halpern-type iterative schemes for finding common solutions of an equilibrium problem and null spaces of a γ-inverse strongly monotone mapping in a 2-uniformly convex Banach space and prove two strong convergence theorems.

Let $g : E \to \mathbb{R}$ be a Legendre function. The *resolvent* of a bifunction $f : C \times C \to \mathbb{R}$ [58] is the operator $\mathrm{Res}^g_f : E \to 2^{E^*}$, defined by

$$\mathrm{Res}^g_f(x) = \{z \in C : f(z, y) + \langle y - z, \nabla g(z) - \nabla g(x)\rangle \ge 0 \;\; for\ all\ y \in C\} \tag{4.2}$$

for all $x \in E$. We also define the mapping $A_f : E \to 2^{E^*}$ in the following way:

$$A_g(x) = \begin{cases} \{\xi \in E^* : f(x, y) \ge \langle \xi, y - x\rangle \ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Lemma 4.1 [33, 39, 58]. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4) and $EP(f) \neq \emptyset$. Then, the following statements hold:*
(1) dom $(\mathrm{Res}^g_f) = E$;
(2) Res^g_f *is single-valued;*
(3) Res^g_f *is a Bregman firmly nonexpansive mapping [52], i.e., for all $x,\ y \in E$,*

$$\langle \mathrm{Res}^g_f(x) - \mathrm{Res}^g_f(y), \nabla g(\mathrm{Res}^g_f(x)) - \nabla g(Res^g_f(y))\rangle \le \langle \mathrm{Res}^g_f(x) - \mathrm{Res}^g_f(y), \nabla g(x) - \nabla g(y)\rangle;$$

(4) the set of fixed points of Res_f^g *is the solution of the corresponding equilibrium problem, i.e.,* $F(\mathrm{Res}_f^g) = EP(f)$*;*
(5) $EP(f)$ *is a closed and convex subset of* C*;*
(6) $D_g(q, \mathrm{Res}_f^g x) + D_g(\mathrm{Res}_f^g x, x) \leq D_g(q, x),\ \forall q \in F(\mathrm{Res}_f^g)$*.*

Lemma 4.2 [33, 58]. *Let* E *be a reflexive Banach space and* $g : E \to \mathbb{R}$ *a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of* E*. Let* C *be a nonempty, closed and convex subset of* E *and* $f : C \times C \to \mathbb{R}$ *a bifunction satisfying (A1)-(A4) and* $EP(f) \neq \emptyset$*. Then, the following statements hold:*
(1) $EP(f) = A_g^{-1}(0^*)$*;*
(2) A_g *is a maximal monotone operator;*
(3) $\mathrm{Res}_f^g = \mathrm{Res}_{A_g}^g$*.*

Theorem 4.1. *Let* E *be a* 2*-uniformly convex Banach space and* $g : E \to \mathbb{R}$ *a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of* E*. Assume that there exists* $c_1 > 0$ *such that* g *is* ρ*-convex with* $\rho(t) := \frac{c_1}{2}t^2$ *for all* $t \geq 0$*. Let* C_i*,* $i = 1, 2, ..., N$ *be* N *nonempty, closed and convex subsets of* E*. Let* $f_i : C_i \times C_i \to \mathbb{R}$*,* $i = 1, 2, ..., N$ *be bifunctions that satisfy conditions (A1)-(A4) such that* $\cap_{i=1}^N EP(f_i) \neq \emptyset$*. Let* $A_i : E \to 2^{E^*}$*,* $i = 1, 2, ..., N$*, be* N *maximal monotone operators such that* $\cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$*. Assume that* $A : C \to E^*$ *is a* γ*-inverse strongly monotone mapping for some* $\gamma > 0$*. Suppose that* $Z := \cap_{i=1}^N (A_i^{-1}(0^*) \cap A^{-1}(0) \cap EP(f_i))$ *is a nonempty subset of* C*, where* $EP(f_i)$ *is the set of solutions to the equilibrium problem (4.1). Let* $\{\alpha_n\}_{n\in\mathbb{N}}$ *and* $\{\beta_n\}_{n\in\mathbb{N}}$ *be two sequences in* $[0, 1]$ *satisfying the following control conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$*;*
(b) $\sum_{n=1}^\infty \alpha_n = \infty$*;*
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*
Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by*

$$\begin{cases} u \in E,\ x_1 \in E \quad \text{chosen arbitrarily,} \\ w_n = \nabla g^*[\nabla g(x_n) - \lambda A x_n], \\ y_n = \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(\mathrm{Res}_{r_N f_N}^g ... \mathrm{Res}_{r_1 f_1}^g (w_n))], \\ x_{n+1} = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] \text{ and } n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where ∇g *is the gradient of* g*. Let* λ *be a constant such that* $0 < \lambda < \frac{c_2^2\gamma}{2}$*, where* c_2 *is the* 2*-uniformly convex constant of* E *satisfying Corollary 2.1 (2). If* $r_i > 0$*, for each* $i = 1, 2, ..., N$*, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined in (4.4) converges strongly to* $\mathrm{proj}_Z^g u$ *as* $n \to \infty$*.*

Proof. We divide the proof into several steps.
Set

$$z = \mathrm{proj}_Z^g u.$$

Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are bounded sequences in C. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in F$ be fixed. In view of (1.9), Lemma 2.3, Lemma 4.2 and (4.1), we obtain

$$\begin{aligned} D_g(p, w_n) &= D_g(p, \nabla g^*[\nabla g(x_n) - \beta A x_n]) \\ &= V(p, \nabla g(x_n) - \lambda A x_n) \\ &\leq V(p, \nabla g(x_n) - \lambda A x_n + \lambda A x_n) - \langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - p, \lambda A x_n \rangle \\ &= V(p, \nabla g(x_n)) - \lambda \langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - p, A x_n \rangle \\ &= D_g(p, x_n) - \lambda \langle x_n - p, A x_n \rangle - \lambda \langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - x_n, A x_n \rangle \\ &\leq D_g(p, x_n) - \lambda\gamma \|A x_n\|^2 + \lambda \|\nabla g^*(\nabla g(x_n) - \lambda A x_n) - \nabla g^* \nabla g(x_n)\| \|A x_n\| \\ &\leq D_g(p, x_n) - \lambda\gamma \|A x_n\|^2 + \frac{4\lambda^2}{c_2^2} \|A x_n\|^2 \\ &\leq D_g(p, x_n) + \lambda \left(\frac{4\lambda}{c_2^2} - \gamma \right) \|A x_n\|^2. \end{aligned} \tag{4.5}$$

This, together with $\frac{4\lambda}{c_2^2} - \gamma < 0$, implies that

$$D_g(p, w_n) \leq D_g(p, x_n).$$

Since S_N is Bregman relatively nonexpansive, for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}D_g(p, y_n) &= D_g(p, \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(S_N w_n)])\\ &= V(p, \beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(S_N w_n))\\ &= g(p) - \langle p, \beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(S_N w_n)\rangle\\ &\quad + g^*(\beta_n \nabla g(w_n) + (1-\alpha_n)\nabla g(S_N w_n))\\ &\le \alpha_n g(p) + (1-\alpha_n) g(p)\\ &\quad + \beta_n g^*(\nabla g(w_n)) + (1-\alpha_n) g^*(\nabla g(S_N w_n))\\ &= \alpha_n V(p, \nabla g(x_n)) + (1-\beta_n) V(p, \nabla g(S_N w_n))\\ &= \beta_n D_g(p, x_n) + (1-\beta_n) D_g(p, S_N w_n)\\ &\le \beta_n D_g(p, x_n) + (1-\beta_n) D_g(p, w_n)\\ &\le \beta_n D_g(p, x_n) + (1-\beta_n) D_g(p, x_n)\\ &= D_g(p, x_n).\end{aligned}$$

This implies that

$$\begin{aligned}D_g(p, x_{n+1}) &= D_g(p, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)])\\ &= V(p, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n))\\ &\le \alpha_n V(p, \nabla g(u)) + (1-\alpha_n) V(p, \nabla g(y_n))\\ &= \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n)\\ &\le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n)\\ &\le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, x_n)\\ &\le \max\{D_g(p, u), D_g(p, x_n)\}.\end{aligned} \tag{4.6}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \le \max\{D_g(p, u), D_g(p, x_1)\} \tag{4.7}$$

for all $n \in \mathbb{N}$. It follows from (4.7) that the sequence $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_3 > 0$ such that

$$D_g(p, x_n) \le M_3, \qquad \forall n \in \mathbb{N}. \tag{4.8}$$

In view of Definition 2.1, we deduce that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_{r_n}\}_{n\in\mathbb{N}}$ is an infinite family of Bregman relatively nonexpansive mappings from E into C, we conclude that

$$D_g(p, S_i w_m) \le D_g(p, w_n) \le D_g(p, x_n), \qquad \forall n \in \mathbb{N} \text{ and } i = 1, 2, ..., N. \tag{4.9}$$

This, together with Definition 2.2 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{S_i w_n\}_{n\in\mathbb{N}}$ is bounded for each $i = 1, 2, ..., N$. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [16](Proposition 1.1.11) for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(S_i w_n)\}_{n\in\mathbb{N}}$ are bounded in E^*. In view of Theorem 2.2 (3), we obtain that dom $g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(S_i w_n)\| : n \in \mathbb{N},\ i = 1, 2, ..., N\}$ and $\rho^*_{s_2} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 2. We prove that for any $n \in \mathbb{N}$

$$D_g(z, y_n) \le D_g(z, x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|). \tag{4.10}$$

Let us show (4.10). For each $n \in \mathbb{N}$, in view of the definition of Bregman distance (see (1.5)), Lemma 2.5

and (4.10), we obtain

$$\begin{aligned}
D_g(z,y_n) &= g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*((1-\beta_n)\nabla g(w_n) + \beta_n \nabla g(S_N w_n)) \\
&\quad - \langle z, (1-\beta_n)\nabla g(w_n) + \beta_n \nabla g(S_N w_n))\rangle \\
&\le (1-\beta_n)g(z) + \beta_n g(z) + (1-\beta_n)g^*(\nabla g(w_n)) + \beta_n g^*(\nabla g(S_N w_n)) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&\quad - (1-\beta_n)\langle z, \nabla g(w_n)\rangle - \beta_n\langle z, \nabla g(S_N w_n)\rangle \\
&= (1-\beta_n)[g(z) + g^*(\nabla g(w_n)) - \langle z, \nabla g(w_n)\rangle] \\
&\quad + \beta_n[g(z) + g^*(\nabla g(S_N w_n)) - \langle z, \nabla g(S_N w_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&= (1-\beta_n)[g(z) - g(x_n) + \langle w_n, \nabla g(w_n)\rangle - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n[g(z) - g(S_N w_n) + \langle S_N w_n, \nabla g(S_N w_n)\rangle - \langle z, \nabla g(S_N w_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&= (1-\beta_n)D_g(z,w_n) + \beta_n D_g(z, S_N w_n) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&\le (1-\beta_n)D_g(z,w_n) + \beta_n D_g(z,w_n) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&= D_g(z,w_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \\
&\le D_g(z,x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|).
\end{aligned}$$

In view of Lemma 2.3 and (4.10), we obtain

$$\begin{aligned}
D_g(z,x_{n+1}) &= D_g(z, \alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)) \\
&= D_g(z, \nabla g^*[\alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)]) \\
&= V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\
&\le \alpha_n V(z, \nabla g(u)) + (1-\alpha_n)V(z, \nabla g(y_n)) \\
&= \alpha_n D_g(z,u) + (1-\alpha_n)D_g(z,y_n) \\
&\le \alpha_n D_g(z,u) + (1-\alpha_n)D_g(z,y_n) \\
&\le \alpha_n D_g(z,u) \\
&\quad + (1-\alpha_n)[D_g(z,x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|)].
\end{aligned} \tag{4.11}$$

Let $M_4 := \sup\{|D_g(z,u) - D_g(z,x_n)| + \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) : n \in \mathbb{N}\}$. It follows from (4.11) that

$$\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(w_n) - \nabla g(S_N w_n)\|) \le D_g(z,x_n) - D_g(z,x_{n+1}) + \alpha_n M_4. \tag{4.12}$$

In view of Lemma 2.3 and (4.10) we obtain

$$\begin{aligned}
D_g(z,x_{n+1}) &= D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\
&= V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\
&\le V(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n) - \alpha_n(\nabla g(u) - \nabla g(z))) \\
&\quad - \langle g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] - z, -\alpha_n(\nabla g(u) - \nabla g(z))\rangle \\
&= V(z, \alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= D_g(z, \nabla g^*[\alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)]) \\
&\quad + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\le \alpha_n D_g(z,z) + (1-\alpha_n)D_g(z,y_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= (1-\alpha_n)D_g(z,x_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle.
\end{aligned} \tag{4.13}$$

Step 3. By the same argument as in the proof of Theorem 3.1 and using (4.12)-(4.13), we conclude that

$$\lim_{n\to\infty}\|w_n - z\| = 0 \ \text{ and } \ \lim_{n\to\infty}\|w_n - S_N w_n\| = 0.$$

Also, the following is obvious:

$$\lim_{n\to\infty}\|x_n - z\| = 0, \quad \text{and} \quad \lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{4.14}$$

In view of Lemma 2.2 and (4.14) we obtain that

$$\lim_{n\to\infty} D_g(S_N w_n, w_n) = 0.$$

This implies that

$$D_g(S_N w_n, y_n) \leq (1-\beta_n)D_g(S_N w_n, w_n) + \beta_n D_g(S_N w_n, S_N w_n) = (1-\beta_n)D_g(S_N w_n, w_n) \to 0 \tag{4.15}$$

as $n \to \infty$. Also, we have

$$D_g(y_n, z_n) \leq \alpha_n D_g(y_n, u) + (1-\alpha_n)D_g(y_n, y_n) = \alpha_n D_g(y_n, u) \to 0 \tag{4.16}$$

as $n \to \infty$ and hence

$$D_g(y_n, x_{n+1}) \leq D_g(y_n, z_n) \to 0 \tag{4.17}$$

as $n \to \infty$. In view of Lemma 2.2 and (4.15)-(4.17), we conclude that

$$\lim_{n\to\infty} \|y_n - w_n\| = \lim_{n\to\infty} \|y_n - S_N w_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{4.18}$$

From (4.15)-(4.18), we deduce that

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{n\to\infty} \|y_n - z\| = 0. \tag{4.19}$$

This, together with Lemma 2.2. implies that

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0. \tag{4.20}$$

A similar argument, as in the proof of Theorem 3.1, we get the desired conclusion.

Let C_i, $i = 1, 2, ..., N$ be N nonempty, closed and convex subsets of a Banach space E. The convex feasibility problem is to find an element in the assumed nonempty intersection $\cap_{i=1}^N C_i$ (see [3]). In the following, we prove a strong convergence theorem concerning convex feasibility problems in a reflexive Banach space.

Theorem 4.2. *Let E be a 2-uniformly convex Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Assume that there exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{2}t^2$ for all $t \geq 0$. Let C_i, $i = 1, 2, ..., N$ be N nonempty, closed and convex subsets of E. Let $f_i : C_i \times C_i \to \mathbb{R}$, $i = 1, 2, ..., N$ be bifunctions that satisfy conditions (A1)-(A4) such that $\cap_{i=1}^N EP(f_i) \neq \emptyset$. Let $A_i : E \to 2^{E^*}$, $i = 1, 2, ..., N$, be N maximal monotone operators such that $\cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Assume that $A : C \to E^*$ is a γ-inverse strongly monotone mapping for some $\gamma > 0$. Suppose that $Z := \cap_{i=1}^N [A_i^{-1}(0^*) \cap EP(f_i)] \cap A^{-1}(0) \neq \emptyset$ is a nonempty subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem (1.2). Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0, 1]$ satisfying the following control conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E, \ x_1 \in E \quad \text{chosen arbitrarily}, \\ w_n = \nabla g^*[\nabla g(x_n) - \lambda A x_n], \\ y_n = \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(\text{proj}^g_{r_N, C_N} ... \text{proj}^g_{r_1, C_1}(w_n)], \\ x_{n+1} = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] \text{ and } n \in \mathbb{N}, \end{cases} \tag{4.21}$$

where ∇g is the gradient of g. Let λ be a constant such that $0 < \lambda < \frac{c_2^2\gamma}{2}$, where c_2 is the 2-uniformly convex constant of E satisfying Corollary 2.1 (2). If $r_i > 0$, for each $i = 1, 2, ..., N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (4.21) converges strongly to $\text{proj}^g_Z u$ as $n \to \infty$.

Remark 4.1. Theorem 3.1 improves Theorems 1.1 and 1.2 in the following aspects.
(1) In Theorem 3.1, we present a strong convergence theorem for products of resolvents of finitely many maximal monotone operators with a new algorithm and new control conditions. This is complementary to Theorem 1.1.
(2) For the algorithm, we remove the sets C_n and Q_n in Theorems 1.1 and 1.2.

5. Applications (Hammerstein-type Equations)

Let E be a real Banach space with the dual space E^*. The generalized formulation of many boundary value problems for ordinary and partial differential equations leads to operator equations of the type

$$\langle z, Ax\rangle = \langle z, b\rangle \quad \forall z \in E,$$

which is equivalent to equality of functionals on E. That is, the equality of the form:

$$Ax = b, \tag{5.1}$$

where A is a monotone-type operator acting from a Banach space E into E^*. Without loss of generality we may assume $b = 0$. It is well known that a solution of the equation $Ax = 0$ (*i.e.*, $\langle z, Ax\rangle = 0\ \forall z \in E$) is a solution of the variational inequality $\langle z - x, Ax\rangle \geq 0\ \forall z \in E$. Therefore, the theory of monotone operators and its applications to nonlinear partial differential equations and variational inequalities are related and have been involved in a substantial topic in nonlinear functional analysis. One important application of solving (5.1) is finding the zeros of the so-called equation of Hammerstein-type (see e.g., [29]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_\Omega k(x, y) f(y, u(y))dy = h(x), \tag{5.2}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel k is defined on $\Omega\times\Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by $Kv(x) = \int_\Omega k(x, y)v(y)dy;\ \ x \in \Omega$, and the so-called superposition or Nemytskii operator by $Qu(y) := f(y, u(y))$, then the integral Eq. (5.2) can be put in operator theoretic form as follows:

$$u + KQu = 0, \tag{5.3}$$

where, without loss of generality, we have taken $h = 0$.

Interest in Eq. (5.2) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts posses Green's functions can, as a rule, be transformed into equations of the form (5.2) (see e.g., [42], chapter IV). Equations of the Hammerstein type play a crucial role in the theory of optimal control systems (see e.g., [25]). Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see e.g., [24, 26]). Very recently, Ofoedu and Malonza in [41] proposed an iterative solution of the operator Hammerstein Eq. (5.1) in a 2-uniformly convex and uniformly smooth Banach space.

Now, we give an application of Theorem 4.1 to an iterative solution of the operator Hammerstein Eq. (5.1).

Theorem 5.1. *Let E be a real Banach space with dual space E^* such that $X = E \times E^*$ (with norm $\|z\|_X^2 = \|u\|_E^2 + \|v\|_{E^*}^2$, $z = (u, v) \in X$) is a 2-uniformly convex and uniformly smooth real Banach space. Let $g : X \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of X. Assume that there exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{2}t^2$ for all $t \geq 0$. Let $Q : E \to E^*$ and $K : E^* \to E$ with $dom\, K = Q(E) = E^*$ be continuous monotone type operators such that Eq. (5.3)*

has a solution in E, and such that the map $A : X \to X^*$ defined by $Az := A(u, v) = (Qu-v, u+Kv)$ is γ-inverse strongly monotone. Let C_i, $i = 1, 2, ..., N$ be N nonempty, closed and convex subsets of X and $f_i : C_i \times C_i \to \mathbb{R}$, $i = 1, 2, ..., N$ be bifunctions that satisfy conditions (A1)-(A4) such that $\cap_{i=1}^{N} EP(f_i) \neq \emptyset$. Let $A_i : E \to 2^{E^*}$, $i = 1, 2, ..., N$, be N maximal monotone operators such that $\cap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0, 1]$ satisfying the following control conditions:
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E,\ x_1 \in C \quad \text{chosen arbitrarily}, \\ w_n = \nabla g^*[\nabla g(x_n) - \beta A x_n], \\ y_n = \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(\mathrm{Res}^g_{r_N f_N} ... \mathrm{Res}^g_{r_1 f_1} w_n)], \\ x_{n+1} = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] \text{ and } n \in \mathbb{N}, \end{cases} \tag{5.4}$$

where ∇g is the gradient of g. Let β be a constant such that $0 < \beta < \frac{c_2^2\gamma}{2}$, where c_2 is the 2-uniformly convex constant of E satisfying Corollary 2.1 (2). If $Z := \cap_{i=1}^{N}[A_i^{-1}(0^*) \cap EP(f_i)] \cap A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (5.4) converges strongly to $\mathrm{proj}^g_Z u$ as $n \to \infty$.

Remark 5.1. Observe that $z_0 \in Z$ implies, in particular, that $z_0 \in A^{-1}(0) \iff Az_0 = 0$. But $z_0 = (u_0, v_0)$ for some $u_0 \in E$ and $v_0 \in E^*$; moreover, $Az_0 = A(u_0, v_0) = (Qu_0 - v_0, u_0 + Kv_0)$. So, $Az_0 = 0$ implies that $(Qu_0 - v_0, u_0 + Kv_0) = (0, 0)$. This is equivalent to $Qu_0 - v_0 = 0$ and $u_0 + Kv_0 = 0$. Thus we have $v_0 = Qu_0$ which in turn implies that $u_0 + Kv_0 = 0$. Therefore, $u_0 \in E$ solves the Hammerstein-type Eq. (5.3).

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