Filomat 31:14 (2017), 4629–4640
https://doi.org/10.2298/FIL1714629T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Krasnoselskii-Mann Method for Multi-valued Non-Self Mappings in CAT(0) Spaces

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Abstract. We define Mann iterative scheme in CAT(0) spaces and obtain $\triangle$-convergence and strong convergence of the iterative scheme to a fixed point of multi-valued nonexpansive non-self mappings. We also obtain strong convergence of the scheme to a fixed point of multi-valued quasi-nonexpansive non-self mappings under appropriate conditions. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let K be a nonempty subset of a CAT(0) space X. The set K is called *proximinal*, if for each $x \in X$ there exists $u \in K$ such that

$$d(x,u) = \inf\{d(x,y) : y \in K\} = d(x,K), \text{where d is a metric on X.}$$

It is well known [4] that every nonempty, closed and convex subset of a complete CAT(0) space is proximinal. We denote the family of nonempty proximinal bounded subsets of K by Prox(K).

Let $CB(X)$ be the family of nonempty, closed and bounded subsets of a CAT(0) space X. For $A, B \in CB(X)$, we shall denote the Hausdorff distance between A and B by $D(A, B)$, i.e,

$$D(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\}.$$

A mapping $T : K \to 2^X$ is said to be *L-Lipschitz* if there exists $L \geq 0$ such that

$$D(Tx,Ty) \leq Ld(x,y), \text{ for all } x,y \in K.$$

T is called *nonexpansive* mapping if $L = 1$ and it is called *contraction* mapping if $L < 1$. It is easy to observe that the class of Lipschitz mappings includes the class of nonexpansive mappings and hence the class of

2010 *Mathematics Subject Classification*. Primary 37C25, 47H09 ; Secondary 47H04
Keywords. CAT(0) space, fixed points, Hausdorff distance, multi-valued mapping, nonexpansive mappings, non-self mappings
Received: 29 April 2016; Accepted: 17 August 2016
Communicated by Naseer Shahzad
Research supported by IMU Break Out Fellowship Program through TWAS and partially supported by Bahir Dar University, Ethiopia and Simons Foundation based at Botswana International Univesity of Science and Technology
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contraction mappings.

A point $x \in K$ is called a *fixed point* of a mapping $T : K \to 2^X$ if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$. A mapping $T : K \to 2^X$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $D(Tx, Tp) \leq d(x, p)$ for all $x \in K$ and all $p \in F(T)$. Clearly, every nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping.

The study of fixed points of nonlinear mappings stems mainly from its applications in convex optimization, control theory, differential inclusions and economics. Consequently, the existence of fixed points and their approximations for contraction and nonexpansive mappings and their generalizations have been studied by several authors (see, e.g. [1, 18, 21-24, 27, 29-31] and the references therein). More specifically, fixed point theorems in a CAT(0) space can be applied to graph theory, biology and computer sciences (see, e.g. [3, 11, 13, 17]).

Mann iteration for multi-valued mappings is first introduced by Sastry and Babu [20] in a Hilbert space settings. Let $X = H$, a real Hilbert space, $T : K \to \text{Prox}(K)$ be a multi-valued mapping and fix $p \in F(T)$. Then they define sequence of Mann iterates as:

$$x_0 \in K, x_{n+1} = \alpha_n y_n + (1 - \alpha_n)x_n, n \geq 0,$$

where $y_n \in Tx_n$ such that $\|y_n - p\| = d(p, Tx_n)$ and $\alpha_n \in [0, 1]$. They obtained strong convergence of the scheme to points in $F(T)$ assuming that K is compact and convex subset of H, T is nonexpansive mapping with $F(T) \neq \emptyset$ and $\alpha_n, \beta_n \in [0, 1]$ satisfying certain conditions.

In 2009, Laowang and Panyanak [16] defined Mann iteration for multi-valued self-mapping T in a CAT(0) space X as follows.

$$x_0 \in K, x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)y_n, n \geq 0, \tag{1}$$

where $y_n \in Tx_n$ such that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1}) + \gamma_n$, $\{\alpha_n\} \subseteq [0, 1)$ and $\gamma_n \in (0, \infty)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$. Then they proved the following result.

Theorem LP. [16] Let K be a nonempty, compact and convex subset of a complete $CAT(0)$ space X. Suppose that $T : K \to CB(K)$ is a multi-valued nonexpansive mapping and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. If $\{x_n\}$ is the sequence of Mann iterates defined by (1) such that one of the following two conditions is satisfied:

1) $\alpha_n \in [0, 1)$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

2) $0 < \liminf \alpha_n \leq \limsup \alpha_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

For approximating fixed points of single-valued non-self mappings, several Mann-type iterative schemes have been studied (see, eg., [25, 26]) with the use of metric projection or sunny nonexpansive retraction mappings in Banach spaces which are not easy to use in practical applications.

Recently, Colao and Marino [7] studied the following iterative scheme which is relatively easy to use in practical applications. Let K be a nonempty, convex and closed subset of a real Hilbert space H and let $T : K \to H$ be a mapping. Let $\{x_n\}$ be generated from arbitrary $x_0 \in K$ by

$$\begin{cases} \alpha_0 = max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ \alpha_{n+1} = max\{\alpha_n, h(x_{n+1})\}, \end{cases} \tag{2}$$

where $h : K \to \mathbb{R}$ is defined by $h(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in K\}$. They obtained strong and weak convergence of the sequence $\{x_n\}$ under certain conditions on $\{\alpha_n\}$, the mapping T and the domain K. The idea of using metric projection and sunny nonexpansive retraction are dispensed with in their setting.

Very recently, Tufa and Zegeye [28], extended the result of Colao and Marino [7] to multi-valued mappings. Precisely, they proved the following theorem.

Theorem TZ. [28] Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to \mathrm{Prox}(H)$ be an inward nonexpansive mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$. Let $\{x_n\}$ be a sequence of Mann-type given by

$$x_1 \in K, \;\; x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n, n \geq 1, \tag{3}$$

where $z_n \in Tx_n$ such that $\|z_n - z_{n+1}\| \leq D(Tx_n, Tx_{n+1})$, $\alpha_1 := max\{\frac{1}{2}, h_{z_1}(x_1)\}$, $\alpha_{n+1} := max\{\alpha_n, h_{z_{n+1}}(x_{n+1})\}$, $h_{z_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda) z_n \in K\}$. Then $\{x_n\}$ weakly converges to a fixed point of T. Moreover, if K is strictly convex and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.

It is our purpose in this paper to construct Mann iterative scheme for multi-valued nonexpansive non-self mappings and obtain $\triangle-$convergence and strong convergence of the scheme to a fixed point of the mappings in CAT(0) spaces. We also obtain strong convergence of the Mann iterative scheme to a fixed point of multi-valued Lipschitz quasi-nonexpansive non-self mappings under appropriate conditions. In the construction, the choice of the coefficient α_n is not made a priori, rather it is constructed step by step and determined by the values of the mapping T and the geometry of the set K.

2. Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is a map $r : [0, l] \subset \mathbb{R} \to X$ such that $r(0) = x, r(l) = y$ and $d(r(t), r(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. We note that r is an isometry and $d(x, y) = l$. The image of r is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The metric space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and it is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A uniquely geodesic space (X, d) is said to be an $\mathbb{R}$*-tree,* if $x, y, z \in X$ with $[x, y] \cap [y, z] = \{y\}$ implies $[x, z] = [x, y] \cup [y, z]$. For the sake of convenience we use X for a geodesic space (X, d) throughout this paper.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X (the vertices of $\triangle$) and three geodesic segments joining each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is the triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space X is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom:

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}) \;\; \forall x, y \in \triangle, \bar{x}, \bar{y} \in \bar{\triangle}, \tag{4}$$

where $\triangle$ is a geodesic triangle in X and $\bar{\triangle}$ is its comparison triangle in $\mathbb{R}^2$. It is well known that a CAT(0) space X is uniquely geodesic. Thus, for each $x, y \in X$ and $t \in [0, 1]$ there exists a unique point $z \in [x, y]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$, and we denote z by $tx \oplus (1-t)y$. A subset K of a CAT(0) space X is said to be convex if K includes every geodesic segment joining any two of its points. Pre-Hilbert spaces, $\mathbb{R}-$trees and Euclidean buildings are examples of $CAT(0)$ spaces. For details we refer the readers to standard texts such as [4, 5, 15].

Given a CAT(0) space X, we denote the pair $(x, y) \in X \times X$ by $\overrightarrow{xy}$ and call it a vector. Then a mapping

$\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by $\left\langle \overrightarrow{xy}, \overrightarrow{wz} \right\rangle = \frac{1}{2}[d^2(x,z) + d^2(y,w) - d^2(x,w) - d^2(y,z)], x, y, w, z \in X$ is said to be *quasilinearization*.

A geodesic space is said to satisfy the *Cauchy-Schwarz* inequality if

$$\left\langle \overrightarrow{xy}, \overrightarrow{wz} \right\rangle \leq d(x,y)d(w,z) \ \ \forall x, y, w, z \in X.$$

It is well known [2] that a geodesic metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup\limits_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

It is known [9] that in a CAT(0) space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subseteq X$ is said to be $\triangle$*-convergent* to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subseteq X$ such that $\{x_n\}$ $\triangle$-converges to x and given $y \in X$ with $y \neq x, \limsup\limits_{n\to\infty} d(x_n, x) < \limsup\limits_{n\to\infty} d(x_n, y)$.

Let K be a nonempty subset of a CAT(0) space X and $T : K \to 2^X$ be a mapping. Then the mapping $I - T$ is said to be *demiclosed* at zero, if for any sequence $\{x_n\} \subseteq K$ such that $\{x_n\}$ $\triangle-$converges to p and $d(x_n, Tx_n) \to 0$, then $p \in Tp$.

Let $x \in K$. The set $I_K(x) = \{w \in X : w = x \ \ or \ \ y = (1 - \frac{1}{\lambda})x \oplus \frac{1}{\lambda}w$, for some $y \in K, \lambda \geq 1\}$ is called an *inward* set at x. A multi-valued mapping T is said to be *inward* on K if $Tx \subseteq I_K(x)$.

A multi-valued mapping $T : K \to CB(X)$ is said to satisfy *Condition (I)* if there exists an increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, T(x)) \geq f(d(x, F(T)))$ for all $x \in K$.

A subset K of a CAT(0) space X is said to be *strictly convex* if it is convex and with the property that $x, y \in \partial K$ and $t \in (0, 1)$ implies that $tx \oplus (1 - t)y \in \mathring{K}$, where ∂K and $\mathring{K}$ denotes boundary and interior of K respectively.

We make use of the following lemmas in the sequel.

Lemma 2.1. *[14] Every bounded sequence in a complete CAT(0) space always has a $\triangle-$convergent subsequence.*

Lemma 2.2. *[6] Let X be a CAT(0) space. Then for each $x, y, z \in X$ and $\lambda \in [0, 1]$, one has*

$$d(\lambda x \oplus (1 - \lambda)y, \mu x \oplus (1 - \mu)y) \leq |\lambda - \mu| d(x, y).$$

Lemma 2.3. *[10] Let X be a CAT(0) space. Then the following inequalities hold true for all $x, y, z \in X$ and $\lambda \in [0, 1]$.*

i) $d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$.

ii) $d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)$.

Lemma 2.4. *[8] If K is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.5. *[12] Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x if and only if $\left\langle \overrightarrow{xx_n}, \overrightarrow{xy} \right\rangle \leq 0$, for all $y \in K$.*

Lemma 2.6. *[19] Let X be a metric space. If $A, B \in Prox(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq D(A, B)$.*

Lemma 2.7. *[18] Let X be a metric space. If $A, B \in CB(X)$ and $a \in A$, then for every $\gamma > 0$, there exists $b \in B$ such that $d(a, b) \leq D(A, B) + \gamma$.*

3. Main Results

Lemma 3.1. *Let K be a nonempty subset of a CAT(0) space X. If a multi-valued mapping $T : K \to CB(X)$ is $L-$Lipschitz quasi-nonexpansive with $Tp = \{p\}, \forall p \in F(T)$, then $F(T)$ is closed and convex.*

Proof. Let $\{x_n\} \subseteq F(T)$ such that $x_n \to x^*$ as $n \to \infty$. Then since T is $L-$Lipschitz, we have that

$$\begin{aligned} d(x^*, Tx^*) &= \lim_{n\to\infty} d(x_n, Tx^*) \\ &\leq \lim_{n\to\infty} D(Tx_n, Tx^*) \\ &\leq \lim_{n\to\infty} Ld(x_n, x^*) = 0. \end{aligned}$$

Thus, $x^* \in Tx^*$ and hence $F(T)$ is closed.
Now, let $p, q \in F(T)$, $x = tp \oplus (1-t)q$ and $u \in Tx$. Then since T is quasi-nonexpansive, by Lemma 2.3, we have

$$\begin{aligned} d^2(x, u) &\leq td^2(p, u) + (1-t)d^2(q, u) - t(1-t)d^2(p, q) \\ &= td^2(Tp, u) + (1-t)d^2(Tq, u) - t(1-t)d^2(p, q) \\ &\leq tD^2(Tp, Tx) + (1-t)D^2(Tq, Tx) - t(1-t)d^2(p, q) \\ &\leq td^2(x, p) + (1-t)d^2(x, q) - t(1-t)d^2(p, q) \\ &= t(1-t)^2d^2(p, q) + (1-t)t^2d^2(p, q) - t(1-t)d^2(p, q) = 0. \end{aligned}$$

Hence, $x = u \in Tx$. Then $x \in F(T)$ and hence $F(T)$ is convex.

Lemma 3.2. *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X. If a multi-valued mapping $T : K \to CB(X)$ is nonexpansive, then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\} \subseteq K$ such that $\{x_n\}$ is $\triangle-$ convergent to x^* and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Then by Lemma 2.4, $x^* \in K$. Now, for each $x \in X$, let $f(x) := \limsup_{n\to\infty} d^2(x_n, x)$. From quasilineariztion, we obtain

$$d^2(x_n, x^*) + d^2(x, x^*) = 2\left\langle \overrightarrow{x^*x_n}, \overrightarrow{x^*x} \right\rangle + d^2(x_n, x).$$

This together with Lemma 2.5 implies that

$$f(x^*) + d^2(x, x^*) \leq f(x), \forall x \in K. \tag{5}$$

Let u be arbitrary but fixed element of Tx^*. Then from (5), we have

$$f(x^*) + d^2(u, x^*) \leq f(u). \tag{6}$$

Now, since T is nonexpansive, we have

$$\begin{aligned} d^2(x_n, u) &\leq (d(x_n, Tx_n) + d(Tx_n, u))^2 \\ &\leq d^2(x_n, Tx_n) + d^2(Tx_n, u) + 2d(x_n, Tx_n)d(Tx_n, u) \\ &\leq d^2(x_n, Tx_n) + D^2(Tx_n, Tx^*) + 2d(x_n, Tx_n)d(Tx_n, u) \\ &\leq d^2(x_n, Tx_n) + d^2(x_n, x^*) + 2d(x_n, Tx_n)d(Tx_n, u). \end{aligned} \tag{7}$$

Then since $d(x_n, Tx_n) \to 0$ as $n \to \infty$, (7) implies

$$f(u) \leq f(x^*). \tag{8}$$

Thus, from (6) and (8), it follows that $x^* = u \in Tx^*$.

Lemma 3.3. *Let K be a nonempty, closed and convex subset of a CAT(0) space X and $T : K \to CB(X)$ be a mapping. Define $h_v : K \to \mathbb{R}$ by $h_v(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1-\lambda)v \in K\}$, where v is an arbitrary fixed element of $T(x)$. Then for any $x \in K$ the following hold:*

(1) $h_v(x) \in [0,1]$ and $h_v(x) = 0$ if and only if $v \in K$;

(2) if $\beta \in [h_v(x), 1]$, then $\beta x \oplus (1-\beta)v \in K$;

(3) if T is inward mapping, then $h_v(x) < 1$;

(4) if $v \notin K$, then $h_v(x)x \oplus (1-h_v(x))v \in \partial K$.

Proof. The proof of (1) and (2) follows directly from the definition of $h_v(x)$.
(3) Suppose that T is inward mapping. Then for any arbitrary fixed element v of Tx, we have $v = x$ or $y = (1-\frac{1}{c})x \oplus \frac{1}{c}v \ \ for \ \ some \ \ y \in K, c \geq 1$. If $v = x$, then $h_v(x) = 0$ and hence the result holds. Assume that $y = (1-\frac{1}{c})x \oplus \frac{1}{c}v$ for some $y \in K$. This implies that

$$h_v(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1-\lambda)v \in K\} \leq 1 - \frac{1}{c} < 1.$$

4) Let $v \notin K$. Then $h_v(x) > 0$. Let $\{w_n\} \subseteq (0, h_v(x))$ be a real sequence such that $w_n \to h_v(x)$. By the definition of h_v, we have $z_n := w_n x \oplus (1-w_n)v \notin K$. Now, since $w_n \to h_v(x)$, Lemma 2.2 implies that

$$\begin{aligned} d(z_n, h_v(x)x \oplus (1-h_v(x))v) &= d(w_n x \oplus (1-w_n)v, h_v(x)x \oplus (1-h_v(x))v) \\ &\leq |w_n - h_v(x)| d(x,v) \to 0 \ \ as \ \ n \to \infty. \end{aligned}$$

Thus, $z_n \to h_v(x)x \oplus (1-h_v(x))v \in K$. But since $z_n = w_n x \oplus (1-w_n)v \notin K$, for all $n \geq 1$, we have

$$h_v(x)x \oplus (1-h_v(x))v \in \partial K.$$

Theorem 3.4. *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X, $T : K \to Prox(X)$ be a nonexpansive inward mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, n \geq 0, \end{cases} \tag{9}$$

where $y_n \in Tx_n$ such that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1})$ and $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in K\}$. Then $\{x_n\}$ $\triangle$-converges to $x^* \in F(T)$. Moreover, if K is strictly convex and $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.

Proof. The existence of $y_n \in Tx_n$ which satisfies the conditions of (9) is guaranteed by Lemma 2.6. Note that $\alpha_n \in [h_{y_n}(x_n), 1)$, for all n. Then $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n \in K$ and hence the algorithm is well defined. Let $p \in F(T)$. Then since T is nonexpansive, from (9) and Lemma 2.3, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n x_n \oplus (1-\alpha_n)y_n, p) \\ &\leq \alpha_n d^2(x_n, p) + (1-\alpha_n)d^2(y_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq \alpha_n d^2(x_n, p) + (1-\alpha_n)D^2(Tx_n, Tp) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq \alpha_n d^2(x_n, p) + (1-\alpha_n)d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \qquad (10) \\ &\leq d^2(x_n, p). \end{aligned}$$

Then $\{d(x_n, p)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, p)$ exists. Moreover, $\{x_n\}$ is bounded and so does $\{y_n\}$. If $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, then since $\alpha_n \geq \frac{1}{2}$, we have that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$. But, from (10), it follows that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)d^2(x_n, y_n) < \infty$. Then we obtain that

$$\liminf_{n\to\infty} d(x_n, y_n) = 0. \tag{11}$$

From Lemma 2.3 and the fact that $d(x_n, x_{n+1}) = (1-\alpha_n)d(x_n, y_n)$ and $d(y_n, y_{n+1}) \leq d(Tx_n, Tx_{n+1})$ for each $n \geq 0$, we have

$$\begin{aligned}
d^2(x_{n+1}, y_{n+1}) &= d^2(\alpha_n x_n \oplus (1-\alpha_n)y_n, y_{n+1}) \\
&\leq \alpha_n d^2(x_n, y_{n+1}) + (1-\alpha_n)d^2(y_n, y_{n+1}) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\
&\leq \alpha_n d^2(x_n, y_{n+1}) + (1-\alpha_n)D^2(Tx_n, Tx_{n+1}) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\
&\leq \alpha_n\Big[d^2(x_n, x_{n+1}) + d^2(x_{n+1}, y_{n+1})\Big] + 2\alpha_n d(x_n, x_{n+1})d(x_{n+1}, y_{n+1}) + (1-\alpha_n)d^2(x_n, x_{n+1}) \\
&\quad -\alpha_n(1-\alpha_n)d^2(x_n, y_n) \\
&\leq \alpha_n(1-\alpha_n)^2 d^2(x_n, y_n) + \alpha_n d^2(x_{n+1}, y_{n+1}) + 2\alpha_n(1-\alpha_n)d(x_n, y_n)d(x_{n+1}, y_{n+1}) \\
&\quad +(1-\alpha_n)^3 d^2(x_n, y_n) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\
&\leq (1-\alpha_n)(1-2\alpha_n)d^2(x_n, y_n) + \alpha_n d^2(x_{n+1}, y_{n+1}) + 2\alpha_n(1-\alpha_n)d(x_n, y_n)d(x_{n+1}, y_{n+1}).
\end{aligned}$$

Putting $\beta_n = d(x_n, y_n)$, we have

$$(1-\alpha_n)\beta_{n+1}^2 \leq (1-\alpha_n)(1-2\alpha_n)\beta_n^2 + 2\alpha_n(1-\alpha_n)\beta_n\beta_{n+1}. \tag{12}$$

Note that $1-\alpha_n > 0$ and we may assume that $\beta_n > 0$ and hence we have

$$\Big[\frac{\beta_{n+1}}{\beta_n}\Big]^2 - 2\alpha_n\frac{\beta_{n+1}}{\beta_n} + 2\alpha_n - 1 \leq 0.$$

Solving this quadratic inequality, we obtain that $\beta_{n+1} \leq \beta_n$. Thus, the sequence $\{d(x_n, y_n)\}$ is decreasing and hence it follows from (11) that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then since $d(x_n, Tx_n) \leq d(x_n, y_n)$, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{13}$$

Moreover, since $\{x_n\}$ is bounded, it follows from Lemma 2.1 that $w_\triangle(x_n) \neq \emptyset$, where $w_\triangle(x_n) := \{x \in X : x_{n_i}\ \triangle-$converges to x for some subsequence $\{x_{n_i}\}$ of $\{x_n\}\}$. Let $x \in w_\triangle(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $\triangle$-converges to x. Now, by Lemma 3.2, $I-T$ is demiclosed at zero, which implies that $x \in F(T)$ and hence $w_\triangle(x_n) \subseteq F(T)$. We show that $w_\triangle(x_n)$ is singleton. Let $x, y \in w_\triangle(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ which $\triangle$-converges to x and y, respectively. If $x \neq y$, then from the fact that $\lim_{n\to} d(x_n, x)$ exists for all $x \in F(T)$ and CAT(0) space satisfies Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x) &= \lim_{i\to\infty} d(x_{n_i}, x) < \lim_{i\to\infty} d(x_{n_i}, y) \\
&= \lim_{n\to\infty} d(x_n, y) = \lim_{j\to\infty} d(x_{n_j}, y) \\
&< \lim_{j\to\infty} d(x_{n_j}, x) = \lim_{n\to\infty} d(x_n, x),
\end{aligned}$$

which is a contradiction and hence $x = y$. Therefore, $\{x_n\}$ $\triangle$-converges to $x \in F(T)$.

If $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, then since $\{x_n\}$ and $\{y_n\}$ are bounded, from the fact that $d(x_n, x_{n+1}) = (1-\alpha_n)d(x_n, y_n)$, we have $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Thus, $\{x_n\}$ is strongly Cauchy sequence and hence $x_n \to x^* \in K$ as $n \to \infty$. In addition from the fact that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1})$, we get that $\{y_n\}$ is strongly Cauchy sequence. Then there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$. Then since $d(y_n, Tx^*) \leq D(Tx_n, Tx^*) \leq d(x_n, x^*) \to 0$ as $n \to \infty$, it follows that $d(y^*, Tx^*) = 0$ and hence $y^* \in Tx^*$. Now, since T is inward mapping, $h_{y^*}(x^*) < 1$. Thus, for any $\beta \in (h_{y^*}(x^*), 1)$, we obtain that $\beta x^* \oplus (1-\beta)y^* \in K$.

On the other hand, since $\lim_{n\to\infty} \alpha_n = 1$ and $\alpha_n = max\{\alpha_{n-1}, h_{y_n}(x_n)\}$, we can choose a subsequence $\{x_{n_i}\}$ such that $\{h_{y_{n_i}}(x_{n_i})\}$ is non-decreasing and $\lim_{i\to\infty} h_{y_{n_i}}(x_{n_i}) = 1$ for some $y_{n_i} \in Tx_{n_i}$. In particular, for any $\beta < 1$, $t_{n_i} := \beta x_{n_i} \oplus (1-\beta)y_{n_i} \notin K$, eventually holds. Now choose $\beta_1, \beta_2 \in (h_{y^*}(x^*), 1)$ such that $\beta_1 \neq \beta_2$ and let $v_1 = \beta_1 x^* \oplus (1-\beta_1)y^*, v_2 = \beta_2 x^* \oplus (1-\beta_2)y^*$. Note that for any β such that $\beta_1 \leq \beta \leq \beta_2$, we have

$$v := \beta x^* \oplus (1-\beta)y^* \in K.$$

Since $x_{n_i} \to x^*$ and $y_{n_i} \to y^*$ as $i \to \infty$, it follows that $t_{n_i} \to v$ as $i \to \infty$ and hence $v \in \partial K$. Furthermore, since β is arbitrary, it follows that $[v_1, v_2] \subseteq \partial K$. Then by strict convexity of K, we have that $v_1 = v_2$. Then $d(v_1, y^*) = d(v_2, y^*)$ which implies that $\beta_1 d(x^*, y^*) = \beta_2 d(x^*, y^*)$. Then since $\beta_1 \neq \beta_2$, we have that $x^* = y^* \in Tx^*$. Therefore, $\{x_n\}$ strongly converges to $x^* \in F(T)$. □

If, in Theorem 3.4, T is assumed to be single-valued mapping we get the following corollary.

Corollary 3.5. *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X, $T : K \to X$ be a nonexpansive inward mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \alpha_0 = max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n, \\ \alpha_{n+1} = max\{\alpha_n, h(x_{n+1})\}, n \geq 0, \end{cases} \tag{14}$$

where $h(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)Tx_n \in K\}$. Then $\{x_n\}$ $\triangle$-converges to $x^* \in F(T)$. Moreover, if K is strictly convex and $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.

To remove the condition $Tp = \{p\}, \forall p \in F(T)$, we define the following Mann iterative scheme. Let $T : K \to \text{Prox}(X)$ be a multi-valued mapping and $P_T x := \{y \in Tx : d(x, y) = d(x, Tx)\}$. The sequence of Mann iterates is defined by

$$\begin{cases} x_0 \in K, \\ \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, n \geq 0, \end{cases} \tag{15}$$

where $y_n \in P_T x_n$ such that $d(y_n, y_{n+1}) \leq D(P_T x_n, P_T x_{n+1})$ and $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in K\}$.

Theorem 3.6. *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X, $T : K \to Prox(X)$ be an inward mapping with $F(T) \neq \emptyset$. Suppose that P_T is nonexpansive. Let $\{x_n\}$ be a sequence of Mann iterates defined by (15). Then $\{x_n\}$ $\triangle$-converges to $x^* \in F(T)$. Moreover, if K is strictly convex and $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.*

Proof. From the definition P_T, one can easily observe that $F(T) = F(P_T)$ and $P_T p = \{p\}$ for all $p \in F(T)$. Then following the method of the proof of Theorem 3.4, we obtain the required result. □

Theorem 3.7. *Let K be a nonempty, closed and strictly convex subset of a complete CAT(0) space X, $T : K \to CB(X)$ be $L-$Lipschitz quasi-nonexpansive inward mapping. Suppose that T satisfies the Condition (I) and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{\gamma_n\} \subseteq (0,1]$ such that $\sum_{n=0}^{\infty} \gamma_n < \infty$ and let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, n \geq 0, \end{cases} \tag{16}$$

where $y_n \in Tx_n$ such that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1}) + \gamma_n$ and $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in K\}$. Then $\{x_n\}$ strongly converges to $x^* \in F(T)$.

Proof. Note that the existence of $y_n \in Tx_n$ which satisfies the conditions of (16) is guaranteed by Lemma 2.7. Let $p \in F(T)$. Then following the method of the proof of Theorem 3.4, we obtain that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leq d^2(x_n, p). \end{aligned} \tag{17}$$

Then $\{d(x_n, p)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, p)$ exists.

If $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$, then it follows that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Thus, from (17), we have $\liminf_{n\to\infty} d(x_n, y_n) = 0$, which implies that $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$. Then since T satisfies Condition (I), we have $\liminf_{n\to\infty} f(d(x_n, F(T))) = 0$ for some increasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$, which implies that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. But since $d(x_{n+1}, p) \leq d(x_n, p)$, taking infimum over all $p \in F(T)$, we have $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$. Then the sequence $\{d(x_n, F(T))\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now for arbitrary $p \in F(T)$ and any $n, m \geq 1$, we have $d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(x_n, p) \leq 2d(x_n, p)$, which implies that $d(x_{n+m}, x_n) \leq 2d(x_n, F(T))$. Then $\{x_n\}$ is a Cauchy sequence and hence $x_n \to x^* \in K$ as $n \to \infty$. But $d(x^*, F(T)) \leq d(x^*, x_n) + d(x_n, F(T)) \to 0$ as $n \to \infty$ and since $F(T)$ is closed by Lemma 3.1, it follows that $x^* \in F(T)$.

If $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, then the required result follows from the method of the proof of Theorem 3.4. □

Theorem 3.8. *Let K be a nonempty, closed and strictly convex subset of a complete CAT(0) space X, $T : K \to CB(X)$ be an inward mapping with $F(T) \neq \emptyset$. Suppose that P_T is $L-$Lipschitz quasi-nonexpansive mapping and satisfies the Condition (I). Let $\{\gamma_n\} \subseteq (0,1]$ such that $\sum_{n=0}^{\infty} \gamma_n < \infty$ and let $\{x_n\}$ be generated from arbitrary initial point $x_0 \in K$ by*

$$\begin{cases} \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, n \geq 0, \end{cases} \tag{18}$$

where $y_n \in P_T x_n$ such that $d(y_n, y_{n+1}) \leq D(P_T x_n, P_T x_{n+1}) + \gamma_n$ and $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in K\}$. Then $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Proof. Following the method of the proof of Theorem 3.7, we get that $\{x_n\}$ converges strongly to $x^* \in F(P_T)$. But from the definition of P_T it follows that $F(T) = F(P_T)$ and hence $x^* \in F(T)$. □

To remove the restriction that T satisfies the Condition (I) in Theorem 3.7, we define the following Mann iterative scheme.

$$\begin{cases} x_0 \in K, \\ \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1}), 1-\frac{1}{(n+1)^2}\}, n \geq 0, \end{cases} \tag{19}$$

where $y_n \in Tx_n$ such that $h_{y_n}(x_n) := \inf\{\lambda \geq 0 : \lambda x_n \oplus (1-\lambda)y_n \in K\}$ and $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1}) + \gamma_n$, for any real sequence $\{\gamma_n\}$ in $(0, 1]$ with $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Theorem 3.9. *Let K be a nonempty, closed and strictly convex subset of a complete CAT(0) space X, $T : K \to CB(X)$ be $L-$Lipschitz quasi-nonexpansive inward mapping with $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a Mann iterative scheme defined by (19). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Proof. Let $p \in F(T)$. Then since T is quasi-nonexpansive, from (19) and Lemma 2.3, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n x_n \oplus (1-\alpha_n)y_n, p) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n)d(y_n, p) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n)D(Tx_n, Tp) \\ &\leq \alpha_n d(x_n, p) + (1-\alpha_n)d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Then $\{d(x_n, p)\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, p)$ exists. Thus, $\{x_n\}$ and $\{y_n\}$ are bounded. Note that from the hypothesis, $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$ and $d(x_n, x_{n+1}) = (1-\alpha_n)d(x_n, y_n)$, which imply that $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Thus, $\{x_n\}$ is strongly Cauchy sequence and hence $x_n \to x^* \in K$ as $n \to \infty$.

Moreover, from the fact that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1}) + \gamma_n \leq Ld(x_n, x_{n+1}) + \gamma_n$, we get that $\{y_n\}$ is strongly Cauchy sequence and hence there exists $y^* \in X$ such that $y_n \to y^*$ as $n \to \infty$. Then since $d(y_n, Tx^*) \leq D(Tx_n, Tx^*) \leq Ld(x_n, x^*) \to 0$ as $n \to \infty$, it follows that $y^* \in Tx^*$. The rest of the proof follows from the proof of Theorem 3.4. □

If, in Theorems 3.7 and 3.9, we assume that T is nonexpansive mapping with $F(T) \neq \emptyset$, then T is $1-$Lipschitz quasi-nonexpansive mapping and hence we have the following corollaries.

Corollary 3.10. *Let K be a nonempty, closed and strictly convex subset of a complete CAT(0) space X, $T : K \to CB(X)$ be nonexpansive inward mapping with $F(T) \neq \emptyset$. Suppose that T satisfies the Condition (I) and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a Mann iterative scheme given by (16). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Corollary 3.11. *Let K be a nonempty, closed and strictly convex subset of a complete CAT(0) space X, $T : K \to CB(X)$ be a nonexpansive inward mapping with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a Mann iterative scheme defined by (19). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Remark 3.12. *Note that all the results hold true if in the Mann iterative schemes, the initial coefficient $\alpha_1 = max\{\frac{1}{2}, h_{y_0}(x_0)\}$ is substituted by $\alpha_1 = max\{a, h_{y_0}(x_0)\}$, where $a \in (0, 1)$ is an arbitrary fixed value.*

Remark 3.13.

1) *Theorem 3.4 extends Theorem 1 of Colao and Marino [7] in two ways. It extends the mapping from single-valued to multi-valued and the space from Hilbert space to CAT(0) space.*

2) Theorem 3.7 extends Theorem 4.2 of Laowang and Panyanak [16] to Lipschitz quasi-nonexpansive mappings more general than nonexpansive mappings.

Now we give an example of our main result, Theorem 3.4.

Example 3.14. *Consider the set $X = \mathbb{R}^2$ with the distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X defined by*

$$d(x,y) = \begin{cases} |x_1| + |y_1| + |x_2 - y_2|, & x_2 \neq y_2, \\ |x_1 - y_1|, & x_2 = y_2. \end{cases} \tag{20}$$

Then, by the arguments of remark (1) of [11], we have that X with this metric is an $\mathbb{R}$-tree. But since every $\mathbb{R}$-tree is a CAT(0) space (see, [4]), (X, d) is CAT(0) space. Note that the geodesic segment joining x to y is a horizontal line segment from x to y, parametrized by arc length in the ordinary Euclidean metric when $x_2 = y_2$. If $x_2 \neq y_2$, then it consists of a horizontal line segment from x to $(0, x_2)$, followed by a vertical line segment from $(0, x_2)$ to $(0, y_2)$, followed by a horizontal line segment from $(0, y_2)$ to y, with each part parametrized by arc length in the ordinary Euclidean metric. To show completeness, let $\{(x_n, y_n)\}$ be a Cauchy sequence in X. We may have two cases. The first case is the situation where $\{y_n\}$ is eventually constant and the sequence $\{x_n\}$ is eventually contained in some open interval not containing 0. In this case the sequence clearly converges. The second case is when $\{y_n\}$ is not eventually constant and that $\lim_{n\to\infty} x_n = 0$. In this case, $\{y_n\}$ is a Cauchy sequence in $\mathbb{R}$ and hence it converges, which implies that $\{(x_n, y_n)\}$ converges. Therefore, X is complete CAT(0) space.

Let $K = \{(x,y) \in \mathbb{R}^2 : d((x,y),(0,0)) \leq 1\}$. Then clearly K is nonempty, closed and strictly convex subset of X. Let $T : K \to Prox(X)$ be defined by $Tx = \{(0, \frac{-1}{2}), (-x_1, -x_2 - 1)\}$. Clearly, T is inward multi-valued mapping and $F(T) = \{(0, \frac{-1}{2})\}$. Next we show that T is nonexpansive mapping. Let $x = (x_1, x_2), y = (y_1, y_2) \in K$. Then $Tx = \{(0, \frac{-1}{2}), (-x_1, -x_2 - 1)\}$ and $Ty = \{(0, \frac{-1}{2}), (-y_1, -y_2 - 1)\}$. Then we have

$$\begin{aligned} D(Tx, Ty) &= \max\{\sup_{a\in Ty} d(a, Tx), \sup_{b\in Tx} d(b, Ty)\} \\ &= \max\{d((-y_1, -y_2 - 1), \{(0, \frac{-1}{2}), (-x_1, -x_2 - 1)\}), d((-x_1, -x_2 - 1), \{(0, \frac{-1}{2}), (-y_1, -y_2 - 1)\})\} \\ &= \max\Big\{\min\{d((-y_1, -y_2 - 1), (0, \frac{-1}{2})), d((-y_1, -y_2 - 1), (-x_1, -x_2 - 1))\}, \\ &\qquad \min\{d((-x_1, -x_2 - 1), (0, \frac{-1}{2})), d((-x_1, -x_2 - 1), (-y_1, -y_2 - 1))\}\Big\} \\ &\leq d((-x_1, -x_2 - 1), (-y_1, -y_2 - 1)) \\ &= d(x, y) \end{aligned}$$

Therefore, T is nonexpansive mapping. Now let $\{x_n\} = \{(x_n^{(1)}, x_n^{(2)})\}$ be a Mann sequence given by:

$$\begin{cases} x_0 = (x_0^{(1)}, x_0^{(2)}) \in K, \\ \alpha_0 = max\{\frac{1}{2}, h_{y_0}(x_0)\}, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) y_n, \\ \alpha_{n+1} = max\{\alpha_n, h_{y_{n+1}}(x_{n+1})\}, n \geq 0, \end{cases}$$

where $y_n \in Tx_n$ such that $d(y_n, y_{n+1}) \leq D(Tx_n, Tx_{n+1})$. Then we have that $Tx_0 = \{(0, \frac{-1}{2}), (-x_0^{(1)}, -x_0^{(2)} - 1)\}$. If $y_0 \in Tx_0$, then one can show that $\frac{1}{2}x_0 \oplus \frac{1}{2}y_0 = (0, \frac{-1}{2}) \in K$ and hence $h_{y_0}(x_0) \leq \frac{1}{2}$. Then $\alpha_0 = \frac{1}{2}$, which implies that $x_1 = (0, \frac{-1}{2})$. Thus, $Tx_1 = \{x_1\}$ and $y_1 = x_1 \in K$. Then $h_{y_1}(x_1) = 0$ and hence $\alpha_1 = \frac{1}{2}$. Then $x_2 = (0, \frac{-1}{2})$. Applying similar computations, we obtain that $\alpha_n = \frac{1}{2}$ and $x_n = (0, \frac{-1}{2})$ for all $n \geq 2$. Therefore, $x_n \to (0, \frac{-1}{2}) \in F(T)$.

Acknowledgement: The authors are grateful to the referee for his/her careful observation and valuable comments and suggestions which lead to the present form of the paper.

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