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Coincidence of Multivalued Mappings on Metric Spaces with a Graph

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Abstract. In this article the coincidence points of a self mapping and a sequence of multivalued mappings are found using the graphic F-contraction. This generalizes Mizoguchi-Takahashi's fixed point theorem for multivalued mappings on a metric space endowed with a graph. As applications we obtain a theorem in homotopy theory, an existence theorem for the solution of a system of Urysohn integral equations, and for the solution of a special type of fractional integral equations.

1. Introduction and Preliminaries

The study of fixed points in metric spaces endowed with a graph was initiated by Jachymski [9]. The famous Banach contraction principle was extended to multivalued mappings by Nadler [14] in 1969. Reich [15] studied the fixed point results for the multivalued mappings on the compact subsets of a complete metric space. Hu [8] in 1980 extended the multivalued fixed point results to locally contractive multivalued mappings in ε-chainable metric space. In 1989, Mizouguchi and Takahashi [13] generalized Nadler's fixed point theorem using the MT-function. Recently, Sultana and Vetrivel [17] used the concept of Jachymski [9] for graphic contraction for multivalued mappings to extend the work of Mizouguchi and Takahashi [13]. Also Frigon and Dinevari [6] considered multivalued mappings on complete metric space endowed with a directed graph.

In 2012, Wardowski [18] introduced the F-contraction and proved the uniqueness of fixed point to extend the famous Banach contraction principle. Batra and Vashistha [4] have used the concept of graphic contraction in connection with F-contraction for the existence of fixed point results. Fixed point results of Hardy-Rogers type for self mappings on ordered complete metric spaces are pursued by Cosentino and Vetro [5] in view of F-contraction. The Hardy-Rogers type fixed point results have been extended for the multivalued mappings by Sgroi and Vetro [16]. For a metric space (X,d), by $\mathsf{CL}(X)$ we mean the set of closed subsets of X, and by $\mathsf{CB}(X)$ we mean the set of all nonempty closed bounded subsets of X. For every $A,B \in \mathsf{CB}(X)$, the generalized Hausdorff metric H induced by the metric d is defined as

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}.$$

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On $\mathsf{CL}(X)$ the generalized Hausdorff metric is defined which is also applicable on $\mathsf{CB}(X)$ [11].

Present article deals with the coincidence points of the sequence of multivalued maps using the concept of F-contraction endowed with a graph. We follow an idea from [2] to show the existence of coincidence points of a sequence of multivalued mappings taking into account the graphic F-contraction. It provides a new way of generalizations of many results existing in the literature [2, 14, 17].

Let us recall some definitions from graph theory which can be found in [10]. For a metric space (X, d) let Δ be the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that $X = V(G)$, where $V(G)$ is the set of vertices of G. The set $E(G)$ of edges of G contains Δ, i.e. $E(G)$ contains all the loops. If G has no parallel edges, then we can identify G with the pair $(V(G), E(G))$. Further, the graph G can be viewed as a weighted graph if to each its edge we assign the distance between its ends. Consider a directed graph G. Then G^{-1} denotes the graph obtained from G by reversing the direction of edges, and if we ignore the direction of edges in the graph G we get an undirected graph $\tilde{G}$. The pair (V', E') will be called a subgraph of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$ and for any edge $(a, b) \in E'$, $a, b \in V'$.

A *path* of length K in G from a vertex p to a vertex q is a sequence $\{v_i\}_{i=0}^{K}$ of $K+1$ vertices such that $v_0 = p$, $v_K = q$ and $(v_{j-1}, v_j) \in E(G)$ for $j = 1, 2, ..., K$. For $v \in V(G)$ and $K \in \mathbb{N} \cup \{0\}$ by $[v]_G^K$ we denote the set

$$[v]_G^K := \{u \in V(G) : \text{ there is a path of lenght } K \text{ from } v \text{ to } u\}.$$

A graph G is called *connected* if there is a path between any two vertices. Graph G is *weakly connected* if $\tilde{G}$ is connected.

The following is the definition of G-contraction by Jachymski [9].

Definition 1.1. ([9]) Let (X, d) be a metric space endowed with a graph G. We say that a mapping $T : X \to X$ is a *G-contraction* if T preserves edges of G, that is if

$$\underset{x,y\in X}{\forall} (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G),$$

and there exists some $\alpha \in [0, 1)$ such that

$$\underset{x,y\in X}{\forall} (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leq \alpha d(x, y).$$

Mizoguchi and Takahashi [13] had defined an MT-function as follows:

Definition 1.2. ([7]) A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an *MT-function* if it satisfies Mizoguchi-Takahashi's condition $\limsup_{r\to t^+} \varphi(r) < 1$ for all $t \in [0, \infty)$.

Clearly, if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then it is an MT-function. By $\mathcal{MT}$ we denote the set of all MT-functions.

Now we state some results from the basic theory of multivalued mappings.

Lemma 1.3. ([11]) *Let (X, d) be a metric space and $B \in \mathsf{CL}(X)$. Then for each $x \in X$ and $q > 1$ there exists an element $b \in B$ such that $d(x, b) \leq qd(x, B)$.*

As mentioned earlier, Wardowski [18] initiated the idea of F-contractions and provided a generalization of the Banach contraction principle. Following Wardowski [18] F denotes the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following three conditions:

(F1) F is strictly increasing;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.4. ([18]) Let (X, d) be a metric space and $F \in \mathcal{F}$. A self mapping T on X is called an *F-contraction* if there exists $\tau > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) > 0 \text{ implies } \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Further Altun, Olgun and Minak [1] used F-contractions in multivalued maps to generalize the constant τ by putting some restriction using the lim inf and prove the fixed point theorems on closed and bounded subsets of complete metric space.

2. Main Results

Definition 2.1. ([17]) A multivalued mapping $F : X \to \mathsf{CB}(X)$ is said to be a *Mizoguchi-Takahashi G-contraction* if for all distinct $x, y \in X$ with $(x, y) \in E(G)$ we have:

(i) $H(F(x), F(y)) \leq \varphi(d(x, y))d(x, y)$, where $\varphi \in \mathcal{MT}$;

(ii) If $u \in F(x)$ and $v \in F(y)$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

Motivated with Definition 2.1 of [17], we define in a more general setting the sequence of multivalued F-G_f-contraction for functions $F \in \mathcal{F}$.

Definition 2.2. Let (X, d) be a metric space endowed with a graph G and let $F \in \mathcal{F}$ be right continuous. A sequence of multivalued mappings $\{T_q\}_{q=1}^{\infty}$ from X into $\mathsf{CB}(X)$ such that for each $u \in X$ and $q \in \mathbb{N}$, $T_q(u) \in \mathsf{CB}(X)$, is said to be a *generalized F-G_f-contraction* if $f : X \to X$ is a surjection such that for $u, v \in X$, $u \neq v$ and $(fu, fv) \in E(G)$ imply

$$2\tau\left(d(fu, fv)\right) + F(H(T_q(u), T_r(v))) \leq F(d(fu, fv)), \quad \text{for all } q, r \in \mathbb{N} \text{ for some } \tau > 0\,, \tag{2.1}$$

where

$$\tau : (0, \infty) \to (0, \infty) \text{ and } \inf \lim_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0.$$

If $fx \in T_q(u)$ and $fy \in T_r(v)$ such that $d(fx, fy) \leq d(fu, fv)$, then $(fx, fy) \in E(G)$.

Theorem 2.3. *Let (X, d) be a complete metric space with a graph G and $\{T_q\}_{q=1}^{\infty}$ be a generalized F-G_f-contraction. Assume there exist $m \in \mathbb{N}$ and $v_0 \in X$ such that:*

(i) $T_1(v_0) \cap [fv_0]_G^m \neq \emptyset$;

(ii) *For any sequence $\{v_n\}$ in X, if $v_n \to v$ and $v_n \in T_n(v_{n-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$.*

Then f and the sequence $\{T_q\}_{q=1}^{\infty}$ have a coincidence point, i.e. there exists $v^* \in X$ such that $fv^* \in \bigcap_{q \in \mathbb{N}} T_q(v^*)$.

Proof. Choose $v_1 \in X$ such that $fv_1 \in T_1(v_0) \cap [fv_0]_G^m$ then there exists a path from fv_0 to fv_1, i.e.

$$fv_0 = fu_0^1, \ldots, fu_m^1 = fv_1 \in T_1(v_0), \text{ and } \left(fu_i^1, fu_{i+1}^1\right) \in E(G)$$

for all $i = 0, 1, 2, \ldots, m-1$. Without any loss of generality, we assume that $fu_k^1 \neq fu_j^1$ for each $k, j \in \{0, 1, 2, ..., m\}$ with $k \neq j$.

Since $(fu_0^1, fu_1^1) \in E(G)$, we get

$$2\tau\left(d(fu_0^1, fu_1^1)\right) + F(H(T_1(u_0^1), T_2(u_1^1))) \leq F(d(fu_0^1, fu_1^1)). \tag{2.2}$$

As F is continuous from right, for $\tau\left(d(fu_0^1, fu_1^1)\right) > 0$ there exists a real number $q > 1$, such that,

$$F(qH(T_1(u_0^1), T_2(u_1^1))) < F(H(T_1(u_0^1), T_2(u_1^1))) + \tau\left(d(fu_0^1, fu_1^1)\right). \tag{2.3}$$

Rename fv_1 as fu_0^2. Using Lemma 1.3, for each $fu_0^2 \in T_1(u_0^1)$ and $q > 1$ we can find some $fu_1^2 \in T_2(u_1^1)$ such that

$$d(fu_0^2, fu_1^2) < qH(T_1(u_0^1), T_2(u_1^1))$$

which implies

$$F(d(fu_0^2, fu_1^2)) < F(qH(T_1(u_0^1), T_2(u_1^1))). \tag{2.4}$$

From (2.2), (2.3) and (2.4) we have

$$\begin{aligned} \tau\left(d(fu_0^1, fu_1^1)\right) + F(d(fu_0^2, fu_1^2)) &< F(H(T_1(u_0^1), T_2(u_1^1))) + 2\tau\left(d(fu_0^1, fu_1^1)\right) \\ &< F(d(fu_0^1, fu_1^1)), \end{aligned}$$

which implies

$$F(d(fu_0^2, fu_1^2)) < F(d(fu_0^1, fu_1^1)) - \tau\left(d(fu_0^1, fu_1^1)\right).$$

So we have

$$d(fu_0^2, fu_1^2) < d(fu_0^1, fu_1^1), \text{ which implies } (fu_0^2, fu_1^2) \in E(G).$$

Since $(fu_1^1, fu_2^1) \in E(G)$, we have

$$2\tau\left(d(fu_1^1, fu_2^1)\right) + F(H(T_2(u_1^1), T_2(u_2^1))) \leq F(d(fu_1^1, fu_2^1)). \tag{2.5}$$

As F is continuous from right, for $\tau\left(d(fu_1^1, fu_2^1)\right) > 0$ there is a real number $q > 1$ such that

$$F(qH(T_2(u_1^1), T_2(u_2^1))) < F(H(T_2(u_1^1), T_2(u_2^1))) + \tau\left(d(fu_1^1, fu_2^1)\right). \tag{2.6}$$

Again by using Lemma 1.3, for each $fu_1^2 \in T_2(u_1^1)$ and $q_1 > 1$ we can find some $fu_2^2 \in T_2(u_2^1)$ such that

$$d(fu_1^2, fu_2^2) < q_1 H(T_2(u_1^1), T_2(u_2^1)).$$

This implies

$$F(d(fu_1^2, fu_2^2)) < F(q_1 H(T_1(u_0^1), T_2(u_1^1))). \tag{2.7}$$

From (2.5), (2.6) and (2.7) we obtain

$$\begin{aligned} \tau\left(d(fu_1^1, fu_2^1)\right) + F(d(fu_1^2, fu_2^2)) &< F(H(T_2(u_1^1), T_2(u_2^1))) + 2\tau\left(d(fu_1^1, fu_2^1)\right) \\ &< F(d(fu_1^1, fu_2^1)), \end{aligned}$$

and from here

$$F(d(fu_1^2, fu_2^2)) < F(d(fu_1^1, fu_2^1)) - \tau\left(d(fu_1^1, fu_2^1)\right).$$

Therefore, we have

$$d(fu_1^2, fu_2^2) < d(fu_1^1, fu_2^1) \text{ which implies } (fu_1^2, fu_2^2) \in E(G).$$

Thus we obtain $m+1$ vertices $\{fu_0^2, fu_1^2, fu_2^2, \ldots, fu_m^2\}$ in X such that $fu_0^2 \in T_1(u_0^1)$ and $fu_s^2 \in T_2(u_s^1)$ for $s = 1, 2, \ldots, m$, with

$$d(fu_s^2, fu_{s+1}^2) < d(fu_s^1, fu_{s+1}^1),$$

for $s = 0, 1, 2, \ldots, m-1$. As $(fu_s^1, fu_{s+1}^1) \in E(G)$ for all $s = 0, 1, 2, \ldots, m-1$, we get $(fu_s^2, fu_{s+1}^2) \in E(G)$ for all $s = 0, 1, 2, \ldots, m-1$.

Let $fu_m^2 = fv_2$. Then the set of points $fv_1 = fu_0^2, fu_1^2, fu_2^2, \ldots, fu_m^2 = fv_2 \in T_2(v_1)$ is a path from fv_1 to fv_2. Rename fv_2 as fu_0^3. Then by the same procedure we obtain a path

$$fv_2 = fu_0^3, fu_1^3, fu_2^3, \ldots, fu_m^3 = fv_3 \in T_3(v_2)$$

from fv_2 to fv_3. Inductively, for some $h \in \mathbb{N}$ we obtain

$$fv_h = fu_0^{h+1}, fu_1^{h+1}, fu_2^{h+1}, \ldots, fu_m^{h+1} = fv_{h+1} \in T_{h+1}(v_h)$$

with

$$2\tau(d(fu_t^h, fu_{t+1}^h)) + F(H(T_{h+1}(u_t^h), T_{h+1}(u_{t+1}^h))) \leq F(d(fu_t^h, fu_{t+1}^h)).$$

Similarly since $fu_t^{h+1} \in T_{h+1}(u_t^h)$, and again using Lemma 1.3, one can find some $fu_{t+1}^{h+1} \in T_{h+1}(u_{t+1}^h)$ such that

$$F(d(fu_t^{h+1}, fu_{t+1}^{h+1})) < F(d(fu_t^h, fu_{t+1}^h)) - \tau(d(fu_t^h, fu_{t+1}^h)), \tag{2.8}$$

which implies that,

$$d(fu_t^{h+1}, fu_{t+1}^{h+1}) < d(fu_t^h, fu_{t+1}^h), \tag{2.9}$$

and hence $(fu_t^{h+1}, fu_{t+1}^{h+1}) \in E(G)$ for $t = 0, 1, 2, \ldots, m-1$.

Consequently, we construct a sequence $\{fv_h\}_{h=1}^{\infty}$ of points of X with

$$\begin{aligned}
fv_1 &= fu_m^1 = fu_0^2 \in T_1(v_0),\\
fv_2 &= fu_m^2 = fu_0^3 \in T_2(v_1),\\
fv_3 &= fu_m^3 = fu_0^4 \in T_3(v_2),\\
&\vdots\\
fv_{h+1} &= fu_m^{h+1} = fu_0^{h+2} \in T_{h+1}(v_h),
\end{aligned}$$

for all $h \in \mathbb{N}$.

Now from (2.8) we have

$$\begin{aligned}
F\left(d(fu_t^{h+1}, fu_{t+1}^{h+1})\right) &< F\left(d(fu_t^h, fu_{t+1}^h)\right) - \tau\left(d(fu_t^h, fu_{t+1}^h)\right)\\
&< F(d(fu_t^{h-1}, fu_{t+1}^{h-1})) - \tau\left(d(fu_t^{h-1}, fu_{t+1}^{h-1})\right) - \tau\left(d(fu_t^h, fu_{t+1}^h)\right)\\
&\ \vdots\\
&< F(d(fu_t^1, fu_{t+1}^1)) - \underbrace{\tau\left(d(fu_t^h, fu_{t+1}^h)\right) - \tau\left(d(fu_t^{h-1}, fu_{t+1}^{h-1})\right) - \ldots - \tau\left(d(fu_t^1, fu_{t+1}^1)\right)}_{h \text{ terms}}\\
&< F(d(fu_t^1, fu_{t+1}^1)) - h\min\left\{\tau\left(d(fu_t^{h-1}, fu_{t+1}^{h-1})\right), \tau\left(d(fu_t^h, fu_{t+1}^h)\right), \ldots, \tau\left(d(fu_t^1, fu_{t+1}^1)\right)\right\}\\
&< F(d(fu_t^1, fu_{t+1}^1)) - h\tau_{\min}
\end{aligned}$$

As $h \to \infty$ we get $\lim_{h\to\infty} F(d(fu_t^{h+1}, fu_{t+1}^{h+1})) \to -\infty$ then from **(F2)** $\lim_{h\to\infty} d(fu_t^{h+1}, fu_{t+1}^{h+1}) = 0$.
Now from **(F3),** there exists some $k \in (0,1)$ such that

$$\lim_{h\to\infty} d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k F\left(d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)\right) = 0.$$

Now from consequences of (2.2) we have

$$F\left(d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)\right) \leq F(d(fu_t^1, fu_{t+1}^1)) - h\tau_{\min} \text{ for all } h \in \mathbb{N},$$

which implies that

$$\begin{aligned} & d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k F\left(d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)\right) - d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k F(d(fu_t^1, fu_{t+1}^1)) \\ \leq\ & d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k \left(F\left(d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)\right) - h\tau_{\min}\right) - d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k F(d(fu_t^1, fu_{t+1}^1)) \\ =\ & -h\tau_{\min} d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k \leq 0. \end{aligned}$$

Letting $h \to \infty$ we deduce that,

$$\lim_{h\to\infty} hd\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k = 0. \tag{2.10}$$

It follows from (2.10) that there exists some $h_1 \in \mathbb{N}$ such that

$$hd\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right)^k \leq 1 \text{ for all } h > h_1. \tag{2.11}$$

This implies that

$$d\left(fu_t^{h+1}, fu_{t+1}^{h+1}\right) \leq \frac{1}{h^{\frac{1}{k}}} \text{ for all } h > h_1.$$

Now for $p > h > h_1$ consider,

$$d(fv_h, fv_p) \leq \sum_{i=h}^{p-1} d\left(fv_i, fv_{i+1}\right) \leq \sum_{i=h}^{p-1} \frac{1}{i^{\frac{1}{k}}}. \tag{2.12}$$

Since $0 < k < 1$ therefore series in (2.12) converges, and so for all $t \in \{0, 1, 2, ..., m-1\}$, it follows that $\{fv_h = fu_m^h\}$ is a Cauchy sequence.

Since (X, d) is complete, there is $v^* \in X$ such that $fv_h \to fv^*$. Since $fv_h \in T_h(v_{h-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, there exists a subsequence $\{fv_{h_k}\}$ such that $(fv_{h_k}, fv^*) \in E(G)$ for all $k \in \mathbb{N}$. Thus

$$\begin{aligned} 2\tau(d(fv_{h_{k-1}}, fv^*)) + F(H(T_{h_k}(v_{h_{k-1}}), T_q(v^*)) &\leq F(d(fv_{h_{k-1}}, fv^*)) \\ F(H(T_{h_k}(v_{h_{k-1}}), T_q(v^*))) &< F(d(fv_{h_{k-1}}, fv^*)). \end{aligned}$$

Since F is an increasing function we have

$$H(T_{h_k}(v_{h_{k-1}}), T_q(v^*)) < d(fv_{h_{k-1}}, fv^*). \tag{2.13}$$

By (2.13), for all $q \in \mathbb{N}$ we have

$$\begin{aligned} d(fv^*, T_q(v^*)) &\leq d(fv^*, fv_{h_k}) + d(fv_{h_k}, T_q(v^*)) \\ &\leq d(fv^*, fv_{h_k}) + H(T_{h_k}(v_{h_{k-1}}), T_q(v^*)) \\ &< d(fv^*, fv_{h+1}) + d(fv_{h_{k-1}}, fv^*). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we get $d(fv^*, T_q(v^*)) \to 0$, which implies $fv^* \in T_q(v^*)$ for all $q \in \mathbb{N}$. Hence, $fv^* \in \bigcap_{q\in\mathbb{N}} T_q(v^*)$ as required. □

Corollary 2.4. *Let (X, d) be a complete metric space with a graph G, $T : X \to \mathsf{CB}(X)$ and $f : X \to X$ a surjection. If $u, v \in X$ are such that $u \neq v$ and $(fu, fv) \in E(G)$ imply*

$$\begin{aligned} \tau\left(d(fu, fv)\right) + F(H(T(u), T(v))) &\leq F(d(fu, fv)), \\ \text{where } \tau &: (0, \infty) \to (0, \infty) \text{ with } \inf \lim_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0. \end{aligned}$$

Also if F is right continuous and there exist $m \in \mathbb{N}$ and $v_0 \in X$ such that:

(*i*) $T(v_0) \cap [fv_0]_G^m \neq \emptyset$;

(*ii*) *For any sequence $\{v_n\}$ in X, if $v_n \to v$ and $v_n \in T(v_{n-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, there exists a subsequence $\{v_{n_k}\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$,*

then f and T have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* \in T(v^*)$.

Corollary 2.5. *Let (X, d) be a complete metric space with a graph G, $T : X \to \mathsf{CB}(X)$. If $u, v \in X$ are distinct elements such that $(u, v) \in E(G)$ implies*

$$\begin{aligned} \tau\left(d(u, v)\right) + F(H(T(u), T(v))) &\leq F(d(u, v)), \\ \text{where } \tau &: (0, \infty) \to (0, \infty) \text{ with } \inf \lim_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0. \end{aligned}$$

Also if F is right continuous and there exist $m \in \mathbb{N}$ and $v_0 \in X$ such that:

(*i*) $T(v_0) \cap [v_0]_G^m \neq \emptyset$;

(*ii*) *For any sequence $\{v_n\}$ in X, if $v_n \to v$ and $v_n \in T(v_{n-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, there exists a subsequence $\{v_{n_k}\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$,*

then T has a fixed point v^*.

If we consider $F(t) = \ln t$ in Corollary 2.5, then we arrive at Theorem 3 on multivalued maps of **[17].**
The following are the consequence for the case of self mappings with τ is taken as a positive real constant in Theorem 2.1.

Corollary 2.6. *Let (X, d) be a complete metric space with a graph G, $\{T_q\}_{q=1}^{\infty}$ be a sequence of the self mappings on X, and $f : X \to X$ a surjection. Suppose that for distinct elements u and v in X, $(fu, fv) \in E(G)$ implies*

$$\tau + F(d(T_q(u), T_r(v))) \leq F(d(fu, fv))$$

for all $q, r \in \mathbb{N}$, and there exist $m \in \mathbb{N}$ and $v_0 \in X$ such that:

(*i*) $T_1(v_0) \cap [fv_0]_G^m \neq \emptyset$;

(*ii*) *For any sequence $\{v_n\}$ in X such that $v_n \to v$ and $v_n = T_n(v_{n-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$.*

Then f and the sequence $\{T_q\}_{q=1}^{\infty}$ have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* = \bigcap_{q \in \mathbb{N}} T_q(v^*)$.

Corollary 2.7. *Let (X, d) be a complete metric space with a graph G, $T : X \to X$, and $f : X \to X$ a surjection. Let u and v be distinct elements in X such that $(fu, fv) \in E(G)$ implies*

$$\tau + F(d(T(u), T(v))) \leq F(d(fu, fv)), \tag{2.14}$$

and let there exist $m \in \mathbb{N}$ and $v_0 \in X$, such that:

(*i*) $T(v_0) \cap [fv_0]_G^m \neq \emptyset$;

(*ii*) *For any sequence* $\{v_n\}$ *in* X *converging to* $v \in X$ *and such that* $v_n = T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *there exists a subsequence* $\{v_{n_k}\}$ *of* $\{v_n\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then f *and* T *have a coincidence point, i.e., there exists* $v^* \in X$ *such that* $fv^* = Tv^*$.

Corollary 2.8. *Let* (X, d) *be a complete metric space with a graph* G, $T : X \to X$. *Assume that for distinct* $u, v \in X$, $(u, v) \in E(G)$ *implies*

$$\tau + F(d(T(u), T(v))) \leq F(d(u, v)), \tag{2.15}$$

and that there exist $m \in \mathbb{N}$ *and* $v_0 \in X$ *such that:*

(*i*) $T(v_0) \cap [v_0]_G^m \neq \emptyset$;

(*ii*) *For any sequence* $\{v_n\}$ *in* X *which converges to* $v \in X$ *and satisfies* $v_n = T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *there is a subsequence* $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then T *has a fixed point* v^*.

Remark 2.9. Using the notion from [9], G_0 is the graph associated with metric space (X, d) with $\mathrm{E}\,(G) = X \times X$. If we assume the graph $G = G_0$ and $F(t) = \ln t$ in Corollary 2.8, then the contractive condition (2.15) is applicable for all u and v in X. Thus Corollary 2.8 reduces to the Banach contraction principle.

3. Applications

A. Homotopy theory

By using some ideas from [12], we give an application in homotopy theory as a consequences of Corollary 2.8 with $G = G_0$.

Theorem 3.1. *Let* (X, d) *be a complete metric space,* W *an open subset of* X, *and* $U : [0, 1] \times \overline{W} \to \mathsf{CB}(X)$ *a multivalued mapping satisfying the following conditions:*

(*a*) $\alpha \notin U(\mu, \alpha)$ *for each* $\alpha \in \partial W$, *where* ∂W *is the boundary of* W, *and* $\mu \in [0, 1]$;

(*b*) $U(\mu, \cdot) : \overline{W} \to \mathsf{CB}(X)$ *is a multivalued map such that;*

$$\tau + F(H(U(\mu, \alpha), U(\mu', \beta)) \leq F(d(\alpha, \beta))$$

for each $\mu,\ \mu' \in [0, 1]$, $\alpha, \beta \in X$ *;*

(*c*) *there exists a continuous increasing function* $\psi : (0, 1] \to \mathbb{R}$ *such that*

$$F(H(U(\lambda, \alpha), U(\mu, \beta))) \leq F(\psi(\lambda) - \psi(\mu)),$$

for all $\lambda, \mu \in [0, 1]$ *and all* $\alpha, \beta \in \overline{W}$.

Then $U(0, \cdot)$ *has a fixed point if and only if* $U(1, \cdot)$ *has a fixed point.*

Proof. Suppose $U(0, \cdot)$ has a fixed point p, so $p \in U(0, p)$; it follows from (*a*), $p \in W$.
Define

$$A := \{(\mu, \alpha) \in [0, 1] \times W : \alpha \in U(\mu, \alpha)\}.$$

Clearly $A \neq \emptyset$. We define the partial ordering in A as follows:

$$(\mu, \alpha) \precsim (\lambda, \beta) \Longleftrightarrow \mu \leq \lambda \text{ and } d(\alpha, \beta) \leq \frac{2}{1 - e^{-\tau}}(\psi(\lambda) - \psi(\mu)) := r.$$

Claim 1. A has a maximal element.

Let L be a totally ordered subset of A and

$$\mu^* = \sup\{\mu : (\mu, \alpha) \in L\}.$$

Consider a sequence $\{(\mu_n, \alpha_n)\}_{n\geq 0}$ in L such that, $(\mu_n, \alpha_n) \precsim (\mu_{n+1}, \alpha_{n+1})$ and $\mu_n \to \mu^*$ as $n \to \infty$. Then for $m > n$, we have

$$d(\alpha_m, \alpha_n) \leq \frac{2}{1 - e^{-\tau}}(\psi(\mu_m) - \psi(\mu_n)) \to \theta, \text{ as } n, m \to \infty,$$

which means that $\{\alpha_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\zeta \in X$, such that $\alpha_n \to \zeta$.

Now consider

$$\tau + F(H(U(\mu_n, \alpha_n), U(\mu^*, \zeta))) \leq F(d(\alpha_n, \zeta))$$

which implies

$$H(U(\mu_n, \alpha_n), U(\mu^*, \zeta)) < d(\alpha_n, \zeta)$$

Since $\alpha_{n+1} \in U(\mu_n, \alpha_n)$ we have

$$d(\alpha_{n_{k+1}}, U(\mu^*, \zeta)) < d(\alpha_n, \zeta).$$

By [8, Lemma 3] there exists $\zeta_{n_k} \in U(\mu^*, \zeta)$ such that

$$d(\alpha_{n+1}, \zeta_{n_k}) < d(\alpha_n, \zeta).$$

Further,

$$\begin{aligned} d(\zeta, \zeta_n) &\leq d(\zeta, \alpha_{n+1}) + d(\alpha_{n+1}, \zeta_n) \\ &< d(\zeta, \alpha_{n_{k+1}}) + ad(\alpha_{n_k}, \zeta) \to 0 \text{ for all } n \to \infty. \end{aligned}$$

Thus $\zeta_n \to \zeta \in U(\mu^*, \zeta)$ and since $U(\mu^*, \zeta) \in \mathsf{CB}(X))$, $\zeta \in W$. From here we get $(\mu^*, \zeta) \in A$. Thus $(\mu, \alpha) \precsim (\mu^*, \zeta)$ for all $(\mu, \alpha) \in \Omega$, which means that (μ^*, ζ) is an upper bound of Ω. Hence by Zorn's Lemma, A has the maximal element (μ^*, ζ). This completes the proof of Claim 1.

Claim 2. $\mu^* = 1$.

Suppose that $\mu^* < 1$ and choose $\mu \geq \mu^*$ such that

$$\overline{B}_d(\zeta, r) \subset W, \text{ where } r = \psi(\mu^*) - \psi(\mu).$$

For any $\xi \in \overline{B}_d(\zeta, r)$, we have

$$d(\alpha, \zeta) < r.$$

Now for any $\xi \in \overline{B}_d(\zeta, r)$ consider

$$\tau + F(H(U(\mu^*, \zeta), U(\mu, \alpha))) \leq F(d(\alpha, \zeta))$$

which implies

$$H(U(\mathring{\mu}, \zeta), U(\mu, \alpha)) < d(\alpha, \zeta) < r.$$

Thus the contractive condition holds for multivalued map $U(\mu,\alpha) : \overline{B}_d(\zeta, r) \to \mathsf{CB}(X)$ in the complete metric space $(\overline{B}_d(\zeta, r), d)$. By Corollary 2.8, for each $\mu \in [0,1]$, there exist some $\alpha \in \overline{B}_d(\zeta, r)$, such that $\alpha \in U(\mu,\alpha)$. As

$$d(\zeta,\alpha) < r = \psi(\mu^*) - \psi(\mu),$$

we have

$$(\mu^*, \zeta) \lesssim (\mu, \alpha),$$

a contradiction. Thus $\mu^* = 1$ and hence $U(\cdot, 1)$ has a fixed point.

Conversely, if $U(1,\cdot)$ has a fixed point, then in a similar way we prove that $U(0,\cdot)$ has a fixed point. □

B. System of integral equations

Consider the system of Urysohn integral equations

$$fx(t) = \int_{\Omega} K_i(t, s, x(s))ds + h_i(t), \ t \in \Omega \text{ and } i \in \mathbb{N}, \tag{3.1}$$

where Ω is a closed and bounded subset of a finite dimensional Euclidean space and x, h_i are in $C[\Omega, \mathbb{R}^n]$.
(1) Suppose that $K_i : \Omega^2 \times \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, 2, \ldots, n$ are such that $F_{i,x} \in C[\Omega, \mathbb{R}^n]$ for each $x \in X$, where

$$F_{i,x}(t) = \int_{\Omega} K_i(t, s, fx(s))ds, \text{ for all } t \in \Omega \text{ and } i \in \mathbb{N}.$$

(2) There is $\tau > 0$ such that for every x, y in $C[\Omega, \mathbb{R}^n]$ it holds

$$\left|F_{m,x}(t) - F_{n,y}(t) + h_n(t) - h_q(t)\right| \leq e^{-\tau} \left|fx(t) - fy(t)\right|, \text{ for all } m, n \in \mathbb{N}.$$

Theorem 3.2. *Under the assumptions* (1) *and* (2) *the system of Urysohn integral equations* (3.1) *have a unique common solution in* $C[\Omega, \mathbb{R}^n]$.

Proof. Consider a space $X = C[\Omega, \mathbb{R}^n]$ with the metric $d_\tau : X \times X \to \mathbb{R}$ defined by:

$$d_\tau(x, y) = \max_{t\in\Omega} |x(t) - y(t)|.$$

For each $i \in \mathbb{N}$ define $S_i : X \to X$ by

$$S_i x = F_{i,x} + h_i.$$

Consider,

$$\begin{aligned}\left|S_m x(t) - S_n y(t)\right| &= \left|F_{m,x}(t) - F_{n,y}(t) + h_m(t) - h_n(t)\right| \ : m \neq n \\ &\leq \max_{t\in\Omega} \left|F_{m,x}(t) - F_{n,y}(t) + h_m(t) - h_n(t)\right| \\ &\leq e^{-\tau} \max_{t\in\Omega} \left|fx(t) - fy(t)\right|.\end{aligned}$$

Equivalently we have

$$d_\tau(S_m x, S_n y) \leq e^{-\tau} d_\tau(fx, fy) \text{ for all } m, n \in \mathbb{N}.$$

Further,

$$\ln(d_\tau(S_m x, S_n y)) \leq -\tau + \ln(d_\tau(fx, fy))$$

or

$$\tau + \ln(d_\tau(S_m x, S_n y)) \leq \ln(d_\tau(fx, fy)).$$

Now we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(t) = \ln t$ for each t in Ω, and $\tau > 0$ is in F. Thus all conditions of Corollary 2.6 of Theorem 2.1 are satisfied, so the system of Urysohn integral equations (3.1) and f have a coincidence point as a solution. □

C. Fractional differential equation

In the next application, we discuss a generalization of a fractional differential equation described in [3]. For the function $g \in C(I)$ and a continuous function $f : I \times \mathbb{R} \to \mathbb{R}$, where $I = [0, 1]$ and $C(I)$ is the Banach space of continuous real-valued functions on I with the uniform topology, consider the fractional differential equation

$$D^\alpha x(t) + f(t, g(x(t))) = 0 \quad (0 \leq t \leq 1,\ \alpha > 1,\ x \in C(I)) \tag{3.2}$$

with boundary conditions $x(0) = x(1) = 0$. Note that the associated Green function with the problem (3.2) is:

$$G(t, s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1. \end{cases}$$

Theorem 3.3. *Let $g : \mathbb{R} \to \mathbb{R}$ and $f : I \times I \to \mathbb{R}$ be continuous functions which satisfy*

(*i*) $\left|(f(s, g(x(s))) - f(s, g(y(s))))\right| \leq \left|g(x(s)) - g(y(s))\right|$ *for all $s \in I$;*

(*ii*) $\sup_{t \in I} \int\limits_0^1 G(t, s) ds \leq e^{-\tau}$ *for some $\tau > 0$.*

Then the problem (3.2) *has a unique solution.*

Proof. For the space $X = C(I)$ we have $d(x, y) = \max\limits_{t \in [0,1]} \left|x(t) - y(t)\right|$ for x and y in X. It is well known that $x \in (X, \mathbb{R})$ is a solution of (3.2) if and only if it is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s) f(s, (gx)(s)) ds \text{ for all } t \in I.$$

Define the operators $F : X \to X$ and $S : X \to X$ by

$$Fx(t) = \int_0^1 G(t, s) f(s, (gx)(s)) ds \ \text{ for all } t \in I,$$

and

$$Sx(t) = (gx)(t) \text{ for } t \in I.$$

Thus, for finding a solution of (3.2) it is sufficient to show that F and g have a coincidence point. Let $x, y \in C(I)$. For all $t, s \in I$ we have

$$\begin{aligned}
\left|Fx(t) - Fy(t)\right| &= \left|\int_0^1 G(t,s)(f(s,(gx)(s)) - f(s,(gy)(s)))ds\right| \\
&\leq \int_0^1 |G(t,s)| \left|(f(s,(gx)(s)) - f(s,(gy)(s)))\right| ds \\
&\leq \int_0^1 |G(t,s)| \left|(gx)(s) - (gy)(s)\right| ds \\
&\leq \left|(Sx)(s) - (Sy)(s)\right| \sup_{t\in I} \int_0^1 |G(t,s)| ds \\
&\leq e^{-\tau} \left|(Sx)(s) - (Sy)(s)\right|.
\end{aligned}$$

This implies that for each $x, y \in X$ we have

$$\ln d(Fx, Fy) \leq -\tau + \ln d(Sx, Sy).$$

Observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(t) = \ln t$, $t \in I$, and $\tau > 0$ is in $\digamma$. Thus by using Corollary 2.7 with graph $G = G_0$ we have $x^* \in X$ such that $Fx^* = Sx^*$ with $(Sx^*)(t) = (gx^*)(t)$ for $t \in I$. Thus x^* is the required coincidence point of F and g. $\square$

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