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Some Extensions of the Prabhu-Srivastava Theorem Involving the (p, q)-Gamma Function

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Abstract. In this paper, we obtain some limit formulas for derivatives of (p, q)-gamma function and (p, q)-digamma function at their poles. These limit formulas extend the Prabhu-Srivastava theorem involving gamma function and digamma function.

1. Introduction

It is well-known that for all complex numbers $x \neq 0, -1, -2, \cdots$, the gamma function and digamma function [1, pp. 255] are defined by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)} \quad \text{and} \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

A. Prabhu and H.M. Srivastava [8] have considered the limits of ratios between two gamma functions and digamma functions at their poles $x = 0, -1, -2, \cdots$, and obtained some nice formulas:

Theorem 1.1. *([8, Theorem 1 and 2]) For non-negative integer k and positive integers n and m, we have*

$$\lim_{x\to -k} \frac{\Gamma(nx)}{\Gamma(mx)} = (-1)^{(n-m)k} \frac{m}{n} \cdot \frac{(mk)!}{(nk)!}, \tag{1}$$

and

$$\lim_{x\to -k} \frac{\psi(nx)}{\psi(mx)} = \frac{m}{n}.$$

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Applying (1) and Gauss-Legendre multiplication formula, they also obtained an interesting product identity for the gamma function:

$$\prod_{j=1}^{n-1} \Gamma(-k + \frac{j}{n}) = (-1)^{(n-1)k} n^{nk-\frac{1}{2}} (2\pi)^{\frac{1}{2}(n-1)} \frac{k!}{(nk)!},$$

for non-negative integer k and positive integer $n \geq 2$.

In 2013, F. Qi [9] considered the limits of ratios between two derivatives of gamma function and digamma function at their poles.

Theorem 1.2. *([9, Theorem 1.2]) For non-negative integers s, k and positive integers n, m, we have*

$$\lim_{x \to -k} \frac{\Gamma^{(s)}(nx)}{\Gamma^{(s)}(mx)} = (-1)^{(n-m)k} \left(\frac{m}{n}\right)^{s+1} \cdot \frac{(mk)!}{(nk)!}, \tag{2}$$

and

$$\lim_{x \to -k} \frac{\psi^{(s)}(nx)}{\psi^{(s)}(mx)} = \left(\frac{m}{n}\right)^{s+1}. \tag{3}$$

Remark. Theorem 1.2 is contained in the Prabhu-Srivastava theorem (Theorem 1.1) by obvious use of the L'Hôpital's rule for limits.

For a non-negative integer p, the p-gamma function is defined by

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\cdots(x+p)}, \tag{4}$$

which was first introduced by Euler. Similarly, the p-digamma function is given by

$$\psi_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}.$$

Note that $\lim_{p\to\infty} \Gamma_p(x) = \Gamma(x)$ and $\lim_{p\to\infty} \psi_p(x) = \psi(x)$, and both $\Gamma_p(x)$ and $\psi_p(x)$ are analytic on the complex plane except for $x = 0, -1, -2, \cdots, -p$.

Recently, L. Yin and L.-G. Huang [5] provided alternative proofs of (1) and (3) by establishing the following results:

Theorem 1.3. *([5, Theorem 2.3 and 2.6]) Let k, p, s be non-negative integers and m, n be positive integers such that $mk, nk \leq p$. Then*

$$\lim_{x \to -k} \frac{\Gamma_p(nx)}{\Gamma_p(mx)} = \frac{m}{n} (-p)^{(m-n)k} \binom{p}{nk} \Big/ \binom{p}{mk}, \tag{5}$$

and

$$\lim_{x \to -k} \frac{\psi_p^{(s)}(nx)}{\psi_p^{(s)}(mx)} = \left(\frac{m}{n}\right)^{s+1}. \tag{6}$$

Letting $p \to \infty$ in (5) and (6) and noting that

$$\lim_{p\to\infty} p^{(m-n)k} \binom{p}{nk} \Big/ \binom{p}{mk} = \frac{(mk)!}{(nk)!},$$

we are led to (1) and (3). They also posed the following conjecture:

Conjecture 1.4. *([5, Conjecture 2.9]) Let s,k and p be non-negative integers and m,n be positive integers such that $mk, nk \leq p$. Then*

$$\lim_{x\to -k}\frac{\Gamma_p^{(s)}(nx)}{\Gamma_p^{(s)}(mx)}=\left(\frac{m}{n}\right)^{s+1}(-p)^{(m-n)k}\binom{p}{nk}\Big/\binom{p}{mk}. \tag{7}$$

It is not hard to see that (7) reduces to (2) when $p\to\infty$.
Remark. Theorem 1.3 and Conjecture 1.4 can be considered as the p-extensions of the Prabhu-Srivastava theorem (Theorem 1.1).

F. H. Jackson defined the following q-gamma functions [4, (I.35), pp.353]:

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}\quad \text{for } 0<q<1,$$

and

$$\Gamma_q(x)=\frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}\quad \text{for } q>1, \tag{8}$$

where $(a;q)_0=1$ and $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$. This function have many analogues of the classical facts about the gamma function [2, 7]. Similarly, the q-digamma function is given by

$$\psi_q(x)=\frac{\Gamma_q'(x)}{\Gamma_q(x)}.$$

It is well-known that $\lim_{q\to1}\Gamma_q(x)=\Gamma(x)$ and $\lim_{q\to1}\psi_q(x)=\psi(x)$, and both $\Gamma_q(x)$ and $\psi_q(x)$ have the poles at $x=0,-1,-2,\cdots$.

V.B. Krasniqi, H.M. Srivastava and S.S. Dragomir[3] considered the following (p,q)-gamma function and (p,q)-digamma function:

$$\Gamma_{p,q}(x)=\frac{q^{\binom{x-1}{2}}[p]_q^x[p]_q!}{[x]_q[x+1]_q\cdots[x+p]_q}\quad \text{for } q>1, \tag{9}$$

and $\psi_{p,q}(x)=\Gamma'_{p,q}(x)/\Gamma_{p,q}(x)$, where p is a non-negative integer and $[x]_q=(1-q^{-x})/(1-q^{-1})$. They have also obtained some complete monotonicity properties of the (p,q)-gamma function. Both $\Gamma_{p,q}(x)$ and $\psi_{p,q}(x)$ have the poles at $x=0,-1,\cdots,-p$. Note that (9) reduces to (4) when $q\to1$, and reduces to (8) when $p\to\infty$.

In this paper, we shall establish some extensions of the Prabhu-Srivastava theorem (Theorem 1.1) involving the (p,q)-gamma function. We will see that all of Theorem 1.1, 1.2, 1.3 and Conjecture 1.4 are special cases of these theorems.

2. Statements of the Results

We can rewrite (9) as

$$\Gamma_{p,q}(x)=\frac{(1-q^{-1})(q^{-1};q^{-1})_p}{(q^{-x};q^{-1})_{p+1}}\left(\frac{1-q^{-p}}{1-q^{-1}}\right)^x q^{\binom{x-1}{2}}\quad \text{for } q>1. \tag{10}$$

It is clear that the definition (10) is equivalent to

$$\Gamma_{p,q}(x)=\frac{(1-q)(q;q)_p[p]_q^x q^{-\binom{x-1}{2}}}{(q^x;q)_{p+1}}\quad \text{for } q<1, \tag{11}$$

where $[p]_q=(1-q^p)/(1-q)$. In what follows we will use the definition (11) for $\Gamma_{p,q}(x)$.

Theorem 2.1. *Let s, k and p be non-negative integers and n, m be positive integers such that $nk, mk \le p$. Then*

$$\lim_{x\to -k} \frac{\psi_{p,q}^{(s)}(nx)}{\psi_{p,q}^{(s)}(mx)} = \left(\frac{m}{n}\right)^{s+1}. \tag{12}$$

Letting $q \to 1$ in (12), we obtain (6).

Theorem 2.2. *Let s, k and p be non-negative integers and n, m be positive integers such that $nk, mk \le p$. Then*

$$\lim_{x\to -k} \frac{\Gamma_{p,q}^{(s)}(nx)}{\Gamma_{p,q}^{(s)}(mx)} = \left(\frac{m}{n}\right)^{s+1} \left(-q[p]_q\right)^{(m-n)k} \begin{bmatrix} p \\ nk \end{bmatrix}_q \Big/ \begin{bmatrix} p \\ mk \end{bmatrix}_q, \tag{13}$$

where $[p]_q = (1-q^p)/(1-q)$ and the q-binomial coefficient is given by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q;q)_a}{(q;q)_b (q;q)_{a-b}}.$$

Letting $q \to 1$ in (13), we obtain (7), and so we confirm Conjecture 1.4.

Theorem 2.3. *Let a, b be positive integers and s, k and p be non-negative integers such that $k \le p$. Then*

$$\lim_{x\to -k} \frac{\Gamma_{p,q^a}^{(s)}(x)}{\Gamma_{p,q^b}^{(s)}(x)} = \frac{b}{a} q^{(a-b)(k+1)^2} \left(\frac{1-q^a}{1-q^b}\right)^{k+1} \left(\frac{1-q^{bp}}{1-q^{ap}}\right)^{k} \begin{bmatrix} p \\ k \end{bmatrix}_{q^a} \Big/ \begin{bmatrix} p \\ k \end{bmatrix}_{q^b}. \tag{14}$$

3. Proof of the Results

In order to prove the results, we need some important lemmas.

Lemma 3.1. *(Faà di Bruno) If g and f are functions with a sufficient number of derivatives, then*

$$\frac{d^s}{dx^s} g(f(x)) = \sum_{\substack{1\cdot r_1 + 2\cdot r_2 + \cdots + s\cdot r_s = s \\ r_1, r_2, \cdots, r_s \ge 0}} \frac{s!}{r_1! \cdot r_2! \cdots r_s!} g^{(r_1+r_2+\cdots+r_s)}(f(x)) \left(\frac{f^{(1)}(x)}{1!}\right)^{r_1} \cdots \left(\frac{f^{(s)}(x)}{s!}\right)^{r_s}. \tag{15}$$

This is the famous Faà di Bruno formula [6].

Lemma 3.2. *Let F and $\log F$ be functions with a sufficient number of derivatives. For any positive integer s, there exist some coefficients $a(r_1, r_2, \cdots, r_s)$ independent of x such that*

$$F^{(s)}(x) = F(x) \sum_{\substack{1\cdot r_1 + 2\cdot r_2 + \cdots + s\cdot r_s = s \\ r_1, r_2, \cdots, r_s \ge 0}} a(r_1, r_2, \cdots, r_s) \left(f^{(1)}(x)\right)^{r_1} \cdots \left(f^{(s)}(x)\right)^{r_s}, \tag{16}$$

where $f(x) = \log F(x)$.

Proof. Letting $g(x) = e^x$ and $f(x) = \log F(x)$ in (15), we immediately get

$$F^{(s)}(x) = F(x) \sum_{\substack{1\cdot r_1 + 2\cdot r_2 + \cdots + s\cdot r_s = s \\ r_1, r_2, \cdots, r_s \ge 0}} \frac{s!}{r_1! \cdots r_s!} \left(\frac{f^{(1)}(x)}{1!}\right)^{r_1} \cdots \left(\frac{f^{(s)}(x)}{s!}\right)^{r_s}.$$

This completes the proof. □

Proof of Theorem 2.1. By (11), we have

$$\psi_{p,q}^{(s-1)}(x) = \frac{d^{s-1}}{dx^{s-1}} \log[p]_q + \frac{d^{s-1}}{dx^{s-1}} \left(\frac{3}{2} - x\right) \log q - \sum_{i=0}^{p} \frac{d^s}{dx^s} \log(1 - q^{x+i}). \tag{17}$$

Letting $g(x) = \log x$ and $f(x) = 1 - q^{i+x}$ in (15) gives

$$\frac{d^s}{dx^s} \log(1 - q^{x+i}) = -(\log q)^s \sum_{\substack{1\cdot r_1+2\cdot r_2+\cdots+s\cdot r_s=s \\ r_1,r_2,\cdots,r_s\geq 0}} \frac{s!}{r_1! \cdot r_2! \cdots r_s!} \cdot \frac{(R-1)!}{(1!)^{r_1}(2!)^{r_2}\cdots(s!)^{r_s}} \cdot \frac{q^{(x+i)R}}{(1-q^{x+i})^R}, \tag{18}$$

where $R = r_1 + r_2 + \cdots + r_s$. Since $1 \cdot r_1 + 2 \cdot r_2 + \cdots + s \cdot r_s = s$, R has the maximum value $R = s$ when $r_1 = s$ and $r_2 = \cdots = r_s = 0$, and so we can write (18) in the form

$$\frac{d^s}{dx^s} \log(1 - q^{x+i}) = -(\log q)^s \sum_{R=0}^{s} C_s(R) \frac{q^{(x+i)R}}{(1-q^{x+i})^R}, \tag{19}$$

where $C_s(R)$ is independent of x and $C_s(s) \neq 0$. Note that (19) has the pole at $x = -i$.

Combining (17) and (19), we have for $s \geq 1$ and $nk, mk \leq p$,

$$\lim_{x\to -k} \frac{\psi_{p,q}^{(s-1)}(nx)}{\psi_{p,q}^{(s-1)}(mx)} = \lim_{x\to -k} \frac{q^{n(x+k)s}}{(1-q^{n(x+k)})^s} \cdot \frac{(1-q^{m(x+k)})^s}{q^{m(x+k)s}}.$$

Noting that

$$\lim_{x\to -k} \frac{1-q^{m(x+k)}}{1-q^{n(x+k)}} = \frac{m}{n}, \tag{20}$$

we obtain

$$\lim_{x\to -k} \frac{\psi_{p,q}^{(s-1)}(nx)}{\psi_{p,q}^{(s-1)}(mx)} = \left(\frac{m}{n}\right)^s \quad \text{for } s \geq 1,$$

which is equivalent to (12). $\square$

Proof of Theorem 2.2. We first prove the case $s = 0$.

$$\begin{aligned} &\lim_{x\to -k} \frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(mx)} \\ &= q^{\binom{mk+2}{2}-\binom{nk+2}{2}} [p]_q^{(m-n)k} \frac{(q^{-mk};q)_{mk}(q;q)_{p-mk}}{(q^{-nk};q)_{nk}(q;q)_{p-nk}} \lim_{z\to -k} \frac{1-q^{m(x+k)}}{1-q^{n(x+k)}}. \end{aligned}$$

Applying (20) and noting that

$$(q^{-i};q)_i = (-1)^i q^{-\binom{i+1}{2}} (q;q)_i,$$

we obtain

$$\lim_{x\to -k} \frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(mx)} = \frac{m}{n} \left(-q[p]_q\right)^{(m-n)k} \begin{bmatrix} p \\ nk \end{bmatrix}_q \Big/ \begin{bmatrix} p \\ mk \end{bmatrix}_q. \tag{21}$$

Let $f_{p,q}(x) = \log \Gamma_{p,q}(x)$. By (16), we have

$$\Gamma_{p,q}^{(s)}(x) = \Gamma_{p,q}(x) \sum_{\substack{1\cdot r_1+2\cdot r_2+\cdots+s\cdot r_s=s \\ r_1,r_2,\cdots,r_s\geq 0}} a(r_1, r_2, \cdots, r_s) \left(f_{p,q}^{(1)}(x)\right)^{r_1} \cdots \left(f_{p,q}^{(s)}(x)\right)^{r_s}. \tag{22}$$

Noting that $f_{p,q}^{(d)}(x) = \psi_{p,q}^{(d-1)}(x)$ and then using (12), we get

$$\lim_{x\to -k} \frac{f_{p,q}^{(d)}(nx)}{f_{p,q}^{(d)}(mx)} = \left(\frac{m}{n}\right)^d \quad \text{for } d \geq 1. \tag{23}$$

It follows from (22) and (23) that

$$\lim_{x\to -k} \frac{\Gamma_{p,q}^{(s)}(nx)}{\Gamma_{p,q}^{(s)}(mx)} = \lim_{x\to -k} \frac{\Gamma_{p,q}(nx)}{\Gamma_{p,q}(mx)} \cdot \left(\frac{m}{n}\right)^s. \tag{24}$$

The proof of (13) then directly follows from (21) and (24). □

Proof of Theorem 2.3. We first prove the case $s = 0$.

$$\lim_{x\to -k} \frac{\Gamma_{p,q^a}(x)}{\Gamma_{p,q^b}(x)} = q^{(a-b)\binom{k+2}{2}} \left(\frac{1-q^a}{1-q^b}\right)^{k+1} \left(\frac{1-q^{bp}}{1-q^{ap}}\right)^k \frac{(q^a;q^a)_p(q^{-bk};q^b)_k(q^b;q^b)_{p-k}}{(q^b;q^b)_p(q^{-ak};q^a)_k(q^a;q^a)_{p-k}} \cdot \lim_{x\to -k} \frac{1-q^{b(x+k)}}{1-q^{a(x+k)}}.$$

Using (20) and noting that

$$\frac{(q^a;q^a)_p(q^{-bk};q^b)_k(q^b;q^b)_{p-k}}{(q^b;q^b)_p(q^{-ak};q^a)_k(q^a;q^a)_{p-k}} = q^{(a-b)\binom{k+1}{2}} \begin{bmatrix} p \\ k \end{bmatrix}_{q^a} \Big/ \begin{bmatrix} p \\ k \end{bmatrix}_{q^b},$$

we obtain

$$\lim_{x\to -k} \frac{\Gamma_{p,q^a}(x)}{\Gamma_{p,q^b}(x)} = \frac{b}{a} q^{(a-b)(k+1)^2} \left(\frac{1-q^a}{1-q^b}\right)^{k+1} \left(\frac{1-q^{bp}}{1-q^{ap}}\right)^k \begin{bmatrix} p \\ k \end{bmatrix}_{q^a} \Big/ \begin{bmatrix} p \\ k \end{bmatrix}_{q^b}. \tag{25}$$

In order to prove (14), by (16) and (25), it suffices to prove that

$$\lim_{x\to -k} \frac{\left(\log \Gamma_{p,q^a}(x)\right)^{(s)}}{\left(\log \Gamma_{p,q^b}(x)\right)^{(s)}} = 1 \quad \text{for } s \geq 1.$$

Replacing q by q^a in (19) yields

$$\frac{d^s}{dx^s} \log(1 - q^{a(x+i)}) = -(a \log q)^s \sum_{R=0}^{s} C_s(R) \frac{q^{a(x+i)R}}{(1-q^{a(x+i)})^R}.$$

Similarly to the proof of Theorem 2.1, we have

$$\lim_{x\to -k} \frac{\left(\log \Gamma_{p,q^a}(x)\right)^{(s)}}{\left(\log \Gamma_{p,q^b}(x)\right)^{(s)}} = \lim_{x\to -k} \left(\frac{a}{b}\right)^s \cdot \frac{q^{a(x+k)s}}{(1-q^{a(x+k)})^s} \cdot \frac{(1-q^{b(x+k)})^s}{q^{b(x+k)s}} = 1 \quad \text{(by (20))}.$$

This completes the proof. □

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