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On a Superclass of $*$-Operfectness

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Abstract. This paper presents $P^\star$-closed sets defined by using the sets in ideal. This concept is a new approach on the sets of ideal spaces. The class of $P^\star$-closed sets is a superclass of $*$-operfect sets and $\star$-open pre$^*_I$-closed sets.

1. Introduction

For many topological properties and major topological subjects, resolvability, compactness, hyperconnectedness, disconnectedness, etc., various set theories via ideal spaces have been studied up to now (see **[5–7, 9, 13, 18]**). So, the sets in ideal spaces have important roles for major topological problems. On the other hand, in 2010, Acikgoz et al. defined the concept of $*$-operfect sets **[1]**. In 2011, Ekici introduced the concept of pre$^*_I$-open sets to establish decompositions of continuity **[11]**. In this paper, a new approach on the sets of ideal spaces called $P^\star$-closed sets are presented. The class of $P^\star$-closed sets is a superclass of $*$-operfect sets and $\star$-open pre$^*_I$-closed sets. Characterizations of $P^\star$-closed sets are gotten.

We consider a space (T, σ) to be a topological space. For (T, σ), the closure and interior of $A \subset T$ will be denoted by $\mathfrak{Cl}(A)$ and $\mathfrak{Int}(A)$, respectively.

A subcollection $\mathfrak{I}$ of the power set $P(T)$ of a set T is called an ideal on T **[16]** if
(i) if $A_1 \subset A_2 \in \mathfrak{I}$ for $A_1, A_2 \subset T$, then $A_1 \in \mathfrak{I}$,
(ii) if $A_1, A_2 \in \mathfrak{I}$, then $A_1 \cup A_2 \in \mathfrak{I}$.

An ideal topological space is a space (T, σ) with an ideal $\mathfrak{I}$ on T and will be denoted by $(T, \sigma, \mathfrak{I})$ **[16]**. For $(T, \sigma, \mathfrak{I})$, the local function of A (with respect to $\mathfrak{I}$ and σ) $(.)^\star : P(T) \to P(T)$ is defined by $A^\star(\mathfrak{I}, \sigma)$ (or $A^\star$) $= \{t \in T : A \cap B \notin \mathfrak{I}$ for every $B \in \sigma$ such that $t \in B\}$ **[16]**. $\mathfrak{Cl}^\star(A) = A \cup A^\star$ is a Kuratowski closure operator for the $\star$-topology which will be denoted by $\sigma^\star$ **[15]**. Recall that a set A in (T, σ) is said to be semi-open **[17]** if $A \subset \mathfrak{Cl}(\mathfrak{Int}(A))$. The complement of a semi-open set will be called semi-closed **[4]**.

Definition 1.1. *A set A in ideal space $(T, \sigma, \mathfrak{I})$ is said to be*
i) *pre$^*_I$-open* **[11]** *if $A \subset \mathfrak{Int}^\star(\mathfrak{Cl}(A))$,*
ii) *pre$^*_I$-closed* **[8, 11]** *if $T \backslash A$ is pre$^*_I$-open,*
iii) *$*$-perfect* **[14]** *if $A = A^\star$,*
iv) *$*$-operfect* **[1]** *if A is open $*$-perfect.*

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2. $P^\star$-Closed Sets

In this Section, a new approach on the sets of ideal spaces called $P^\star$-closed sets are presented. The class of $P^\star$-closed sets is a superclass of $*$-operfect sets and $\star$-open pre^*_I-closed sets.

Definition 2.1. *For any set A in any ideal space $(T, \sigma, \Im)$, A is called*
(i) a $P^\star$-closed set if there exists a $C \in \Im$ such that $(\mathfrak{Int}(A))^\star \subset \mathfrak{Int}^\star(B) \cup C$ for each semi-open set B with $A \subset B$,
(ii) a $P^\star$-open set if $T \setminus A$ is $P^\star$-closed.

Theorem 2.2. *The following conditions are equivalent for any set A in any ideal space $(T, \sigma, \Im)$:*
(a) A is $P^\star$-closed,
(b) There exists a $C \in \Im$ such that $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \subset \mathfrak{Int}^\star(B) \cup C$ for each semi-open set B with $A \subset B$,
(c) $(\mathfrak{Int}(A))^\star \setminus \mathfrak{Int}^\star(B) \in \Im$ for each semi-open set B with $A \subset B$,
(d) $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(B) \in \Im$ for each semi-open set B with $A \subset B$.

Proof. $(c) \Rightarrow (d)$ and $(d) \Rightarrow (c)$: Suppose that $A \subset B \subset T$ and B is a semi-open set. Then

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}^\star(B))$$

is the union of $(\mathfrak{Int}(A))^\star \cap (T \setminus \mathfrak{Int}^\star(B))$ and $\mathfrak{Int}(A) \cap (T \setminus \mathfrak{Int}^\star(B)) = \varnothing$. Then

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}^\star(B)) \in \Im$$

and

$$(\mathfrak{Int}(A))^\star \cap (T \setminus \mathfrak{Int}^\star(B)) \in \Im.$$

The others follows by the ideal and $\mathfrak{Cl}^\star(A) = A^\star \cup A$ for a set A in T. □

Remark 2.3. *We have the following implications for subsets of an ideal space $(T, \sigma, \Im)$ where none of the implications is reversible:*

$$\begin{array}{ccc}
P^\star\text{-closed} & & \\
\uparrow & & \\
\star\text{-open and pre}^*_I\text{-closed} & & \\
\uparrow & & \\
\star\text{-open and } \star\text{-closed} & \longleftarrow & \text{open and } \star\text{-closed} \\
\uparrow & & \uparrow \\
\star\text{-open and closed} & \longleftarrow & *\text{-operfect}
\end{array}$$

Example 2.4. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{n\}, \{k, l\}, \emptyset, T\}$ and $\Im = \{\{m\}, \emptyset\}$. In this ideal space, the set $A = \{l, n\}$ is a $P^\star$-closed set, A is not a $\star$-open set and A is not pre^*_I-closed.*

Example 2.5. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{n\}, \{k, l\}, \emptyset, T\}$ and $\Im = \{\{k\}, \emptyset\}$. The set $A = \{l\}$ is a $\star$-open and pre^*_I-closed set, A is not a $\star$-closed set. The set $B = \{l, m, n\}$ is a $\star$-open and $\star$-closed set, B is not an open set.*

Example 2.6. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{n\}, \{k, l\}, \emptyset, T\}$ and $\Im = \{\{k, m\}, \{m\}, \emptyset, \{k\}\}$. In this ideal space, the set $A = \{k\}$ is an open and $\star$-closed set, A is not closed and $A^\star \neq A$. The set $B = \{l, m, n\}$ is a $\star$-open closed set, B is not open. The set T is a $\star$-open and closed set and $T^\star \neq T$.*

Theorem 2.7. *For a set A in an ideal space $(T, \sigma, \Im)$, the properties (i) and (ii) are equivalent:*
(i) A is $P^\star$-open,
(ii) There exists a set C in $\Im$ such that $\mathfrak{Cl}^\star(B) \setminus C \subset \mathfrak{Int}^\star(\mathfrak{Cl}(A))$ for each semi-closed set B with $B \subset A$.

Proof. $(i) \Rightarrow (ii)$: Assume that A is $P^\star$-open and B is a semi-closed set with $B \subset A$. Then $T \setminus A \subset T \setminus B$, $T \setminus B$ is semi-open and $T \setminus A$ is $P^\star$-closed. We have

$$\mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \setminus \mathfrak{Int}^\star(T \setminus B) \in \mathfrak{I}$$

and

$$\mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I}.$$

Put $C = \mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \setminus (T \setminus \mathfrak{Cl}^\star(B))$. Then $C \in \mathfrak{I}$ and

$$\mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \subset (T \setminus \mathfrak{Cl}^\star(B)) \cup C.$$

Therefore, the intersection of $T \setminus ((T \setminus \mathfrak{Cl}^\star(B)))$ and $T \setminus C$ is a subset of

$$T \setminus (\mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A))).$$

Thus,

$$\mathfrak{Cl}^\star(B) \cap (T \setminus C)$$

is a subset of $\mathfrak{Int}^\star(\mathfrak{Cl}(A))$. Hence, $\mathfrak{Cl}^\star(B) \setminus C \subset \mathfrak{Int}^\star(\mathfrak{Cl}(A))$.

$(ii) \Rightarrow (i)$: Suppose that $\mathfrak{Cl}^\star(B) \setminus C \subset \mathfrak{Int}^\star(\mathfrak{Cl}(A))$ for a set C in $\mathfrak{I}$ and for each semi-closed set B with $B \subset A$. Let $T \setminus A \subset D \subset T$ and D be a semi-open set. So $T \setminus D \subset A$ and $T \setminus D$ is a semi-closed set. There exists a set C in $\mathfrak{I}$ such that

$$\mathfrak{Cl}^\star(T \setminus D) \setminus C \subset \mathfrak{Int}^\star(\mathfrak{Cl}(A))$$

and therefore the intersection of $T \setminus \mathfrak{Int}^\star(D)$ and $T \setminus C$ is a subset of

$$\mathfrak{Int}^\star(\mathfrak{Cl}(A)).$$

Then

$$T \setminus \mathfrak{Int}^\star(\mathfrak{Cl}(A))$$

is a subset of $T \setminus ((T \setminus \mathfrak{Int}^\star(D)) \setminus C)$. Therefore, $T \setminus (T \setminus \mathfrak{Cl}^\star(T \setminus \mathfrak{Cl}(A))) \subset \mathfrak{Int}^\star(D) \cup C$. So $\mathfrak{Cl}^\star(T \setminus \mathfrak{Cl}(A)) \subset \mathfrak{Int}^\star(D) \cup C$ and

$$\mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \setminus \mathfrak{Int}^\star(D) \subset C \in \mathfrak{I}.$$

As a result, $T \setminus A$ is a $P^\star$-closed. Thus, A is a $P^\star$-open set. □

Definition 2.8. *([9]) Let $(T, \sigma, \mathfrak{I})$ be an ideal space and $A \subset T$. Then A is called a $\star$-nowhere dense set if $\mathfrak{Int}(\mathfrak{Cl}^\star(A)) = \emptyset$.*

Theorem 2.9. *For any $\star$-nowhere dense set A in an ideal space $(T, \sigma, \mathfrak{I})$, A is $P^\star$-closed.*

Proof. Let A be a $\star$-nowhere dense set, $A \subset B$ and B be semi-open. Since $\mathfrak{Int}(A) = \emptyset$, then

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(B) \in \mathfrak{I}.$$

So A is a $P^\star$-closed set. □

Remark 2.10. *For any ideal space $(T, \sigma, \mathfrak{I})$, there exists a $P^\star$-closed set which fails to be $\star$-nowhere dense.*

Example 2.11. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{n\}, \{k, l\}, \emptyset, T\}$ and $\mathfrak{I} = \{\{m\}, \emptyset\}$. Then $A = \{l, n\}$ is a $P^\star$-closed set but A is not $\star$-nowhere dense.*

Theorem 2.12. *For any ideal space $(T, \sigma, \mathfrak{I})$, the properties (i), (ii) and (iii) are equivalent:*
(i) Each set in T is a $P^\star$-closed set,
(ii) $(\mathfrak{Int}(A))^\star \setminus \mathfrak{Int}(A) \in \mathfrak{I}$ for each semi-open set A in T,
(iii) $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}(A) \in \mathfrak{I}$ for each semi-open set A in T.

Proof. $(i) \Rightarrow (iii)$: Suppose that every set in T is a $P^\star$-closed set. Let $A \subset T$ be a semi-open set. Since $\mathfrak{Int}(A)$ is a $P^\star$-closed set and $\mathfrak{Int}(A)$ is a semi-open set, then

$$\begin{aligned} & \mathfrak{Cl}^\star(\mathfrak{Int}(\mathfrak{Int}(A))) \cap (T \setminus \mathfrak{Int}^\star(\mathfrak{Int}(A))) \\ = \; & \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}(A)) \in \mathfrak{I}. \end{aligned}$$

$(iii) \Rightarrow (i)$: Let $B \subset A \subset T$ and A be a semi-open set. Then

$$\begin{aligned} & \mathfrak{Cl}^\star(\mathfrak{Int}(B)) \cap (T \setminus \mathfrak{Int}^\star(A)) \\ \subset \; & \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}^\star(A)) \\ \subset \; & \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}(A)). \end{aligned}$$

Therefore, $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus \mathfrak{Int}(A)) \in \mathfrak{I}$. Hence, $\mathfrak{Cl}^\star(\mathfrak{Int}(B)) \cap (T \setminus \mathfrak{Int}^\star(A)) \in \mathfrak{I}$. As a result, B is $P^\star$-closed.
$(i) \Leftrightarrow (ii)$: Similar to that of $(i) \Leftrightarrow (iii)$. □

Theorem 2.13. *For a $P^\star$-closed set A in any ideal space $(T, \sigma, \mathfrak{I})$, if B is a semi-closed set such that $B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$, then $\mathfrak{Cl}^\star(B) \in \mathfrak{I}$.*

Proof. Suppose that $A \subset T$ is $P^\star$-closed. Let B be a semi-closed set in T with $B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$. Then $B \subset T \setminus A$ and so $A \subset T \setminus B$ and $T \setminus B$ is semi-open. Since A is $P^\star$-closed, then the intersection of $\mathfrak{Cl}^\star(\mathfrak{Int}(A))$ and $(T \setminus \mathfrak{Int}^\star(T \setminus B))$ is an element of $\mathfrak{I}$. We have

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I}.$$

Since

$$\mathfrak{Cl}^\star(B) \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I},$$

then $\mathfrak{Cl}^\star(B) \in \mathfrak{I}$. □

Corollary 2.14. *For a $P^\star$-closed set A in any ideal space $(T, \sigma, \mathfrak{I})$, if B is a semi-closed set such that $B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$, then $B \in \mathfrak{I}$.*

Proof. It follows by Theorem 2.13. □

Theorem 2.15. *For a $P^\star$-closed set A in any ideal space $(T, \sigma, \mathfrak{I})$, if B is a semi-closed set such that $B \subset (\mathfrak{Int}(A))^\star \setminus A$, then $B \in \mathfrak{I}$.*

Proof. Suppose that A is a $P^\star$-closed set in T. Let B be a semi-closed set in T with $B \subset (\mathfrak{Int}(A))^\star \setminus A$. So $B \subset T \setminus A$. Then $A \subset T \setminus B$ and $T \setminus B$ is a semi-open set. Since A is a $P^\star$-closed set,

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(T \setminus B) \in \mathfrak{I}.$$

Therefore, $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I}$ and

$$(\mathfrak{Int}(A))^\star \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I}.$$

Since

$$B \subset (\mathfrak{Int}(A))^\star \setminus (T \setminus \mathfrak{Cl}^\star(B)) \in \mathfrak{I},$$

then $B \in \mathfrak{I}$. □

Remark 2.16. *Example 2.17 enable us to realize that Theorem 2.13 and 2.15 are not true without $P^\star$-closedness.*

Example 2.17. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{k, l\}, \{n\}, \emptyset, T\}$ and $\mathfrak{I} = \{\{l, n\}, \{n\}, \{l\}, \emptyset\}$. Put $A = \{k\}$ and $B = \{l, m\}$. Then A is not a $P^\star$-closed set, B is a semi-closed set, $B \subset (\mathfrak{Int}(A))^\star \setminus A$ and $B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$. But $B \notin \mathfrak{I}$ and $\mathfrak{Cl}^\star(B) \notin \mathfrak{I}$.*

Theorem 2.18. *For any $P^\star$-closed set A in any ideal space $(T, \sigma, \mathfrak{I})$, $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$ is $P^\star$-open.*

Proof. Assume that $A \subset T$ is $P^\star$-closed. Let $B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus A$ and B be a semi-closed set. By Theorem 2.13, $\mathfrak{Cl}^\star(B) \in \mathfrak{I}$. So, there exists a set $C = \mathfrak{Cl}^\star(B) \in \mathfrak{I}$ such that

$$\mathfrak{Cl}^\star(B) \setminus C \subset \mathfrak{Int}^\star(\mathfrak{Cl}(\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus A))).$$

By Theorem 2.7, $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \cap (T \setminus A)$ is $P^\star$-open. □

Theorem 2.19. *Let $(T, \sigma, \mathfrak{I})$ be an ideal space and $A \subset T$ be a $P^\star$-open set. If $\mathfrak{Int}^\star(\mathfrak{Cl}(A)) \cup (T \setminus A) \subset B$ and B is semi-open in T, then $T \setminus B \in \mathfrak{I}$.*

Proof. Assume that $A \subset T$ is $P^\star$-open. Let $\mathfrak{Int}^\star(\mathfrak{Cl}(A)) \cup (T \setminus A) \subset B$ and B be semi-open in T. Therefore,

$$\begin{aligned} & T \setminus [\mathfrak{Int}^\star(\mathfrak{Cl}(A)) \cup (T \setminus A)] \\ = \; & (T \setminus \mathfrak{Int}^\star(\mathfrak{Cl}(A))) \cap A \\ = \; & \mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \cap A. \end{aligned}$$

Then $T \setminus B$ is a semi-closed set and

$$T \setminus B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A)) \setminus (T \setminus A)$$

and $T \setminus A$ is a $P^\star$-closed set. By Theorem 2.13, $\mathfrak{Cl}^\star(T \setminus B) \in \mathfrak{I}$ and therefore $T \setminus B \in \mathfrak{I}$. □

Theorem 2.20. *$\{t\}$ is semi-closed or $\{t\}$ is a $P^\star$-open set for each $t \in T$ in any ideal space $(T, \sigma, \mathfrak{I})$.*

Proof. Assume that $\{t\}$ is not a semi-closed set for $t \in T$. Let $B \subset \{t\}$ and B be a semi-closed set in T. Then $B = \emptyset$ and so there exists a $B \in \mathfrak{I}$ such that

$$\mathfrak{Cl}^\star(B) \setminus B \subset \mathfrak{Int}^\star(\mathfrak{Cl}(\{t\})).$$

By Theorem 2.7, $\{t\}$ is a $P^\star$-open set. □

Definition 2.21. *([2]) Let $(T, \sigma, \mathfrak{I})$ be an ideal space. $(T, \sigma, \mathfrak{I})$ is called an F^*-space if each open set in $(T, \sigma, \mathfrak{I})$ is $\star$-closed.*

Theorem 2.22. *Each set is $P^\star$-closed in any F^*-ideal space $(T, \sigma, \mathfrak{I})$.*

Proof. Let $C \subset B \subset T$ and B be a semi-open set. Then

$$\begin{aligned} & \mathfrak{Cl}^\star(\mathfrak{Int}(C)) \cap (T \setminus \mathfrak{Int}^\star(B)) \\ = \; & \mathfrak{Int}(C) \cap (T \setminus \mathfrak{Int}^\star(B)) \\ = \; & \emptyset \in \mathfrak{I}. \end{aligned}$$

Therefore, C is a $P^\star$-closed set. As a result, each set C in T is a $P^\star$-closed set. □

3. Further Properties

In this Section, further properties of $P^\star$-closed sets are studied.

Theorem 3.1. ***([15])*** *Let $(T, \sigma, \mathfrak{I})$ be an ideal space and A and B be sets in T.*
(i) if $A \subset B$, $A^\star \subset B^\star$,
(ii) $(A^\star)^\star \subset A^\star$

Theorem 3.2. *Let $A \subset B \subset (\mathfrak{Int}(A))^\star$ for $P^\star$-closed A in any ideal space $(T, \sigma, \mathfrak{I})$. Then B is $P^\star$-closed.*

Proof. Assume that $A \subset B \subset (\mathfrak{Int}(A))^\star$ and C is a semi-open set in T such that $B \subset C$. So $A \subset C$. Since A is a $P^\star$-closed set, $(\mathfrak{Int}(A))^\star \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}$. Since $B \subset (\mathfrak{Int}(A))^\star$,

$$(\mathfrak{Int}(B))^\star \subset ((\mathfrak{Int}(A))^\star)^\star \subset (\mathfrak{Int}(A))^\star.$$

We have

$$\begin{aligned} & (\mathfrak{Int}(B))^\star \setminus \mathfrak{Int}^\star(C) \\ \subset \quad & (\mathfrak{Int}(A))^\star \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}. \end{aligned}$$

Then $(\mathfrak{Int}(B))^\star \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}$. As a result, B is a $P^\star$-closed set. $\square$

Theorem 3.3. *If $A \subset B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A))$ for $P^\star$-closed A in any ideal space $(T, \sigma, \mathfrak{I})$, B is $P^\star$-closed.*

Proof. Assume that $A \subset B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(A))$ and C is a semi-open set in T such that $B \subset C$. Since A is $P^\star$-closed, $\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}$. Then

$$\begin{aligned} & \mathfrak{Cl}^\star(\mathfrak{Int}(B)) \setminus \mathfrak{Int}^\star(C) \\ \subset \quad & \mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}. \end{aligned}$$

Therefore, $\mathfrak{Cl}^\star(\mathfrak{Int}(B)) \setminus \mathfrak{Int}^\star(C) \in \mathfrak{I}$ and B is a $P^\star$-closed set. $\square$

Theorem 3.4. *For any open $P^\star$-closed set A in any ideal space $(T, \sigma, \mathfrak{I})$, $\mathfrak{Cl}^\star(A)$ is $P^\star$-closed.*

Proof. By Theorem 3.3, $\mathfrak{Cl}^\star(A)$ is a $P^\star$-closed set. $\square$

Theorem 3.5. *Let $A \subset T$ be $P^\star$-open in any ideal space $(T, \sigma, \mathfrak{I})$. Assume that $\mathfrak{Int}^\star(\mathfrak{Cl}(A)) \subset B$ and $B \subset A$. Then B is $P^\star$-open.*

Proof. Assume that $\mathfrak{Int}^\star(\mathfrak{Cl}(A)) \subset B \subset A$ for any $P^\star$-open A. So $T \setminus A \subset T \setminus B \subset T \setminus \mathfrak{Int}^\star(\mathfrak{Cl}(A))$ and $T \setminus A$ is a $P^\star$-closed set. Since $T \setminus A \subset T \setminus B \subset \mathfrak{Cl}^\star(\mathfrak{Int}(T \setminus A))$, by Theorem 3.3, $T \setminus B$ is $P^\star$-closed. Therefore, B is $P^\star$-open. $\square$

Theorem 3.6. *For any ideal space $(T, \sigma, \mathfrak{I})$, assume that $A \subset T$ is a closed $P^\star$-open set. Then $\mathfrak{Int}^\star(A)$ is $P^\star$-open.*

Proof. By Theorem 3.5, $\mathfrak{Int}^\star(A)$ is $P^\star$-open. $\square$

Remark 3.7. *For any ideal topological space $(T, \sigma, \mathfrak{I})$, there exist $P^\star$-closed sets A and B but $A \cup B$ and $A \cap B$ fail to be $P^\star$-closed.*

Example 3.8. *Let $T = \{k, l, m, n\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{k\}, \{k, l\}, \{n\}, \emptyset, T\}$ and $\mathfrak{I} = \{\{k\}, \emptyset\}$. Then $A = \{k\}$ and $B = \{l\}$ are $P^\star$-closed sets, $A \cup B$ is not a $P^\star$-closed set.*

Example 3.9. *Let* $T = \{k, l, m, n, o\}$, $\sigma = \{\{k, l, n\}, \{k, n\}, \{n\}, \{k, l\}, \{k\}, \emptyset, T\}$ *and* $\mathfrak{I} = \{\{l\}, \emptyset\}$. *Then* $A = \{k, m, n, o\}$ *and* $B = \{l, m, n, o\}$ *are* $P^\star$*-closed sets,* $A \cap B$ *is not a* $P^\star$*-closed set.*

Definition 3.10. *([12]) A function* $f : (T_1, \sigma, \mathfrak{I}) \to (T_2, \rho, \mathfrak{I})$ *is called* $\star$*-closed if* $f(A)$ *is* $\star$*-closed for every* $\star$*-closed set* A *in* $(T_1, \sigma, \mathfrak{I})$.

Definition 3.11. *([3]) A function* $f : T_1 \to T_2$ *is called s-continuous if for each* $t \in T_1$ *and each semi-open set* A *containing* $f(t)$*, there exists an open set* B *in* T_1 *containing* t *such that* $f(B) \subset A$.

Theorem 3.12. *Let* $f : (T_1, \sigma, \mathfrak{I}) \to (T_2, \rho, f(\mathfrak{I}))$ *be a function where* $f(\mathfrak{I}) = \{f(I) : I \in \mathfrak{I}\}$. *If* f *is bijection,* $\star$*-closed and s-continuous, then* $f(A)$ *is* $P^\star$*-closed for* $P^\star$*-closed* A *in* T_1.

Proof. Let f be a bijection, $\star$-closed and s-continuous function and $A \subset T_1$ be $P^\star$-closed. Let $f(A) \subset B$ such that B is semi-open in T_2. Therefore, $A \subset f^{-1}(B)$ and

$$\mathfrak{Cl}^\star(\mathfrak{Int}(A)) \setminus \mathfrak{Int}^\star(f^{-1}(B)) \in \mathfrak{I}.$$

Therefore,

$$f(\mathfrak{Cl}^\star(\mathfrak{Int}(A))) \setminus f(\mathfrak{Int}^\star(f^{-1}(B))) \in f(\mathfrak{I}).$$

Since f is bijective, $\star$-closed function and s-continuous, then $f(\mathfrak{Cl}^\star(\mathfrak{Int}(A))) \setminus \mathfrak{Int}^\star(B) \in f(\mathfrak{I})$ and $\mathfrak{Cl}^\star(\mathfrak{Int}(f(A)))$ is a subset of $f(\mathfrak{Cl}^\star(\mathfrak{Int}(A)))$. Therefore, $\mathfrak{Cl}^\star(\mathfrak{Int}(f(A))) \setminus \mathfrak{Int}^\star(B) \in f(\mathfrak{I})$ and $f(A)$ is a $P^\star$-closed set. □

Definition 3.13. *([10]) A function* $f : (T_1, \sigma, \mathfrak{I}) \to (T_2, \rho, \mathfrak{I})$ *is called* $\star$*-open if* $f(A)$ *is* $\star$*-open for every* $\star$*-open set* A *in* $(T_1, \sigma, \mathfrak{I})$.

Corollary 3.14. *Let* $f : (T_1, \sigma, \mathfrak{I}) \to (T_2, \rho, f(\mathfrak{I}))$ *be a function where* $f(\mathfrak{I}) = \{f(I) : I \in \mathfrak{I}\}$. *If* f *is bijection,* $\star$*-open and s-continuous, then* $f(A)$ *is a* $P^\star$*-closed set for each* $P^\star$*-closed set* A *in* T_1.

Proof. It follows by Theorem 3.12. □

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