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Twin Signed k-Domination Numbers in Directed Graphs

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Abstract. Let $D = (V, A)$ be a finite simple directed graph (digraph). A function $f : V \longrightarrow \{-1, 1\}$ is called a twin signed k-dominating function (TSkDF) if $f(N^-[v]) \geq k$ and $f(N^+[v]) \geq k$ for each vertex $v \in V$. The twin signed k-domination number of D is $\gamma^*_{sk}(D) = \min\{\omega(f) \mid f \text{ is a TS}k\text{DF of } D\}$. In this paper, we initiate the study of twin signed k-domination in digraphs and present some bounds on $\gamma^*_{sk}(D)$ in terms of the order, size and maximum and minimum indegrees and outdegrees, generalising some of the existing bounds for the twin signed domination numbers in digraphs and the signed k-domination numbers in graphs. In addition, we determine the twin signed k-domination numbers of some classes of digraphs.

1. Introduction

Throughout this paper, D is a finite simple directed graph (digraph) with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). A digraph without directed cycles of length 2 is an *oriented graph*. If (u, v) is an arc of D, we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex v, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Let $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+_D(v)$ for the outdegree of a vertex v and $d^-_D(v)$ for its indegree. The *minimum* and *maximum indegrees* and *minimum* and *maximum outdegrees* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called *regular* or *r-regular* if $\delta^-(D) = \delta^+(D) = \Delta^-(D) = \Delta^+(D) = r$. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from X to v. We denote by $A(X, Y)$ the set of arcs from a subset X to a subset Y. The notation D^{-1} is used for the digraph obtained from D by reversing the arcs of D . The complete digraph of order n, K^*_n, is a digraph D such that $(u, v), (v, u) \in A(D)$ for any two distinct vertices $u, v \in V(D)$. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Consult [16] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer and let $D = (V, A)$ be a finite simple digraph with $\delta^-(D) \geq k - 1$. A *signed k-dominating function* (abbreviated SkDF) of D is defined in [6] as a function $f : V \to \{-1, 1\}$ such that $f(N^-[v]) \geq k$ for every $v \in V$. The signed k-domination number for a directed graph D is

$$\gamma_{sk}(D) = \min\{\omega(f) \mid f \text{ is a S}k\text{DF of } D\}.$$

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A $\gamma_{sk}(D)$-function is a SkDF of D of weight $\gamma_{sk}(D)$. When $k = 1$, the signed k-domination number $\gamma_{sk}(D)$ is the usual signed domination number $\gamma_s(D)$, which was introduced by Zelinka in [17] and has been studied by several authors (see for example [14]).

Let $k \geq 1$ be an integer and let D be a digraph with $\min\{\delta^-(D), \delta^+(D)\} \geq k - 1$. we define the *twin signed k-dominating function* (briefly TSkDF) as a signed k-dominating function of D which is also a signed k-dominating function of D^{-1}, i.e., $f(N^+[v]) \geq k$ and $f(N^-[v]) \geq k$ for every $v \in V$. The twin signed k-domination number for a digraph D is $\gamma^*_{sk}(D) = \min\{\omega(f) \mid f$ is a TSkDF of $D\}$. As the assumption $\min\{\delta^-(D), \delta^+(D)\} \geq k-1$ is necessary, we always assume that when we discuss $\gamma^*_{sk}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k - 1$ and $\delta^+(D) \geq k - 1$ and thus the order of D, $n(D) \geq k$. When $k = 1$, the twin signed k-domination number $\gamma^*_{sk}(D)$ is the usual twin signed domination number $\gamma^*_s(D)$, which was introduced by Atapour et al. [5].

For any function $f : V \to \{-1, 1\}$, we define $P = P_f = \{v \in V \mid f(v) = 1\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every TS$k$DF of D is a SkDF on both D and D^{-1}, and since the constant function 1 is a TSkDF of D, we have

$$\max\{\gamma_{sk}(D), \gamma_{sk}(D^{-1})\} \leq \gamma^*_{sk}(D) \leq |V(D)|. \tag{1}$$

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$. Let $N[v] = N(v) \cup \{v\}$. A function $f : V \to \{-1, 1\}$ is called a *signed dominating function* (SDF) of G if $f(N[v]) \geq 1$ for every $v \in V$. The *signed domination number* of G, denoted by $\gamma_s(G)$, is the minimum weight of a signed dominating function on G. The signed domination number of a graph was introduced by Dunbar et al. [11] and has been studied by several authors [12, 13].

The signed k-dominating function of a graph G is defined in [15] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq k$ for all $v \in V(G)$. The *signed k-domination number* of G, denoted by $\gamma_{sk}(G)$, is the minimum weight of a signed k-dominating function on G.

In this paper, we initiate the study of the twin signed k-domination numbers of digraphs and establish some sharp bounds on this parameter. Some of our results are extensions of well-known bounds of the twin signed domination numbers of digraphs proved in [5].

2. Basic properties of twin signed k-domination numbers

In this section, we present basic properties of the twin signed k-domination number of digraphs. By (1), $\gamma^*_{sk}(D) \leq n$. The next proposition provides conditions to establish the equality.

Proposition 2.1. Let D be a digraph of order n. Then $\gamma^*_{sk}(D) = n$ if and only if $d^-(u) \leq k$ for some $u \in N^+[v]$ or $d^+(w) \leq k$ for some $w \in N^-[v]$.

Proof. The sufficiency is clear. Thus, we verify the necessity of the condition. Assume that $\gamma^*_{sk}(D) = n$. Suppose to the contrary that there exists a vertex $v \in V(D)$ such that $d^-(u) \geq k + 1$ for each $u \in N^+[v]$ and $d^+(w) \geq k + 1$ for each $w \in N^-[v]$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a twin signed k-dominating function of D of weight less than n, a contradiction. This completes the proof. □

A tournament is a digraph D in which for every pair u and v of distinct vertices, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but not both. Next we determine the exact value of the twin signed k-domination number for particular type of tournament. Let $n = 2r + 1$ for some positive integer r. We define the circulant tournament CT(n) with n vertices as follows. The vertex set of CT(n) is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$ and for each i, the arcs go from u_i to the vertices $u_{i+1}, \ldots, u_{i+r}$, where the indices are taken modulo n. The proof of the next result can be found in [6].

Proposition 2.2. Let $r \geq k \geq 1$ be integers and $n \geq 2k + 1$. Then

$$\gamma_{sk}(\mathrm{CT}(n)) = \begin{cases} 2k + 1 & \text{if } r \equiv k + 1 \pmod 2 \\ 2k + 3 & \text{if } r \equiv k \pmod 2. \end{cases}$$

The next result shows that $\gamma^*_{sk}(\mathrm{CT}(n)) = \gamma_{sk}(\mathrm{CT}(n))$

Proposition 2.3. Let $r \geq k \geq 1$ be integers and $n = 2r + 1$. Then $\gamma^*_{sk}(\mathrm{CT}(n)) = \gamma_{sk}(\mathrm{CT}(n))$.

Proof. By (1) and Proposition 2.2, we have

$$\gamma^*_{sk}(\mathrm{CT}(n)) \geq \begin{cases} 2k+1 & \text{if } r \equiv k+1 \pmod 2 \\ 2k+3 & \text{if } r \equiv k \pmod 2). \end{cases}$$

Assume that $s = \lfloor \frac{r-k-1}{2} \rfloor$, $V^- = \{u_0, u_1, \ldots, u_s, u_{r+1}, \ldots, u_{r+s}\}$ and $V^+ = V(\mathrm{CT}(n)) - V^-$. For any vertex $v \in V(\mathrm{CT}(n))$, we have $|N^-[v]| = r+1$, $|N^+[v]| = r+1$, $|N^+[v] \cap V^-| \leq s+1$ and $|N^-[v] \cap V^-| \leq s+1$. Define $f : V(\mathrm{CT}(n)) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in V^+$ and $f(v) = -1$ when $v \in V^-$. Clearly, $f(N^-[v]) \geq r - 2s - 1 \geq k$ and $f(N^+[v]) \geq r - 2s - 1 \geq k$ for each $v \in V$. Therefore f is a TSkDF on $\mathrm{CT}(n)$ of weight $2k+1$ if $r \equiv k \pmod 2$ and $2k+3$ when $r \equiv k+1 \pmod 2$. Thus

$$\gamma^*_{sk}(\mathrm{CT}(n)) \leq \omega(f) = \begin{cases} 2k+1 & \text{if } r \equiv k+1 \pmod 2 \\ 2k+3 & \text{if } r \equiv k \pmod 2). \end{cases}$$

and the proof is complete. □

As we observed in (1), $\gamma^*_{sk}(D) \geq \max\{\gamma_{sk}(D), \gamma_{sk}(D^{-1})\}$. It was proved in [5] that the difference $\gamma^*_s(D) - \max\{\gamma_s(D), \gamma_s(D^{-1})\}$ can be arbitrarily large. Now we show that for $k \geq 2$, the difference $\gamma^*_{sk}(D) - \max\{\gamma_{sk}(D), \gamma_{sk}(D^{-1})\}$ can also be arbitrarily large.

Theorem 2.4. Let $k \geq 2$ and $t \geq 1$ be integers. Then there exists a digraph D such that

$$\gamma^*_{sk}(D) - \max\{\gamma_{sk}(D), \gamma_{sk}(D^{-1})\} \geq 2t.$$

Proof. For $1 \leq i \leq 2t+1$, let D_i be a circulant tournament of order $2k-1$ with vertex set $\{u^i_0 \ldots u^i_{2k-2}\}$. Let D be obtained from the disjoint union of D_i's, $1 \leq i \leq 2t+1$, by adding the set $\{w^i \mid 1 \leq i \leq 2t\}$ of new vertices and the set

$$\begin{aligned} &\{(u^{2t+1}_j, u^i_j), (u^\ell_j, u^{2t+1}_j) \mid 0 \leq j \leq 2k-2,\ 1 \leq i \leq t \text{ and } t+1 \leq \ell \leq 2t\} \\ &\cup\{(w^i, u^i_s), (u^i_{s+k-1}, w^i), \mid 1 \leq i \leq 2t \text{ and } 1 \leq s \leq k-1\} \\ &\cup\{(u^i_0, w^i), (u^{2t+1}_0, w^i), (w^{i+t}, u^{i+t}_0), (w^{i+t}, u^{2t+1}_0) \mid 1 \leq i \leq t\} \end{aligned}$$

of new arcs. Then the order of D is $n = 4kt + 2k - 1$. Obviously, $D \cong D^{-1}$ and so, $\gamma_{sk}(D) = \gamma_{sk}(D^{-1})$. By Proposition 2.1, $\gamma^*_{sk}(D) = n$. On the other hand, it is easy to verify that the function $f : V(D) \to \{-1, 1\}$ defined by $f(x) = -1$, for $x \in \{w^i \mid 1 \leq i \leq t\}$ and $f(x) = +1$ otherwise, is a SkDF of D and so $\gamma_{sk}(D) \leq n - 2t$. Thus $\gamma^*_{sk}(D) - \max\{\gamma_{sk}(D), \gamma_{sk}(D^{-1})\} \geq n - (n - 2t) = 2t$, and the proof is complete. □

Now we show that the twin signed k-domination number of digraphs can be arbitrary small.

Theorem 2.5. For any positive integers $k, t \geq 1$, there exists a digraph D such that

$$\gamma^*_{sk}(D) \leq 4kt + 2t - 4(k+1)t^2$$

Proof. Let $k, t \geq 1$ be integers and D be a digraph obtained from a complete digraph of order $2(k+1)t$ with vertex set $V(K^*_{2(k+1)t}) = \{u^i_1, \ldots, u^i_{2k+2} \mid 1 \leq i \leq t\}$ by adding the set $\{v^i_j, w^i_j \mid 1 \leq i \leq t \text{ and } 1 \leq j \leq 2kt + 2t - k\}$ of new vertices and the set $\{(u^i_j, v^i_\ell), (v^i_l, u^i_{j+k+1}), (w^i_\ell, u^i_j), (u^i_{j+k+1}, w^i_\ell) \mid 1 \leq i \leq t, 1 \leq j \leq k+1,\ 1 \leq \ell \leq 2kt + 2t - k\}$ of new arcs. It is easy to see that the function $f : V(D) \to \{-1, 1\}$ defined by $f(x) = -1$, for $x \in \{v^i_j, w^i_j \mid 1 \leq i \leq t,\ 1 \leq \ell \leq 2kt + 2t - k\}$ and $f(x) = +1$ otherwise, is a TSkDF of D and so $\gamma^*_{sk}(D) \leq w(f) = 2kt + 2t - 2t(2kt + 2t - k) = 4kt + 2t - 4(k+1)t^2$. □

3. Bounds on twin signed k-domination in digraphs

In this section we establish bounds for $\gamma^*_{sk}(D)$ in terms of the order, size, the maximum and minimum indegrees and outdegrees of D.

Proposition 3.1. If D is a digraph of order n with $\delta^+ \geq \delta^- \geq k+1$, then

$$\gamma^*_{sk}(D) \leq n - 2\lfloor \frac{\delta^- - k + 1}{2} \rfloor.$$

Proof. Define $t = \lfloor \frac{\delta^- - k+1}{2} \rfloor$. Let $v \in V(D)$ be a vertex, and let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of t out-neighbors of v. Define the function $f : V(D) \to \{-1, 1\}$ by $f(x) = -1$ for $x \in \{u_1 \ldots, u_t\}$ and $f(x) = 1$ otherwise. Then

$$f(N^-[x]) \geq (\delta^- + 1) - 2t = \delta^- - 2t + 1 = \delta^- - 2\lfloor \frac{\delta^- - k + 1}{2} \rfloor + 1 \geq k$$

and

$$f(N^+[x]) \geq (\delta^+ + 1) - 2t = \delta^+ - 2t + 1 \geq \delta^- - 2\lfloor \frac{\delta^- - k + 1}{2} \rfloor + 1 \geq k$$

for each vertex $x \in V(D)$. Therefore f is an TSkDF on D of weight $1 - t + (n - t - 1) = n - 2t$ and thus $\gamma^*_{sk}(D) \leq n - 2t = n - 2\lfloor \frac{\delta^- - k+1}{2} \rfloor$. □

Letting $t = \lfloor \frac{\delta^+ - k+1}{2} \rfloor$, in the proof of Proposition 3.1, we obtain the following proposition.

Proposition 3.2. If D is a digraph of order n with $\delta^- \geq \delta^+ \geq k+1$, then

$$\gamma^*_{sk}(D) \leq n - 2\lfloor \frac{\delta^+ - k + 1}{2} \rfloor.$$

Lemma 3.3. Let D be a digraph of order n and let f be a $\gamma^*_{sk}(D)$-function. Then

(a) $\lceil \frac{\delta^- + k+1}{2} \rceil |M| \leq |A(P, M)| \leq \lfloor \frac{\Delta^+ - k+1}{2} \rfloor |P|$.

(b) $\lceil \frac{\delta^+ + k+1}{2} \rceil |M| \leq |A(M, P)| \leq \lfloor \frac{\Delta^- - k+1}{2} \rfloor |P|$.

(c) $|A(P, P)| \geq \max\{\lceil \frac{\delta^- + k-1}{2} \rceil |P|, \lceil \frac{\delta^+ + k-1}{2} \rceil |P|\}$.

Proof. (a) Let $v \in M$. Since $f(N^-[v]) \geq k$, we deduce that $|A(P, v)| \geq \lceil \frac{d^-(v)+k+1}{2} \rceil \geq \lceil \frac{\delta^- + k+1}{2} \rceil$. It follows that $|A(P, M)| \geq \lceil \frac{\delta^- + k+1}{2} \rceil |M|$. Assume now that $v \in P$. Since $f(N^+[v]) \geq k$, $|A(v, M)| \leq \lfloor \frac{d^+(v) - k+1}{2} \rfloor \leq \lfloor \frac{\Delta^+ - k+1}{2} \rfloor$ and so $|A(P, M)| \leq \lfloor \frac{\Delta^+ - k+1}{2} \rfloor |P|$. Combining the inequalities, we obtain (a).

(b) The proof is similar to the proof of (a).

(c) Let $v \in P$. Since $f(N^+[v]) \geq k$ and $f(N^-[v]) \geq k$, then

$$|A(v, P)| \geq \lceil \frac{d^+(v) + k - 1}{2} \rceil \geq \lceil \frac{\delta^+ + k - 1}{2} \rceil,$$

and

$$|A(P, v)| \geq \lceil \frac{d^-(v) + k - 1}{2} \rceil \geq \lceil \frac{\delta^- + k - 1}{2} \rceil$$

Thus

$$|A(P, P)| \geq \max\{\lceil \frac{\delta^- + k - 1}{2} \rceil |P|, \lceil \frac{\delta^+ + k - 1}{2} \rceil |P|\},$$

and the proof is complete. □

Theorem 3.4. Let D be a digraph of order n, minimum indegree δ^-, minimum outdegree δ^+, maximum indegree Δ^- and maximum outdegree Δ^+. Then

$$\gamma^*_{sk}(D) \geq \max \left\{ \frac{\lceil \frac{\delta^- + k+1}{2} \rceil - \lfloor \frac{\Delta^+ - k+1}{2} \rfloor}{\lceil \frac{\delta^- + k+1}{2} \rceil + \lfloor \frac{\Delta^+ - k+1}{2} \rfloor} n, \frac{\lceil \frac{\delta^+ + k+1}{2} \rceil - \lfloor \frac{\Delta^- - k+1}{2} \rfloor}{\lceil \frac{\delta^+ + k+1}{2} \rceil + \lfloor \frac{\Delta^- - k+1}{2} \rfloor} n \right\}.$$

Proof. Let f be a minimum TSkDF of D. Using Lemma 3.3 and replacing $|M|$ and $|P|$ by $\frac{n-\gamma^*_{sk}(D)}{2}$ and $\frac{n+\gamma^*_{sk}(D)}{2}$ in (a) and (b), the desired inequality follows. □

The next corollary is a consequence of Theorem 3.4.

Corollary 3.5. If D is an r-regular digraph with $r \geq k-1$, then $\gamma^*_{sk}(D) \geq (k+1)n/(r+1)$ when $r+k$ is even and $\gamma^*_{sk}(D) \geq kn/(r+1)$ when $r+k$ is odd.

Example 3.6. If K^*_n is the complete digraph of order n, then $\gamma^*_{sk}(K^*_n) = k$ when $n+k$ is even and $\gamma^*_{sk}(K^*_n) = k+1$ when $n+k$ is odd.

Proof. According to Corollary 3.5, we have $\gamma^*_{sk}(K^*_n) \geq k+1$ when $n+k$ is odd and $\gamma^*_{sk}(K^*_n) \geq k$ when $n+k$ is even. On the other hand, if $n+k$ is odd, then the function $f : V(D) \to \{-1, 1\}$ which assigns to $\frac{n+k+1}{2}$ vertices the value $+1$ and to $\frac{n-k-1}{2}$ vertices the value -1 is a TSkDF of K^*_n of weight $k+1$ and so $\gamma^*_{sk}(K^*_n) = k+1$ when $n+k$ is odd. If $n+k$ is even, then the function $f : V(D) \to \{-1, 1\}$ which assigns to $\frac{n+k}{2}$ vertices the value $+1$ and to $\frac{n-k}{2}$ vertices the value -1 is a TSkDF of K^*_n of weight k and so $\gamma^*_{sk}(K^*_n) = k$ when $n+k$ is even. □

Example 3.6 shows that Propositions 3.1, 3.2 and Theorem 3.4 are sharp.

Theorem 3.7. If D is a digraph of order n and maximum indegree Δ^-, then

$$\gamma^*_{sk}(D) \geq 2\lceil \frac{\Delta^- + k + 1}{2} \rceil - n.$$

Proof. Let $u \in V(D)$ be a vertex of maximum indegree $d^-(u) = \Delta^-$, and let f be a $\gamma^*_{sk}(D)$-function. Assume first that $u \in M$. Since $f(N^-[u]) \geq k$, we deduce that $|A(P, u)| \geq \lceil \frac{\Delta^- + k + 1}{2} \rceil$. It follows that

$$\frac{n + \gamma^*_{sk}(D)}{2} = |P| \geq |A(P, u)| \geq \lceil \frac{\Delta^- + k + 1}{2} \rceil,$$

and this leads to the desired inequality. If $u \in P$, then $f(N^-[u]) \geq k$ implies that $|A(P, u)| \geq \lceil \frac{\Delta^- + k - 1}{2} \rceil$. We conclude that

$$\frac{n + \gamma^*_{s}(D)}{2} = |P| \geq |A(P, u)| + 1 \geq 1 + \lceil \frac{\Delta^- + k - 1}{2} \rceil = \lceil \frac{\Delta^- + k + 1}{2} \rceil,$$

and this leads to the desired inequality. □

The condition $f(N^+[v]) \geq k$ for each vertex v yields analogously the next result.

Theorem 3.8. If D is a digraph of order n and maximum outdegree Δ^+, then $\gamma^*_{sk}(D) \geq 2\lceil \frac{\Delta^+ + k + 1}{2} \rceil - n$.

Example 3.6 demonstrates that Theorems 3.7 and 3.8 are sharp.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same vertices as e. Since $N^-_{D(G)}[v] = N^+_{D(G)}[v] = N_G[v]$ for each $v \in V(G) = V(D(G))$, the following useful observation is valid.

Notation 3.9. If $D(G)$ is the associated digraph of a graph G, then $\gamma^*_{sk}(D(G)) = \gamma_{sk}(G)$.

There are many interesting applications of Observation 3.9, such as the following results.

Proposition 3.10. If G is a graph of order n and maximum degree Δ, then $\gamma_{sk}(G) \geq 2\lceil \frac{\Delta + k + 1}{2} \rceil - n$.

Proof. Since $\Delta(G) = \Delta^-(D(G))$ and $n = n(D(G))$, it follows from Theorem 3.7 and Observation 3.9 that

$$\gamma_{sk}(G) = \gamma^*_{sk}(D(G)) \geq 2\lceil \frac{\Delta^- + k + 1}{2} \rceil - n = 2\lceil \frac{\Delta + k + 1}{2} \rceil - n.$$

□

Corollary 3.11. Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then

$$\gamma_{sk}(G) \geq \frac{\lceil\frac{\delta+k+1}{2}\rceil - \lfloor\frac{\Delta-k+1}{2}\rfloor}{\lceil\frac{\delta+k+1}{2}\rceil + \lfloor\frac{\Delta-k+1}{2}\rfloor}n.$$

Since

$$\frac{\lceil\frac{\delta+k+1}{2}\rceil - \lfloor\frac{\Delta-k+1}{2}\rfloor}{\lceil\frac{\delta+k+1}{2}\rceil + \lfloor\frac{\Delta-k+1}{2}\rfloor}n \geq \frac{\delta+2k-\Delta}{\delta+2+\Delta}n,$$

Corollary 3.11 implies the following known bound.

Corollary 3.12. ([6]) If G is a graph of order n, minimum degree δ and maximum degree Δ, then

$$\gamma_{sk}(G) \geq (\frac{\delta+2k-\Delta}{\delta+2+\Delta})n.$$

Theorem 3.13. For any digraph D of order n, size m, minimum indegree δ^- and minimum outdegree δ^+,

$$\gamma^*_{sk}(D) \geq \frac{n(2+2\lceil\frac{\delta^++k-1}{2}\rceil + \lceil\frac{\delta^-+k-1}{2}\rceil) - 2m}{2+\lceil\frac{\delta^-+k-1}{2}\rceil}.$$

Proof. Let f be a $\gamma^*_{sk}(D)$-function. By Lemma 3.3, we have

$$\begin{aligned} m &\geq |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geq (1+\lceil\tfrac{\delta^++k-1}{2}\rceil)|M| + (1+\lceil\tfrac{\delta^-+k-1}{2}\rceil)|M| + \lceil\tfrac{\delta^++k-1}{2}\rceil|P| \\ &= \lceil\tfrac{\delta^++k-1}{2}\rceil n + (2+\lceil\tfrac{\delta^-+k-1}{2}\rceil)(\tfrac{n-\gamma^*_{sk}(D)}{2}). \end{aligned}$$

This leads to the desired inequality. □

Using $|A(P,P)| \geq \lceil\frac{\delta^-+k-1}{2}\rceil|P|$ in the proof of Theorem 3.13, we obtain the following theorem.

Theorem 3.14. For any digraph D of order n, size m, minimum indegree δ^- and minimum outdegree δ^+,

$$\gamma^*_{sk}(D) \geq \frac{n(2+2\lceil\frac{\delta^-+k-1}{2}\rceil + \lceil\frac{\delta^++k-1}{2}\rceil) - 2m}{2+\lceil\frac{\delta^++k-1}{2}\rceil}.$$

Theorem 3.15. Let D be a digraph of order n, maximum indegree Δ^- and maximum outdegree Δ^+. Then

$$\gamma^*_{sk}(D) \geq \frac{2k+2-\lfloor\frac{\Delta^--k+1}{2}\rfloor - \lfloor\frac{\Delta^+-k+1}{2}\rfloor}{2k+2+\lfloor\frac{\Delta^--k+1}{2}\rfloor + \lfloor\frac{\Delta^+-k+1}{2}\rfloor}n.$$

Proof. Let f be a $\gamma^*_{sk}(D)$-function and let $v \in M$. Since $f(N^+[v]) \geq k$ and $f(N^-[v]) \geq k$, it follows that $|A(v,P)| \geq k+1$ and $|A(P,v)| \geq k+1$ and thus $|A(M,P)| + |A(P,M)| \geq (2k+2)|M|$. Using Lemma 3.3 (Parts a, b), it follows that

$$|P|(\lfloor\frac{\Delta^--k+1}{2}\rfloor + \lfloor\frac{\Delta^+-k+1}{2}\rfloor) \geq (2k+2)|M|. \tag{2}$$

Replacing $|M|$ and $|P|$ by $\frac{n-\gamma^*_{sk}(D)}{2}$ and $\frac{n+\gamma^*_{sk}(D)}{2}$ in (2), we obtain the desired bound. □

Theorem 3.16. For any digraph D of order n and size m,

$$\gamma^*_{sk}(D) \geq \frac{(2k+1)n-m}{k+2}.$$

Proof. Let f be a $\gamma^*_{sk}(D)$-function. In view of the proof of Theorem 3.15, $|A(P,M)| \geq (k+1)|M|$ and $|A(M,P)| \geq (k+1)|M|$. If $x \in P$, then it follows from $f(N^+[x]) \geq k$ that $|A(x,P)| \geq |A(x,M)| + k - 1$. This implies that

$$|A(P,P)| \geq |A(P,M)| + (k-1)|P| \geq (k+1)|M| + (k-1)(n-|M|).$$

Hence,

$$\begin{aligned} m &\geq |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geq (k+1)|M| + (k+1)|M| + (k+1)|M| + (k-1)(n-|M|) \\ &= (2k+4)|M| + (k-1)n. \end{aligned}$$

Since $n = |P| + |M|$, we deduce that $\gamma^*_{sk}(D) = |P| - |M| = n - 2|M| \geq \frac{(2k+1)n-m}{k+2}$. □

Theorem 3.16 and Observation 3.9 lead to the next well-known result.

Corollary 3.17. ([15]) If G is a graph of order n and size m, then

$$\gamma_{sk}(G) \geq \frac{(2k+1)n - 2m}{k+2}.$$

Theorem 3.18. Let D be a digraph of order n. Then

$$\gamma^*_{sk}(D) \geq 2\left\lceil \frac{-1 + \sqrt{4n(k+1)+1}}{2} \right\rceil - n.$$

Proof. Let f be a $\gamma^*_{sk}(D)$-function. In view of the proof of Theorem 3.16, $|A(P,P)| \geq (k-1)n+2|M| = (k+1)n-2|P|$. On the other hand, $|A(P,P)| \leq |P|(|P|-1)$. It follows that $|P|(|P|-1) \geq (k+1)n-2|P|$ and so $|P|^2+|P|-(k+1)n \geq 0$. This implies that

$$|P| \geq \frac{-1 + \sqrt{4(k+1)n+1}}{2},$$

and thus we obtain

$$\gamma^*_{sk}(D) = 2|P| - n \geq 2\left\lceil \frac{-1 + \sqrt{4(k+1)n+1}}{2} \right\rceil - n.$$

□

Theorem 3.19. Let D be a bipartite digraph of order n. Then

$$\gamma^*_{sk}(D) \geq 2\left\lceil \sqrt{2(k+1)n+4} \right\rceil - n - 4.$$

Proof. Let f be a $\gamma^*_{sk}(D)$-function. In view of the proof of Theorem 3.16, $|A(P,P)| \geq (k+1)n - 2|P|$. On the other hand, $|A(P,P)| \leq |P|^2/2$. It follows that $|P|^2/2 \geq (k+1)n - 2|P|$ and so $|P| \geq \sqrt{2(k+1)n+4} - 2$. Therefore

$$\gamma^*_{sk}(D) = 2|P| - n \geq 2\left\lceil \sqrt{2(k+1)n+4} \right\rceil - n - 4.$$

□

Theorems 3.18, 3.19 and Observation 3.9 lead to the next well-known result.

Corollary 3.20. ([15]) If G is a graph of order n, then $\gamma_{sk}(G) \geq 2\left\lceil \frac{-1+\sqrt{4n(k+1)+1}}{2} \right\rceil - n$.
If G is a bipartite graph of order n, then $\gamma_{sk}(G) \geq 2\left\lceil \sqrt{2(k+1)n+4} \right\rceil - n - 4$.

Wang [15] presents examples which show that the bounds given in Corollaries 3.17 and 3.20 are sharp. The associated digraphs of these examples show that Theorems 3.16, 3.18 and 3.19 are sharp. Note that our proof of Corollary 3.20 is shorter than the one given in [15].

With any digraph D, we can associate a graph G with the same vertex set simply by replacing each arc by an edge with the same vertices. This graph is the underlying graph of D, denoted $G(D)$.

Theorem 3.21. Let D be a digraph of order n and let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of the underlying graph G of D. If s is the smallest positive integer for which $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geq (2k+2)n - 4s$, then

$$\gamma^*_{sk}(D) \geq 2s - n.$$

Furthermore, this bound is sharp.

Proof. Let f be a $\gamma^*_{sk}(D)$-function and $p = |P|$. Since $f([N^+_D[v]) \geq k$ and $f(N^-_D[v]) \geq k$ for each $v \in V(D)$, we have

$$\begin{array}{rcl} kn & \leq & \sum_{v\in V} f(N^+_D[v]) = \sum_{v\in V}(d^-_D(v)+1)f(v) \\ & = & |P| - |M| + \sum_{v\in P} d^+_D(v) - \sum_{v\in M} d^+_D(v) \end{array}$$

and

$$\begin{array}{rcl} kn & \leq & \sum_{v\in V} f(N^-_D[v]) = \sum_{v\in V}(d^+_D(v)+1)f(v) \\ & = & |P| - |M| + \sum_{v\in P} d^-_D(v) - \sum_{v\in M} d^-_D(v). \end{array}$$

Summing the above inequalities, we deduce that

$$\begin{aligned} 2kn &\leq 2(|P| - |M|) + \sum_{v\in P}(d^+_D(v) + d^-_D(v)) - \sum_{v\in M}(d^+_D(v) + d^-_D(v)) \\ &= 2(2p - n) + \sum_{v\in P} \deg_G(v) - \sum_{v\in M} \deg_G(v) \\ &\leq 4p - 2n + \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i. \end{aligned}$$

Thus $(2k+2)n - 4p \leq \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i$. By the assumption on s, we must have $p \geq s$. This implies that $\gamma^*_{sk}(D) = 2p - n \geq 2s - n$.

In order to prove sharpness, suppose that D is the digraph obtained from the union of $k+1$ tournament $\mathrm{CT}^i(2k+1)$, and $V(\mathrm{CT}^i(2k+1)) = \{v^i_1, \ldots, v^i_{2k+1}\}$, $1 \leq i \leq k+1$ by adding $2k+1$ new vertices $w_1, \ldots, w_{2k+1}$ and adding arcs (v^i_j, w_j) and (w_j, v^i_{j+1}) for each $1 \leq i \leq k+1$ and $1 \leq j \leq t$, where $2k+1+1$ is identified with 1. Obviously, D is $(2k+2)$-regular of order $n = (2k+1)(k+2)$. Hence,

$$\sum_{i=1}^{(2k+1)(k+1)} d_i - \sum_{(2k+1)(k+1)+1}^{n} d_i = k(2k+1)(2k+2) = (2k+2)n - 4(2k+1)(k+1).$$

It follows that $s = (2k+1)(k+1)$ is the smallest positive integer s such that $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geq (2k+2)n - 4s$ and so $\gamma^*_{sk}(D) \geq k(2k+1)$. Now define $f : V(D) \to \{-1, 1\}$ which assigns -1 to w_j for $1 \leq j \leq 2k+1$ and $+1$ to the other vertices. Obviously, f is a TSkDF of D and $\omega(f) = k(2k+1)$. This completes the proof. □

The special case $k = 1$ of Theorems 3.4, 3.13, 3.16 and 3.21 was recently proved in [5].

4. Twin Signed k-Domination in Oriented Graphs

Let G be the complete bipartite graph $K_{2k+2,2k+2}$ with bipartite sets $\{u_1, \ldots, u_{2k+2}\}$ and $\{v_1, \ldots, v_{2k+2}\}$. Let D_1 and D_2 be the orientations of G such that

$$A(D_1) = \{(u_i, v_r), (v_i, u_j), (v_j, u_i), (u_j, v_s) \mid 1 \leq i, r \leq k-1,\ k \leq j, s \leq 2k+2\}$$

and

$$A(D_2) = \{(u_i, v_r), (v_i, u_j), (v_j, u_i), (u_j, v_s), (u_i, v_t), (u_t, v_i), (v_t, u_j), (v_j, u_t), (u_t, v_\ell) \mid$$
$$1 \le i, r \le k,\ k+1 \le j, s \le 2k,\ 2k+1 \le t, \ell \le 2k+2\}.$$

It is easy to see that $\gamma^*_{sk}(D_1) = 4k + 4$ and $\gamma^*_{sk}(D_2) = 4k$. Thus two distinct orientations of a graph can have distinct twin signed k-domination numbers. Motivated by this observation, we define lower orientable twin signed k-domination number $\mathrm{dom}^*_{sk}(G)$ and upper orientable twin signed k-domination number $\mathrm{Dom}^*_{sk}(G)$ of a graph G as follows:

$$\mathrm{dom}^*_{sk}(G) = \min\{\gamma^*_{sk}(D) \mid \text{D is an orientation of } G\},$$

and

$$\mathrm{Dom}^*_{sk}(G) = \max\{\gamma^*_{sk}(D) \mid \text{D is an orientation of } G\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) [8], twin domination number [9], twin signed domination number [5], twin signed total domination number [2], twin signed total k-domination number [3], twin minus domination number [4], twin minus total domination number [10], twin signed Roman domination number [7] and twin signed total Roman domination number [1]. Note that the definitions are well-defined because every graph G with $\delta(G) \ge 2k - 2$, has an orientation D such that $\delta^-(D), \delta^+(D) \ge k - 1$.

Proposition 4.1. For any graph G of order n, $\gamma_{sk}(G) \le \mathrm{dom}^*_{sk}(G)$.

Proof. Let D be an orientation of G such that $\gamma^*_{sk}(D) = \mathrm{dom}^*_{sk}(G)$, and let f be a $\gamma^*_{sk}(D)$-function. Then $f(N_G[v]) = f(N^+_D[v]) + f(N^-_D[v]) - f(v)$ for each $v \in V$. Since $f(N^+_D[v]) \ge k$ and $f(N^-_D[v]) \ge k$, we have $f(N_G[v]) \ge 2k - 1$ for each $v \in V$, and so f is a SkDF of G. Therefore $\gamma_{sk}(G) \le \omega(f) = \mathrm{dom}^*_{sk}(G)$ as desired. □

In the rest of this section, we determine the lower orientable twin signed k-domination numbers of complete graphs and complete bipartite graphs.

Lemma 4.2. For $n \ge 2k + 1$,

$$\mathrm{dom}^*_{sk}(K_n) \ge \begin{cases} 2k+1 & \text{if } n \text{ is odd} \\ 2k+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let D be an orientation of K_n such that $\gamma^*_{sk}(D) = \mathrm{dom}^*_{sk}(K_n)$ and let f be a $\gamma^*_{sk}(D)$-function. If $M_f = \emptyset$, then $\omega(f) = n$ and the proof is complete. Assume that $v \in M_f$. We consider two cases.

Case 1. n is odd.
Since $f(N^+[v]) \ge k$ and $f(N^-[v]) \ge k$ and since $N^+(v) \cup N^-(v)$ is a partition of $V(K_n) \setminus \{v\}$, we deduce that $\mathrm{dom}^*_{sk}(K_n) = \omega(f) = f(N^+[v]) + f(N^-[v]) - f(v) \ge 2k + 1$.

Case 2. n is even.
Since $n - 1$ is odd and since $N^+(v) \cup N^-(v)$ is a partition of $V(K_n) \setminus \{v\}$, one of the $d^+(v)$ or $d^-(v)$ must be odd. Assume, without loss of generality, that $d^+(v)$ is odd. Then we must have $f(N^+[v]) \ge k + 1$ and $f(N^-[v]) \ge k$. Proceeding as above, we obtain $\mathrm{dom}^*_s(K_n) \ge 2k + 2$. □

Theorem 4.3. For $n \ge 2k + 1$,

$$\mathrm{dom}^*_{sk}(K_n) = \begin{cases} 2k+1 & \text{if } n \text{ is odd} \\ 2k+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. The result is trivial for $n = 2k + 1, 2k + 2$, so assume $n \ge 2k + 3$. Let

$$V(K_n) = \{u_i, v_i, w_j \mid 1 \le i \le \lceil \frac{n}{2} \rceil - (k+1) \ \text{ and } \ 1 \le j \le n - 2\lceil \frac{n}{2} \rceil + 2k + 2\}.$$

We consider two cases.

Case 1. n is odd. Let D be an orientation of K_n such that

$$\begin{array}{rcl} A(D) & = & \{(u_t,u_\ell),(u_t,v_\ell),(v_t,v_\ell),(v_r,u_s) \mid 1\le t<\ell\le\lceil\frac{n}{2}\rceil-(k+1) \\ & & \text{and } 1\le r\le s\le\lceil\frac{n}{2}\rceil-(k+1)\} \\ & \cup & \{(u_i,w_j),(v_i,w_j)\mid 1\le i\le\lceil\frac{n}{2}\rceil-(k+1),\ 1\le j\le k+1\} \\ & \cup & \{(w_q,u_i),(w_q,v_i)\mid 1\le i\le\lceil\frac{n}{2}\rceil-(k+1),\ k+2\le q\le 2k+1\} \\ & \cup & \{(w_t,w_{t+\ell})\mid 1\le\ell\le k,\ 1\le t\le 2k+1\} \end{array}$$

where we identify $2k+1+i$ with i.

Case 2. n is even. Let D be an orientation of K_n such that

$$\begin{array}{rcl} A(D) & = & \{(u_t,u_\ell),(u_t,v_\ell),(v_t,v_\ell),(v_r,u_s) \mid 1\le t<\ell\le\lceil\frac{n}{2}\rceil-(k+1) \\ & & \text{and } 1\le r\le s\le\lceil\frac{n}{2}\rceil-(k+1)\} \\ & \cup & \{(u_i,w_j),(v_i,w_j)\mid 1\le i\le\lceil\frac{n}{2}\rceil-(k+1),\ 1\le j\le k+1\} \\ & \cup & \{(w_q,u_i),(w_q,v_i)\mid 1\le i\le\lceil\frac{n}{2}\rceil-(k+1),\ k+2\le q\le 2k+2\} \\ & \cup & \{(w_t,w_{t+\ell})\mid 1\le\ell\le k,\ 1\le t\le 2k+1\} \\ & \cup & \{(w_{2k+2},w_i),(w_j,w_{2k+2})\mid 1\le i\le k,\ k+1\le j\le 2k+1\} \end{array}$$

where we identify $2k+1+i$ with i.

It is easy to see that the function $f: V(D)\to\{-1,+1\}$ defined by $f(u_i)=-1$ for $1\le i\le\lceil\frac{n}{2}\rceil-(k+1)$ and $f(x)=+1$ otherwise, is a TSkDF of D of weight $2k+1$ when n is odd and wight $2k+2$ when n is even. This implies that

$$\mathrm{dom}^*_{sk}(K_n)\le\omega(f)=\begin{cases} 2k+1 & \text{if } n \text{ is odd}\\ 2k+2 & \text{if } n \text{ is even.}\end{cases}$$

Now the result follows from Lemma 4.2. □

Let $m\le n$ and $K_{m,n}$ be the bipartite graph with bipartite sets V_1 and V_2 such that $|V_1|=m$ and $|V_2|=n$.

Lemma 4.4. Let D be an orientation of $K_{m,n}$ with $n\ge m\ge 2k+2$. If f is a TSkDF of D such that $V_i\cap M_f\ne\emptyset$ for $i=1,2$, then

$$\omega(f)\ge\begin{cases} 4k+4 & \text{if } n \text{ and } m \text{ are both even}\\ 4k+5 & \text{if } n \text{ and } m \text{ have different parity}\\ 4k+6 & \text{if } n \text{ and } m \text{ are both odd .}\end{cases}$$

Proof. Let $u\in V_1\cap M_f$ and $v\in V_2\cap M_f$. We consider three cases.

Case 1. m and n are both even.

Since $f(N^+[u])\ge k$ and $f(N^-[u])\ge k$, we must have

$$|N^+(u)\cap P_f\cap V_2|\ge|N^+(u)\cap M_f\cap V_2|+k+1$$

and

$$|N^-(u)\cap P_f\cap V_2|\ge|N^-(u)\cap M_f\cap V_2|+k+1.$$

Since $V_2=N^+(u)\cup N^-(u)$, we deduce that

$$|V_2\cap P_f|\ge|V_2\cap M_f|+2k+2. \tag{3}$$

Similarly, we have

$$|V_1\cap P_f|\ge|V_1\cap M_f|+2k+2. \tag{4}$$

Adding (3) and (4), we obtain $|P_f|\ge|M_f|+4k+4$ and so $\omega(f)=|P_f|-|M_f|\ge 4k+4$ as desired.

Case 2. m and n have different parity.

Assume, without loss of generality, that m is even and n is odd. Since $d^+(u)+d^-(u)=n$ is odd, we may assume that $d^+(u)$ is odd. It follows that $f(N^+[u])\ge k+1$ and hence

$$|N^+(u)\cap P_f\cap V_2|\ge|N^+(u)\cap M_f\cap V_2|+k+2.$$

Using an argument similar to that described in Case 1, we obtain $\omega(f) = |P_f| - |M_f| \geq 4k + 5$.

Case 3. m and n are both odd.

Since $d^+(u) + d^-(u) = n$ and $d^+(v) + d^-(v) = m$ are both odd, we may assume, without loss of generality, that $d^+(u)$ and $d^+(v)$ are both odd. As Cases 1, 2, we have

$$|N^+(u) \cap P_f \cap V_2| \geq |N^+(u) \cap M_f \cap V_2| + k + 2$$

$$|N^-(u) \cap P_f \cap V_2| \geq |N^-(u) \cap M_f \cap V_2| + k + 1$$

$$|N^+(v) \cap P_f \cap V_1| \geq |N^+(v) \cap M_f \cap V_1| + k + 2$$

$$|N^-(v) \cap P_f \cap V_1| \geq |N^-(v) \cap M_f \cap V_1| + k + 1.$$

Summing the above inequalities, we deduce that $|P_f| \geq |M_f| + 4k + 6$ and so $\omega(f) \geq 4k + 6$ as desired. □

Lemma 4.5. Let $2k \leq m \leq n$ and D be an orientation of $K_{m,n}$ and f be a TSkDF of D. If $V_1 \cap M_f = \emptyset$, then

$$\omega(f) \geq \begin{cases} m + 2k - 2 & \text{if } n \text{ is even} \\ m + 2k - 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $u \in V_1$. If n is even, then it follows from $f(N^+[u]) \geq k$ and $f(N^-[u]) \geq k$ that $|N^+(u) \cap P_f| \geq |N^+(u) \cap M_f| + k - 1$ and $|N^-(u) \cap P_f| \geq |N^-(u) \cap M_f| + k - 1$. This implies that $|V_2 \cap P_f| \geq |V_2 \cap M_f| + 2k - 2$ and hence $\omega(f) = |P_f| - |M_f| = |V_1| + |V_2 \cap P_f| - |V_2 \cap M_f| \geq m + 2k - 2$.

Assume that n is odd. Since $d^+(u) + d^-(u) = n$ is odd, we may assume, without loss of generality, that $d^+(u)$ is odd. This implies that $f(N^+[u]) \geq k + 1$. As above we have $|N^+(u) \cap P_f| \geq |N^+(u) \cap M_f| + k$ and $|N^-(u) \cap P_f| \geq |N^-(u) \cap M_f| + k - 1$, which implies that $|V_2 \cap P_f| \geq |V_2 \cap M_f| + 2k - 1$. Therefore $\omega(f) = |P_f| - |M_f| = |V_1| + |V_2 \cap P_f| - |V_2 \cap M_f| \geq m + 2k - 1$ as desired. □

The next result is an immediate consequence of Lemmas 4.4 and 4.5.

Corollary 4.6. For $2k + 2 \leq m \leq n$,

$$\mathrm{dom}^*_{sk}(K_{m,n}) \geq \min\{m + 2k - 2 + (2\lceil \frac{n}{2} \rceil - n), 4k + 4 + (2\lceil \frac{m}{2} \rceil - m) + (2\lceil \frac{n}{2} \rceil - n)\}.$$

Theorem 4.7. For $2k + 2 \leq m \leq n$,

$$\mathrm{dom}^*_{sk}(K_{m,n}) = \min\{m + 2k - 2 + (2\lceil \frac{n}{2} \rceil - n), 4k + 4 + (2\lceil \frac{m}{2} \rceil - m) + (2\lceil \frac{n}{2} \rceil - n)\}.$$

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of $K_{m,n}$. First we consider the cases $m = 2k + 2$ and $m = 2k + 3$. Partition the sets U and V according to Table 1. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = [U_1, V_1 \cup V_3] \cup [V_1 \cup V_3, U_2] \cup [V_2, U_1] \cup [U_2, V_2],$$

where $[X, Y] = \{(x, y) \mid x \in X,\ y \in Y\}$. Define $f : V(G) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in U \cup \{v_1, \ldots, v_{\lceil \frac{n}{2} \rceil + k - 1}\}$ and $f(x) = -1$ otherwise. It is easy to see that f is an TSkDF of D, so $\mathrm{dom}^*_{sk}(K_{m,n}) \leq \omega(f) = m + 2k - 2 + (2\lceil \frac{n}{2} \rceil - n)$.

$m = 2k + 2$	$U_1 = \{u_1, \ldots, u_{k+1}\}$, $U_2 = \{u_{k+2}, \ldots, u_{2k+2}\}$
$m = 2k + 3$	$U_1 = \{u_1, \ldots, u_{k+2}\}$, $U_2 = \{u_{k+3}, \ldots, u_{2k+3}\}$
n even	$V_1 = \{v_1, \ldots, v_{k-1}\}$, $V_2 = \{v_k, \ldots, v_{2k-2}\}$, $V_3 = \{v_{2k-1}, \ldots, v_n\}$
n odd	$V_1 = \{v_1, \ldots, v_k\}$, $V_2 = \{v_{k+1}, \ldots, v_{2k-1}\}$, $V_3 = \{v_{2k}, \ldots, v_n\}$

Table 1: $m = 2k + 2, 2k + 3$

We now deal with the case $m \geq 2k + 4$. Partition the sets U and V according to Table 2. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = [U_1, V_1 \cup V_3] \cup [U_2 \cup U_3, V_2] \cup [V_1, U_2 \cup U_3] \cup [V_2, U_1] \cup [V_3, U_2 \cup U_3].$$

It is easy to verify that the function $f : V(G) \to \{-1,+1\}$ defined by $f(x) = +1$ for $x \in \{u_1,\ldots,u_{\lceil \frac{m}{2}\rceil+k+1}\} \cup \{v_1,\ldots,v_{\lceil \frac{n}{2}\rceil+k+1}\}$ and $f(x) = -1$ otherwise, is a TSkDF of D, so $\mathrm{dom}^*_{sk}(K_{m,n}) \le \omega(f) = 4k+4+(2\lceil \frac{m}{2}\rceil - m) + (2\lceil \frac{n}{2}\rceil - n)$. Now the result follows by Corollary 4.6.

m even	$U_1 = \{u_1,\ldots,u_{k+1}\}$, $U_2 = \{u_{k+2},\ldots,u_{2k+2}\}$,$U_3 = \{u_{2k+3},\ldots,u_m\}$
m odd	$U_1 = \{u_1,\ldots,u_{k+2}\}$,$U_2 = \{u_{k+3},\ldots,u_{2k+3}\}$, $U_3 = \{u_{2k+4},\ldots,u_m\}$
n even	$V_1 = \{v_1,\ldots,v_{k+1}\}$,$V_2 = \{v_{k+2},\ldots,v_{2k+2}\}$, $V_3 = \{v_{2k+3},\ldots,v_n\}$
n odd	$V_1 = \{v_1,\ldots,v_{k+2}\}$, $V_2 = \{v_{k+3},\ldots,v_{2k+3}\}$ $V_3 = \{v_{2k+4},\ldots,v_n\}$

Table 2: $m \ge 2k+4$

□

The special case $k = 1$ of Theorems 4.3 and 4.7 was recently proved in [5].

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