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On Strictly Quasi-Fredholm Linear Relations and Semi-B-Fredholm Linear Relation Perturbations

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Abstract. In this paper we introduce the set of strictly quasi-Fredholm linear relations and we give some of its properties. Furthermore, we study the connection between this set and some classes of linear relations related to the notions of ascent, essentially ascent, descent and essentially descent. The obtained results are used to study the stability of upper semi-B-Fredholm and lower semi-B-Fredholm linear relations under perturbation by finite rank operators.

1. Introduction

Let X and Y be two Banach spaces. A linear relation $T : X \to Y$ is a mapping from a subspace $D(T) \subset X$ called the domain of T, into the collection of nonempty subsets of Y such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all nonzero α, β scalars and x, $y \in D(T)$. We denote the set class of linear relations from X to Y by $LR(X, Y)$ and abbreviate $LR(X, X)$ to $LR(X)$. The graph of a relation $T \in LR(X, Y)$ is the subset $G(T)$ of $X \times Y$ defined by $G(T) = \left\{(x, y)/\ y \in Tx\right\}$. Let $T \in LR(X, Y)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) = \left\{(y, x)/\ (x, y) \in G(T)\right\}$. The range and kernel part of T, denoted $R(T)$ and $N(T)$ are defined respectively by $R(T) = \bigcup_{x \in D(T)} Tx$ and $N(T) = T^{-1}(0)$. We say that T is injective, if $N(T) = \{0\}$, surjective if $R(T) = Y$ and bijective if T is both injective and surjective. If $M \subset X$ then the image of M under T is defined to be the set

$$T(M) = \bigcup_{x \in D(T) \cap M} Tx$$

and if $N \subset Y$, then the inverse image of N under T is defined to be the set

$$T^{-1}(N) := \left\{x \in D(T) : Tx \cap N \neq \emptyset\right\}.$$

In particular, for any $y \in R(T)$

$$T^{-1}y := \left\{x \in D(T) : y \in Tx\right\}.$$

The adjoint T^* of T is defined by $G(T^*) = G(-T^{-1})^{\perp}$, that is, $(y', x') \in G(T^*)$ if and only if, for all $(x, y) \in G(T)$, $y'y - x'x = 0$. For a linear relation T, the root manifold $N^{\infty}(T)$ is defined by $N^{\infty}(T) = \cup_{n=1}^{\infty} N(T^n)$.

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Similarly, the root manifold $R^{\infty}(T)$ is defined by $R^{\infty}(T) = \cap_{n=1}^{\infty} R(T^n)$. The singular chain manifold of T, denoted by $R_c(T)$, is defined by $R_c(T) = N^{\infty}(T) \cap R_{\infty}(T)$ where $R_{\infty}(T) = \cup_{i=1}^{\infty} T^i(0)$.
Let $T \in LR(X, Y)$. The nullity and the deficiency of T are defined respectively as follows: $\alpha(T) := \dim N(T)$ *and* $\beta(T) := \dim Y/R(T)$. If $\alpha(T) < \infty$ and $R(T)$ is closed then T is called upper-semi Fredholm linear relation and if $\beta(T) < \infty$ and $R(T)$ is closed then T is called lower-semi Fredholm linear relation. If either $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is called a Fredholm linear relation and we define the index of T by $ind(T) := \alpha(T) - \beta(T)$.
For a given closed subspace E of X, let Q_E^X or simply Q_E denoted the natural quotient map from X onto X/E. We shall denote $Q^X_{\overline{T(0)}}$ by Q_T. Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\| Tx \| := \| Q_T Tx \|$ and the norm of T is defined by $\| T \| := \| Q_T T \|$. We note that this quantity is not a true norm since $\| T \| = 0$ does not imply $T = 0$.
Let T and $S \in LR(X)$. The linear relations $T+S$ and TS are defined respectively by $G(T+S) = \Big\{(x, y+z) \in X \times X : (x, y) \in G(T) \text{ and } (x, z) \in G(S)\Big\}$ and $G(TS) = \Big\{(x, y) \in X \times X : \exists z \in X \text{ such that } (x, z) \in G(S) \text{ and } (z, y) \in G(T)\Big\}$.
We say that T commutes with S, if $TS \subseteq ST$, and T and S commute mutually if $TS = ST$. We say that a linear relation is closed if its graph is a closed subspace of $X \times X$, continuous if for each neighborhood V in $R(T)$, $T^{-1}(V)$ is neighborhood in $D(T)$ and open if its inverse is continuous. Continuous everywhere defined linear relation on X is referred to be a bounded linear relation. We denote by $CR(X)$ (resp. by $BR(X)$) the set of all closed (resp. bounded) linear relations on X. The class of all bounded and closed linear relations on X is denoted by $BCR(X)$.

The kernels and the ranges of the iterates $T^n, n \in \mathbb{N}$, of a linear relation T defined on a vector space X, form two increasing and decreasing chains, respectively.

$$N(T^0) = \{0\} \subseteq N(T) \subseteq N(T^2) \subseteq ...$$

$$R(T^0) = X \supseteq R(T) \supseteq R(T^2)...$$

Let $T \in LR(X)$, $n \in \mathbb{N}$ and let $c_n(T) = \dim R(T^n)/R(T^{n+1})$. The descent of T is defined by:

$$\delta(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$$

and the essential descent of T is defined by:

$$\delta_e(T) = \inf\{n : c_n(T) < \infty\}.$$

Let $T \in LR(X)$, $n \in \mathbb{N}$ and let $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. The ascent of T is defined by:

$$a(T) = \inf\{n : c'_n(T) = 0\} = \inf\{n : N(T^n) = N(T^{n+1})\}$$

and the essential ascent of T is defined by:

$$a_e(T) = \inf\{n : c'_n(T) < \infty\}.$$

If T is a bounded linear relation on a Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the null space of the induced map. $k_n(T)$ is called the difference sequence of T. The following definition describes the class of linear relations that we will study. Let $T \in BCR(X)$. If there exists a nonnegative integer d for which $k_n(T) = 0$ for all $n \geq d$ and $R(T^{d+1})$ is closed, we say that T is Strictly quasi-Fredholm linear relation. We denote the set of these linear relations by $Sq\phi(X)$.
Recall that $T \in BCR(X)$ is said to be quasi-Fredholm $\Big($resp. relatively quasi-Fredholm$\Big)$ linear relation if there exists $d \in \mathbb{N}$ such that:
1) $k_n(T) = 0$ for all $n \geq d$.
2) $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are closed $\Big($resp. complemented$\Big)$ in X.
The class of all quasi-Fredholm $\Big($resp. relatively quasi-Fredholm$\Big)$ linear relations is denoted by $q\phi(X)$ $\Big($resp.

$Rq\phi(X)\Big)$.
We have the following relationship $Rq\phi(X) \subset Sq\phi(X) \subset q\phi(X)$ and where X is a Hilbert space we have $Rq\phi(X) = Sq\phi(X) = q\phi(X)$. Moreover, when X is a Banach space and $\rho(T) \neq \emptyset$ we have $T \in q\phi(X)$ if and only if $T \in Sq\phi(X)$.
Trivial examples of Strictly quasi-Fredholm linear relations are upper semi-Fredholm linear relations, lower semi-Fredholm linear relations as well as semi-regular linear relations, essential-semi regular linear relations and bounded below linear relations.

To make the paper easily accessible, some algebraic properties of linear relations are given in Section 2. In particular, generalization of Lemma 10 in [12] and Lemma 11 in [12] are given in the context of linear relations. In the Section 3, we investigate the properties of the difference sequence $(k_n(T))_n$ of bounded linear relation T. The main result of this section is Proposition 3.2, wish gives some relations between $k_n(T)$, $c_n(T)$ and $c'_n(T)$. In the Section 4, we introduce and study the class of strictly quasi-Fredholm linear relations. Section 5, is devoted to give the connection between the class of strictly quasi-Fredholm linear relations and a various classes of relations defined by means of ascent, descent, essentially ascent and essentially descent. After that, we apply the results given, to study in Section 6, the stability of the upper semi-B-Fredholm and lower semi-B-Fredholm linear relations under commuting finite rank operator perturbation.

2. Some Algebraic Properties for Linear Relations.

In this section, we collect some algebraic properties of linear relations defined in Banach spaces and we give some technical lemmas which we will need repeatedly in the sequel.

Definition 2.1. *A subset U of X $\Big($resp. a subset V of $Y\Big)$ is said to be a neighbourhood of a point $x \in X$ $\Big($resp. of $Tx\Big)$ if U contains an open set containing x $\Big($resp. V contains an open set containing $Tx\Big)$. Let $T \in LR(X, Y)$. Then T is said to be continuous at a point $x \in D(T)$ if the inverse image of any neighbourhood of Tx is a neighbourhood of x. T is said to be continuous if it is continuous at every point in its domain. A continuous linear relation of domain $D(T) = X$ is called bounded. We denote $BR(X, Y)$ the class of all bounded linear relations and as useful we write $BR(X, X) := BR(X)$.*

Definition 2.2. *Let $T \in LR(X, Y)$. T is called open if whenever U is a neighbourhood in $D(T)$, the image $T(U)$ is a neighbourhood in $R(T)$. Clearly*

T is open if and only if T^{-1} is continuous.

Definition 2.3. *Let $T \in LR(X, Y)$. The closure of T is the relation $\overline{T}$ defined by $G(\overline{T}) = \overline{G(T)}$. The relation T is called closed if $G(T)$ is closed in $X \times Y$ or, equivalently, $\overline{T} = T$. We denote the class of all closed linear relations from X to Y by $CR(X, Y)$ and as useful we write $CR(X, X) := CR(X)$.*

Definition 2.4. *Let X be a Banach space and $T \in CR(X)$. For $\lambda \in \mathbb{C}$, the linear relation $(T - \lambda)^{-1}$ is called the resolvent of T (corresponding to λ). The resolvent set of T is the set:*

$$\rho(T) = \Big\{\lambda \in \mathbb{C} : (T - \lambda) \text{ is injective and surjective}\Big\}.$$

The spectrum of T is the set $\sigma(T) := \mathbb{C} \backslash \rho(T)$.

Lemma 2.1. *([8], Corollary 3.1.) Let X and Y be two Banach spaces and let $T \in CR(X)$. If $\beta(T) < \infty$, then $R(T)$ is closed.*

Definition 2.5. *Let A and $B \in LR(X)$. We say that A commutes with B, if $AB \subset BA$. And we say that A and B commute mutually if $AB = BA$.*

Lemma 2.2. *([14], Lemma 22.2) Let U, V and W be three subspaces of a Banach space X such that $U \subset W$. Then*

$$(U + V) \cap W = U + (V \cap W).$$

Definition 2.6. *[16] Let M be a subspace of a Banach space Y. M is said to be a range subspace, if there exists a Banach space X and a bounded linear operator T defined from X to Y where the range of T is M.*

The following proposition gives a characterization of range subspaces.

Proposition 2.1. *([16], Proposition 2.1) Let M be a linear subspace of a Banach space $(Y, \| \, . \, \|)$. The following properties are equivalent:*

i) *M is a range subspace.*

ii) *M is a domain of a closed operator defined in Y.*

iii) *There is a norm $\| \, . \, \|_1$ on M such that $(M, \| \, . \, \|_1)$ is a Banach space and $\| y \|_1 \geq \| y \|$ for all $y \in M$.*

Remark 2.1. **i)** *Any closed subspace of a Banach space X is a range subspace of X.*

ii) *M and N are range subspaces in X if and only if $M \times N$ is a range subspace in $X \times X$.*

iii) *The sum of two range subspaces is a range subspace.*

iv) *The intersection of two range subspaces is a range subspace.*

Lemma 2.3. *Let T be a closed linear relation in a Banach space X. Then $R(T)$ and $N(T)$ are range subspaces of X.*

Proof

Let T be a closed linear relation. Then $G(T)$ is a closed subspace of $X \times X$ and so a range subspace of $X \times X$. In other hand we have

$$N(T) \times \{0\} = G(T) \cap (X \times \{0\}) \text{ and } G(T) + (X + \{0\}) = X \times R(T).$$

So using Remark 2.1, we get that $N(T)$ and $R(T)$ are range subspaces of X. □

Lemma 2.4. *([11], Theorem 2.4) Let M and N be two range subspaces of a Banach space X such that $M + N$ and $M \cap N$ are closed. Then M and N are closed.*

The next lemma is a generalization of Lemma 10 in [12].

Lemma 2.5. *Let $T \in BCR(X)$, $m \geq 0$ and $n \geq i \geq 1$. If $R(T^n) + N(T^m)$ is closed then $R(T^{n-i}) + N(T^{m+i})$ is closed .*

Proof

Since T is bounded hence $Q_T T$ is a bounded operator. It's clear that

$$Q_T(R(T^n) + N(T^m)) = R(T^n) + N(T^m) + T(0)/T(0) = R(T^n) + N(T^m)/T(0).$$

As T is closed so $T(0)$ is closed then $Q_T(R(T^n) + N(T^m))$ is closed. On the other hand we have

$$\begin{array}{rcl} (Q_T T)^{-1}(Q_T(R(T^n) + N(T^m))) & = & T^{-1}(R(T^n) + N(T^m) + N(Q_T)) \\ & = & T^{-1}(R(T^n) + N(T^m) + T(0)) \\ & = & T^{-1}(R(T^n) + N(T^m)) \\ & = & R(T^{n-1}) + N(T^{m+1}). \end{array}$$

Thus $R(T^{n-1}) + N(T^{m+1})$ is closed. By repeating the same technique we get the result.

□

The next lemma is a generalization of Lemma 11 in [12].

Lemma 2.6. *Let $T \in BCR(X)$, $\rho(T) \neq \emptyset$ and $n \geq 0$. If $R(T^n)$ and $R(T) + N(T^n)$ are closed then $R(T^{n+1})$ is closed.*

Proof

Let $x_j \in R(T^{n+1})$, such that $x_j \longrightarrow x$. It suffices to show that $x \in R(T^{n+1})$. Since $R(T^n)$ is closed and $x_j \in R(T^{n+1}) \subset R(T^n)$, then $x \in R(T^n)$ and there exist $u_j \in X$ and $u \in X$ such that $x_j \in T^{n+1}u_j$ and $x \in T^n u$. It follows that $x_j - x \in T^{n+1}u_j - T^n u = T^n(Tu_j - u)$. Then, there exists $z_j \in Tu_j$ such that $x_j - x \in T^n(z_j - u)$. Thus, $x_j - x + T^n(0) = T^n(z_j - u)$.
We consider

$$\widehat{T^n} : X/N(T^n) \longrightarrow X/T^n(0)$$

$$\widehat{T^n}\overline{x} = \{\widetilde{y}, y \in T^n x\}.$$

We have $\widehat{T^n}(\overline{0}) = \{\widetilde{y}, y \in T^n(0)\} = \{\widetilde{0}\}$. This implies that $\widehat{T^n}$ is an operator.

$$\begin{array}{rcl}\widehat{T^n}(\overline{u - z_j}) & = & \Big\{\widetilde{y},\ y \in T^n(u - z_j)\Big\} \\ & = & \Big\{\widetilde{y},\ y \in x_j - x + T^n(0)\Big\} \\ & = & \widetilde{x_j - x} = \widetilde{x_j} - \widetilde{x} \longrightarrow \widetilde{0},\ \Big(\text{ since, } x_j \longrightarrow x\Big).\end{array}$$

The operator $\widehat{T^n}$ is injective. In fact, let $\overline{x} \in X/N(T^n)$ such that $\widehat{T^n}(\overline{x}) = \{\widetilde{y},\ y \in T^n x\} = \widetilde{0}$. It follows that $T^n x \subset T^n(0)$ and hence $\overline{x} = \overline{0}$.
Furthermore, $R(\widehat{T^n})$ is closed. In fact:
$R(\widehat{T^n}) = \{\widehat{T^n}\overline{x},\ \overline{x} \in X/N(T^n)\} = \{\widetilde{y},\ y \in T^n(x)\} = \{\widetilde{y},\ y \in R(T^n)\} = R(T^n)/T^n(0)$.
Since T is closed and $\rho(T) \neq \emptyset$, then by Lemma 3.1 in [8], T^n is closed. Thus $T^n(0)$ is closed and hence $R(T^n)/T^n(0)$ is closed in the Banach space $X/T^n(0)$. Thus $R(T^n)/T^n(0)$ is a Banach space.
Then $\widehat{T^n} :\ X/N(T^n) \longrightarrow R(T^n)/T^n(0)$ is bijective and continuous, with $R(T^n)/T^n(0)$ and $X/N(T^n)$ are Banach spaces. Thus, by the Banach Theorem $(\widehat{T^n})^{-1}$ is continuous.
Now, since, $\widehat{T^n}(\overline{u - z_j}) \to \widetilde{0}$ and $(\widehat{T^n})^{-1}$ is continuous, this implies that, $\overline{u - z_j} \to \overline{0}$ in $X/N(T^n)$. So there exists $v_j \in N(T^n)$ such that $z_j + v_j \to u$. Since $z_j + v_j \in Tu_j + N(T^n) \subset R(T) + N(T^n)$, and $R(T) + N(T^n)$ is closed, then $u \in R(T) + N(T^n)$. On the other hand, we have

$$\begin{array}{rcl}T^n u & \subset & T^n\Big(R(T) + N(T^n)\Big) \\ & = & T^n\Big(R(T)\Big) + T^n(N(T^n)\Big) \\ & = & T^n\Big(R(T)\Big) + T^n T^{-n}(0),\ \Big(\text{ by Poroposition } I.3.1 \text{ in [4] }\Big) \\ & = & R(T^{n+1}) + T^n(0) = R(T^{n+1}),\ \Big(\text{ since } T^n(0) \subset R(T^{n+1})\ \Big).\end{array}$$

So, $x \in T^n u \subset R(T^{n+1})$. Hence $R(T^{n+1})$ is closed. □

3. Some Properties of the Difference Sequence $(k_n(T))_n$.

This section is devoted to the study of the difference sequence $\Big(k_n(T)\Big)_n$, where $T \in BR(X)$. We first recall the following definition.

Definition 3.1. *If T is a bounded linear relation on a Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the null space of the induced map and $k_{-1}(T) = \infty$.*

Proposition 3.1. *Let $T \in BR(X)$. Then*

$$k_n(T) = \dim\Big(R(T^n) \cap N(T)\Big)\Big/\Big(R(T^{n+1}) \cap N(T)\Big) \text{ for each nonnegative integer } n.$$

Proof

Let $n \in \mathbb{N}$. We consider $\widehat{T}$ the linear transformation induced by T defined by:

$$\widehat{T} : R(T^n)\big/R(T^{n+1}) \longrightarrow R(T^{n+1})\big/R(T^{n+2}) \quad \overline{x} \longmapsto \widehat{Tx}$$

where $\widehat{Tx} = \{\widetilde{y},\ y \in Tx\}$. We have $\widehat{T0} = \{\widetilde{y},\ y \in T(0)\} = \widetilde{0}$. Hence $\widehat{T}$ is an operator.

$$\begin{array}{rcl} N(\widehat{T}) & = & \{\ \overline{x} \in R(T^n)\big/R(T^{n+1}),\ \widehat{Tx} = \widetilde{0}\ \} \\ & = & \{\ \overline{x} \in R(T^n)\big/R(T^{n+1}),\ Tx \subset R(T^{n+2})\} \\ & = & \{\ \overline{x} \in R(T^n)\big/R(T^{n+1}),\ x \in T^{-1}(R(T^{n+2})\ \} \\ & = & \{\ \overline{x} \in R(T^n)\big/R(T^{n+1}),\ x \in R(T^{n+1}) + N(T)\ \} \\ & = & \Big[R(T^n) \cap \Big(R(T^{n+1}) + N(T)\Big)\Big]\Big/R(T^{n+1}). \end{array}$$

So by Lemma 2.2, we get

$$N(\widehat{T}) = \Big[R(T^n) \cap \Big(R(T^{n+1}) + N(T)\Big)\Big]\Big/R(T^{n+1}) = \Big[R(T^{n+1}) + \Big(R(T^n) \cap N(T)\Big)\Big]\Big/R(T^{n+1}).$$

Using (a) in Lemma 2.1 in [15], we get:

$$\begin{array}{rcl} k_n(T) = \dim N(\widehat{T}) & = & \dim \Big[R(T^{n+1}) + \Big(R(T^n) \cap N(T)\Big)\Big]\Big/R(T^{n+1}) \\ & = & \dim \Big[R(T^n) \cap N(T)\Big]\Big/\Big[R(T^{n+1}) \cap N(T)\Big]. \end{array}$$

□

The next proposition provides the relationships between the sequences: $\Big(k_n(T)\Big)_n$, $\Big(c_n(T)\Big)_n$ and $\Big(c'_n(T)\Big)_n$.

Proposition 3.2. *Let $T \in BR(X)$. The sequence $\Big(k_n(T)\Big)_n$ satisfies the following relations:*

i) *If $c_n(T) < \infty$ for some $n \in \mathbb{N}$, then $k_n(T) = c_n(T) - c_{n+1}(T)$.*

ii) *If $c'_n(T) < \infty$ for some $n \in \mathbb{N}$ and $R_c(T) = \{0\}$, then $k_n(T) = c'_n(T) - c'_{n+1}(T)$.*

Proof

i) We consider :

$$\widehat{T} : R(T^n)\big/R(T^{n+1}) \longrightarrow R(T^{n+1})\big/R(T^{n+2})$$
$$\overline{x} \longmapsto \widehat{Tx}.$$

If $c_n(T) = \dim R(T^n)\big/R(T^{n+1}) < \infty$ for some $n \in \mathbb{N}$, then, applying the rank theorem to $\widehat{T}$, we get that $\dim R(\widehat{T}) + \dim N(\widehat{T}) = \dim R(T^n)\big/R(T^{n+1})$. Hence

$$\begin{array}{rcl} k_n(T) = \dim N(\widehat{T}) & = & \dim R(T^n)\big/R(T^{n+1}) - \dim R(\widehat{T}) \\ & = & \dim R(T^n)\big/R(T^{n+1}) - \dim R(T^{n+1})\big/R(T^{n+2}) \\ & = & c_n(T) - c_{n+1}(T). \end{array}$$

ii) If $R_c(T) = \{0\}$ then by Lemma 4.4 in [15], we have $N(T^{i+k})\big/N(T^i) \simeq N(T^k) \cap R(T^i)$. So,

$$\begin{array}{rcl} k_n(T) & = & \dim R(T^n) \cap N(T)\big/R(T^{n+1}) \cap N(T) \\ & = & \dim R(T^n) \cap N(T) - \dim R(T^{n+1}) \cap N(T) \\ & = & \dim N(T^{n+1})\big/N(T^n) - \dim N(T^{n+2})\big/N(T^{n+1}) \\ & = & c'_n(T) - c'_{n+1}(T). \end{array}$$

□

Using Lemma 3.1 in [7] and Lemma 4.2.2 in [16], we get:

Lemma 3.1. *Let $T \in BR(X)$, $\rho(T) \neq \emptyset$, $p, n \in \mathbb{N}$ and $T_n = T_{/R(T^n)}$. Then*

i) $c_p(T_n) = c_{p+n}(T)$. *In particular* $c_0(T_n) = \beta(T_n) = c_n(T)$.

ii) $c'_p(T_n) = c'_{p+n}(T)$. *In particular* $c'_0(T_n) = \alpha(T_n) = c'_n(T)$.

iii) $k_p(T_n) = k_{p+n}(T)$.

Lemma 3.2. *Let X be a Banach space and $T \in BR(X)$. Then we have*
$\dim\Big(N(T)/(N(T)\cap R(T^n)\Big) = \sum\limits_{i=0}^{n-1} \dim\Big(N(T)\cap R(T^i)\Big/N(T)\cap R(T^{i+1})\Big).$

Proof

Observe that $N(T)\cap R(T^n) \subset N(T)\cap R(T^{n-1}) \subset N(T)\cap R(T^{n-2}) \subset \cdots \subset N(T)$.
Using Lemma 2.1 in [15], we get:
$\dim N(T)/N(T)\cap R(T^n) = \dim N(T)/N(T)\cap R(T^{n-1}) + \dim N(T)\cap R(T^{n-1})/N(T)\cap R(T^n)$.
As we have
$\dim N(T)/N(T)\cap R(T^{n-1}) = \dim N(T)/N(T)\cap R(T^{n-2}) + \dim N(T)\cap R(T^{n-2})/N(T)\cap R(T^{n-1})$, by a repeated application of Lemma 2.1 in [15], we get

$$\dim\Big(N(T)/(N(T)\cap R(T^n)\Big) = \sum_{i=0}^{n-1} \dim\Big(N(T)\cap R(T^i)\Big/N(T)\cap R(T^{i+1})\Big).$$

□

As a consequence of Lemma 3.2, we get:

Proposition 3.3. *Let $T \in BR(X)$. Then*

$$k(T) = \sum_{i=0}^{\infty} k_i(T) = \dim[N(T)/(N(T)\cap R^{\infty}(T))].$$

4. Strictly Quasi-Fredholm Linear Relations.

The goal of this section is to introduce and study the class of strictly quasi-Fredholm linear relations.

Definition 4.1. *Let X be a Banach space and $T \in BCR(X)$. We say that T is strictly quasi-Fredholm relation of degree $d \in \mathbb{N}$, if $k_n(T) = 0$ for all $n \geq d$, $k_{d-1}(T) \neq 0$ and $R(T^{d+1})$ is closed. We denote by $Sq\phi(d)(X)$, the set of all strictly quasi-Fredholm linear relations of degree d and by $Sq\phi(X)$ the set of all strictly quasi-Fredholm linear relations for some degree $d \in \mathbb{N}$.*

We start by giving this lemma. Which is useful to the proof of the following propositions.

Lemma 4.1. *Let $T \in BCR(X)$ and $d \in \mathbb{N}$ such that $\rho(T) \neq \emptyset$ and $k_i(T) < \infty$ for every $i \geq d$. Then the following statements are equivalent:*

i) *there exists $n_0 \geq d+1$ such that $R(T^{n_0})$ is closed.*

ii) *$R(T^n)$ is closed for every $n \geq d$.*

iii) *$R(T^n) + N(T^m)$ is closed for all n, m with $n + m \geq d$.*

Proof
It is clear that: iii) $\Rightarrow$ ii) $\Rightarrow$ i).
ii)$\Rightarrow$iii): Let $n \geq d$. We have $R(T^n)$ is closed. If $n = 0$, then there is nothing to prove. If $n \geq 1$, then from Lemma 2.5, for all $1 \leq i \leq n$ we have $R(T^{n-i}) + N(T^i)$ is closed. Thus $R(T^n) + N(T^m)$ is closed for all $n,\ m \in \mathbb{N}$ with $n + m \geq d$.
i) $\Rightarrow$ ii): If there exits $n_0 \geq d+1$ such that $R(T^{n_0})$ is closed then, by Lemma 2.5, $R(T)+N(T^{n_0-1})$ is closed. Since $k_i(T) < \infty$ for all $i \geq d$, it follows that,
$k_{n_0-1}(T) = \dim R(T^{n_0-1}) \cap N(T)/R(T^{n_0}) \cap N(T) = \dim R(T) + N(T^{n_0})\Big/R(T) + N(T^{n_0-1}) < \infty$. This implies that $R(T) + N(T^{n_0})$ is closed. So, by Lemma 2.6, $R(T^{n_0+1})$ is closed. Consequently $R(T^n)$ is closed for all $n \geq n_0$. On the other hand, by Lemma 2.5, we have $R(T^{n_0-1}) + N(T)$ is closed. And as, $R(T^{n_0}) \cap N(T)$ is closed and $k_{n_0-1}(T) = \dim R(T^{n_0-1}) \cap N(T)/R(T^{n_0}) \cap N(T) < \infty$ then $R(T^{n_0-1}) \cap N(T)$ is closed. By Lemma 2.3 and Lemma 2.4, it follows that $R(T^{n_0-1})$ is closed. By repeating these considerations, we can prove that $R(T^n)$ is closed for all n with $d \leq n \leq n_0$. □

In the next we give some characterizations of strictly quasi-Fredholm linear relations.

Proposition 4.1. *Let $T \in BCR(X)$ and $d \in \mathbb{N}$. Then, $T \in Sq\phi(d)(X)$ if and only if $R(T) + N(T^d) = R(T) + N^\infty(T)$ and $R(T^{d+1})$ is closed.*

Proof

The result follows immediately from the equivalence :

$$\Big[R(T) + N(T^{n+1})\Big]/\Big[R(T) + N(T^n)\Big] \simeq \Big[N(T) \cap R(T^n)\Big]/\Big[N(T) \cap R(T^{n+1})\Big].$$

□

The following proposition is another characterization of the strictly quasi-Fredholm linear relations.

Proposition 4.2. *Let $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$ and $d \in \mathbb{N}$. Then, $T \in Sq\phi(d)(X)$ if and only if*

i) *$k_n(T) = 0$ for all $n \geq d$ and $k_{d-1}(T) \neq 0$.*

ii) *$R(T^n)$ is a closed subspace of X for each integer $n \geq d$.*

iii) *$R(T) + N(T^d)$ is a closed subspace of X.*

Proof

Let $T \in Sq\phi(d)(X)$. Then $R(T^{d+1})$ is closed, $k_n(T) = 0$ for all $n \geq d$ and $k_{d-1}(T) \neq 0$. It follows from Lemma 4.1, that $R(T^n)$ is closed for all $n \geq d$. The statement iii) follows immediately from Lemma 2.5.
Conversely, by i) we have $k_n(T) = 0$ for all $n \geq d$ and $k_{d-1}(T) \neq 0$ and by ii) we have $R(T^{d+1})$ is closed. Then $T \in Sq\phi(d)(X)$. □

Proposition 4.3. *Let $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$. Then $T \in Sq\phi(X)$ if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is semi-regular.*

Proof

Suppose that $T \in Sq\phi(X)$. There exists $d \in \mathbb{N}$, such that $T \in Sq\phi(d)(X)$. So by Proposition 4.2, we have $R(T^n)$ is closed for all $n \geq d$. By Lemma 3.1, we have $k_i(T_d) = k_{i+d}(T)$, for all $i \in \mathbb{N}$. Hence $k_i(T_d) = 0$ for all $i \geq 0$. Then T_d is semi-regular.
Conversely, suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is semi-regular. Then $R(T_n)$ is closed and $k_i(T_n) = 0$ for all $i \geq 0$. It follows that $k_m(T) = 0$ for all $m \geq n$. and $R(T^{n+1}) = R(T_n)$ is closed. Thus $T \in Sq\phi(n)(X)$. □

Next, we recall some well known classes of linear relations and we give some connection between them and the class of strictly quasi-Fredholm linear relations.
Let X be a Banach space and $T \in BCR(X)$. We say that T is upper semi-Fredholm linear relation if it has finite dimensional null space and closed range. We denote the set of all upper semi-Fredholm linear relations by $\phi_+(X)$. We say that T is lower semi-Fredholm if its range is closed and has a finite codimension. We denote the set of all lower semi-Fredholm linear relations by $\phi_-(X)$. We say that T is semi-regular if $N(T) \subset R^\infty(T)$ and $R(T)$ is closed. We denote the set of all semi-regular linear relations by $SR(X)$. We say that T is essentially semi-regular if $N(T) \subset_e R^\infty(T)$ and $R(T)$ is closed. We denote the set of all essentially semi-regular linear relations by $ESR(X)$.
As trivial examples of strictly quasi-Fredholm linear relations we have:

Example 4.1. *Let X be a Banach space and $T \in BCR(X)$ with $\rho(T) \neq \emptyset$. If T is semi-Fredholm then T is strictly quasi-Fredholm for some degree $d \in \mathbb{N}$.*
Indeed, first if T is upper semi-Fredholm then $c'_0(T) < \infty$. Since the sequence $\left(c'_n(T)\right)_n$ is decreasing then $c'_n(T) < \infty$ for all $n \in \mathbb{N}$. Thus $k_n(T) = c'_n(T) - c'_{n+1}(T) < \infty$ for all $n \in \mathbb{N}$. As $R(T)$ is closed, then by Lemma 4.1, $R(T^n)$ is closed for all $n \in \mathbb{N}$. The sequence $\left(c'_n(T)\right)_n$ is stationary for n large enough. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. Then $d < \infty$ and $R(T^{d+1})$ is closed. So $T \in Sq\phi(d)(X)$.
Secondly, if T is lower semi-Fredholm then $c_0(T) < \infty$. Since $\left(c_n(T)\right)_n$ is decreasing, it follows, by Proposition 3.2, that there exists $n_0 \in \mathbb{N}$ such that $k_n(T) = c_n(T) - c_{n+1}(T) = 0$ for all $n \geq n_0$. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. Then clearly $k_n(T) = 0$ for all $n \geq d$. Furthermore, Since $T \in \phi_-(X)$ and $\rho(T) \neq \emptyset$, then by Proposition 3.1 in [8], we have $T^{d+1} \in \phi_-(X)$ and so $R(T^{d+1})$ is closed. Thus $T \in Sq\phi(d)(X)$.

Example 4.2. *Let X be a Banach space and $T \in BCR(X)$. If T is onto or bounded below then T is strictly quasi-Fredholm of degree 0. In fact, if T is onto then $R(T) = X$ is closed and $k_n(T) = c_n(T) - c_{n+1}(T) = 0$ for all $n \in \mathbb{N}$. It follows that T is strictly quasi-Fredholm of degree 0. On the other hand, if T is bounded below then $R(T)$ is closed and $c'_0(T) = \dim N(T) = 0$. Since $\left(c'_n(T)\right)_n$ is decreasing then $c'_n(T) = 0$ for all $n \in \mathbb{N}$. So $k_n(T) = 0$ for all $n \in \mathbb{N}$. Hence T is strictly quasi-Fredholm of degree 0.*

Example 4.3. *Let X be a Banach space and $T \in BCR(X)$. If T is semi-regular or essentially semi-regular then T is strictly quasi-Fredholm. Indeed, if T is essentially semi-regular then $N(T) \subset_e R^\infty(T)$ so $\dim N(T)\big/N(T)\cap R^\infty(T) < \infty$. This implies that $\sum_{i=0}^{\infty} k_i(T) < \infty$. Thus, there exists $d \in \mathbb{N}$ such that $k_n(T) = 0$ for all $n \geq d$ and $k_{d-1} \neq 0$. As $R(T)$ is closed, then by Lemma 4.1, $R(T^{d+1})$ is closed. Thus $T \in Sq\phi(d)(X)$.*

Afterwards, we give the main theorem of this section where we investigate the relations between the following sets: the set of relatively quasi-Fredholm linear relations, the set of strictly quasi-Fredholm linear relations and the set of quasi-Fredholm linear relations. The following theorem is a generalization of Proposition 3 in [13].

Theorem 4.1. *Let X be a Banach space. We have:*

i) *$Rq\phi(X) \subset Sq\phi(X) \subset q\phi(X)$.*

ii) *If $T \in BCR(X)$ be such that $\rho(T) \neq \emptyset$, then $T \in q\phi(X)$ if and only if $T \in Sq\phi(X)$.*

Proof
i) Let $T \in Rq\phi(X)$. By Theorem 2.1 in [7], there exist X_1 and X_2 two closed subspaces of X such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$ and $T = T_1 \oplus T_2$ where $T_1 = T \cap (X_1 \times X_1)$ is nilpotent with degree d and $T_2 = T \cap (X_2 \times X_2)$ is semi-regular. Hence $R(T^{d+1}) = R(T_1^{d+1}) + R(T_2^{d+1}) = R(T_2^{d+1})$. Since T_2 is semi-regular then $R(T_2^{d+1})$ is closed. Thus $R(T^{d+1})$ is closed. As $k_n(T) = 0$ for all $n \geq d$, it follows that $T \in Sq\phi(d)(X)$.
Now, let $T \in Sq\phi(d)(X)$. So $k_n(T) = 0$ for all $n \geq d$ and by Proposition 4.2, we have $N(T^d) + R(T)$ and $R(T^d)$ are closed. As $N(T)$ is closed then $R(T^d) \cap N(T)$ is also closed. Hence $T \in q\phi(d)(X)$.

ii) Let $T \in BCR(X)$ be a quasi-Fredholm linear relation of degree d with $\rho(T) \neq \emptyset$. First we claim that $T(0) + N(T^d)$ is closed. Indeed, using Lemma 3.1 in [8], we get $T^d \in BCR(X)$. So, $Q_{T^d}T^d$ is a bounded operator and $Q_{T^d}(T^{d+1}(0))$ is closed. Hence, $(Q_{T^d}T^d)^{-1}(Q_{T^d}(T^{d+1}(0))) = T(0) + N(T^d)$ is closed.
To show that $R(T^{d+1})$ is closed, it is therefore sufficient to prove that $R(T^{d+1}) + N(T^d)$ and $R(T^{d+1}) \cap N(T^d)$ are closed. First, we prove by induction on j that for all $j \geq 1$, we have

$$N(T^j) \cap R(T^d) = N(T^j) \cap R(T^{d+1}). \tag{1}$$

This is true for $j = 1$. Let $j \geq 1$ and suppose that $N(T^j) \cap R(T^d) = N(T^j) \cap R(T^{d+1})$. We will show that $N(T^{j+1}) \cap R(T^d) = N(T^{j+1}) \cap R(T^{d+1})$. It is clear that $N(T^{j+1}) \cap R(T^{d+1}) \subset N(T^{j+1}) \cap R(T^d)$ so it suffices to show that $N(T^{j+1}) \cap R(T^d) \subset N(T^{j+1}) \cap R(T^{d+1})$. Let $x \in N(T^{j+1}) \cap R(T^d)$. Then by Lemma 2.2 in [7], we obtain $x \in \bigcap_{n=0}^{\infty} R(T^n)$ and so $x \in R(T^{d+1})$. It follows that $x \in N(T^{j+1}) \cap R(T^{d+1})$. Hence, $N(T^{j+1}) \cap R(T^d) = N(T^{j+1}) \cap R(T^{d+1})$. This proves Eq. (1).
We now, prove that

$$N(T^j) \cap R(T^d) \quad \text{is closed for all } 1 \leq j \leq d. \tag{2}$$

This is true for $j = 1$. Suppose that $N(T^j) \cap R(T^d)$ is closed for all $1 \leq j < d$. We have

$$T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big) = N(T) + N(T^{j+1}) \cap R(T^d).$$

Indeed, let $x \in T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big)$. It follows that $Tx \cap \Big(N(T^j) \cap R(T^{d+1})\Big) \neq \emptyset$. Hence there exists $z \in Tx \cap \Big(N(T^j) \cap R(T^{d+1})\Big)$ such that $Tx = z + T(0)$. Therefore $x + N(T) = T^{-1}(z) + N(T)$ and so $x \in T^{-1}(z) + N(T)$. On the other hand $z \in N(T^j)$ and $z \in R(T^{d+1})$ hence $x \in T^{-1}(N(T^j)) \cap T^{-1}(R(T^{d+1})) + N(T) = N(T^{j+1}) \cap \Big(R(T^d) + N(T)\Big) + N(T)$. This means that $x \in N(T^{j+1}) \cap R(T^d) + N(T)$. It follows that $T^{-1}\Big(N(T^j) \cap R(T^d)\Big) \subset N(T) + N(T^{j+1}) \cap R(T^d)$. It remains to show that $N(T) + N(T^{j+1}) \cap R(T^d) \subset T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big)$. Let $x \in N(T) + N(T^{j+1}) \cap R(T^d)$. Then $x = z_1 + z_2$ with $z_1 \in N(T)$ and $z_2 \in N(T^{j+1}) \cap R(T^d)$. Hence $Tz_1 \subset T(0)$, $Tz_2 \subset T\Big(N(T^{j+1})\Big) = N(T^j) \cap R(T) + T(0) \subset N(T^j) + T(0)$ and $Tz_2 \subset T\Big(R(T^d)\Big) = R(T^{d+1})$. It follows that $Tx = Tz_1 + Tz_2 \subset T(0) + \Big(N(T^j) + T(0)\Big) \cap R(T^{d+1}) = T(0) + N(T^j) \cap R(T^{d+1})$. Hence $x \in T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big) + N(T)$. This means that $N(T) \subset T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big)$. Hence $x \in T^{-1}\Big(N(T^j) \cap R(T^{d+1})\Big)$.
We prove now that $N(T) + N(T^{j+1}) \cap R(T^d)$ is closed. Indeed, we have $Q_T\Big(N(T^j) \cap R(T^{d+1})\Big) = \Big(N(T^j) \cap R(T^{d+1}) + T(0)\Big)\Big/T(0)$. Then,

$$\begin{aligned}
(Q_T T)^{-1}\Big(Q_T(N(T^j) \cap R(T^{d+1}))\Big) &= T^{-1}\Big(N(T^j) \cap R(T^{d+1}) + N(Q_T)\Big) \\
&= T^{-1}\Big(N(T^j) \cap R(T^{d+1}) + T(0)\Big) \\
&= T^{-1}\Big(N(T^j) \cap R(T^{d+1})) + T^{-1}(0).
\end{aligned}$$

Hence $(Q_T T)^{-1}\Big(Q_T(N(T^j) \cap R(T^{d+1}))\Big) = N(T) + N(T^{j+1}) \cap R(T^d)$. Since $Q_T T$ is a bounded operator, it suffices to prove that $Q_T\Big(N(T^j) \cap R(T^{d+1})\Big)$ is closed. For this we recall that $Q_T\Big(N(T^j) \cap R(T^{d+1})\Big) = N(T^j) \cap R(T^{d+1}) + T(0)\Big/T(0)$. As $j < d$ then $N(T^j) \cap R(T^{d+1}) \subset N(T^d)$. It follows that, $(T(0) + N(T^j) \cap R(T^{d+1})) + N(T^d) = T(0) + N(T^d)$ is closed. On the other hand and by Lemma 2.2, we have $(T(0) + N(T^j) \cap R(T^{d+1})) \cap N(T^d) = N(T^j) \cap R(T^{d+1})$ is closed. Thus by Lemma 2.4, $T(0) + N(T^j) \cap R(T^{d+1})$ is closed. As $T(0)$ is closed then $N(T^j) \cap R(T^{d+1}) + T(0)\Big/T(0)$ is closed. Further, $N(T) \cap N(T^{j+1}) \cap R(T^d) = N(T) \cap R(T^d)$ is closed. By Lemma 2.4, $N(T^{j+1}) \cap R(T^d)$ is closed. This proves Eq. (2).
Using Eq. (1) and Eq. (2) we get $N(T^d) \cap R(T^d) = N(T^d) \cap R(T^{d+1})$ is closed. It remains to show that $N(T^d) + R(T^{d+1})$ is closed. We show first that

$$N(T^{d+1}) \subset R(T^j) + N(T^d) \text{ for each } j \in \mathbb{N}. \tag{3}$$

Let $x \in N(T^{d+1})$ hence $0 \in T^{d+1}(x)$ and so $T^{d+1}(x) = T^{d+1}(0)$. This implies that $T^d(x) + N(T) = T^d(0) + N(T) \subset N(T) + R(T^d)$. Thus $T^d(x) \subset N(T) + T^d(0)$. It follows that $T^d(x) \subset R(T^d) \cap \big(N(T) + T^d(0)\big) = R(T^d) \cap N(T) + T^d(0)$. Since $R(T^d) \cap N(T) = R(T^{d+j}) \cap N(T)$ for all $j \in \mathbb{N}$. This implies that $T^d(x) \subset R(T^{d+j}) \cap N(T) + T^d(0)$ for all $j \in \mathbb{N}$. Hence there exists $y \in R(T^{d+j}) \cap N(T)$ such that $T^d x \subset T^{d+j} y + T^d(0)$, so $T^d(x - T^j y) \subset T^{d+j}(0) + T^d(0) = T^{d+j}(0)$. It follows that $x - T^j y \subset N(T^d) + T^j(0)$ so there exists $z \in T^j y$ such that $x - z \in \big(N(T^d) + T^j(0)\big)$. This implies that $x \in z + N(T^d) + T^j(0) \subset N(T^d) + R(T^j)$. Thus $N(T^{d+1}) \subset N(T^d) + R(T^j)$.
We consider now

$$\widehat{T} : X/N(T^d) \longrightarrow X/N(T^d)$$
$$\widehat{T}\widetilde{x} = \{\widetilde{y}, y \in Tx\}.$$

$\widehat{T}$ is a linear relation. Let $\widetilde{x} \in N(\widehat{T})$. Then $\widetilde{0} \in \widehat{T}(\widetilde{x}) = \{\widetilde{y} : y \in Tx\}$. Hence there exists $y \in Tx \cap N(T^d)$ such that $Tx = y + T(0)$. Therefore, $T^{d+1}x = T^d y + T^{d+1}(0) = T^d(0) + T^{d+1}(0) = T^{d+1}(0)$. Thus $x \in N(T^{d+1})$ and so $N(\widehat{T}) \subset \{\widetilde{x}, x \in N(T^{d+1})\}$. On the other hand we have $R(\widehat{T}) = \{\widehat{T}\widetilde{x}, x \in X\} = \{\widetilde{y}, y \in R(T)\}$ and for all $j \in \mathbb{N}$ we have $R(\widehat{T^j}) = \{\widetilde{y}, y \in R(T^j)\}$. Using Eq. (3), we get $N(\widehat{T}) \subset \bigcap_{j=0}^{\infty} R(\widehat{T^j})$. Further $R(T) + N(T^d)$ is closed and thus $R(\widehat{T})$ is a closed subspace of $X/N(T^d)$, then $\widehat{T}$ is semi regular. Consequently $k_n(\widehat{T}) = 0$ for all $n \in \mathbb{N}$. As $R(\widehat{T})$ is closed, using Lemma 4.1, we get $R(\widehat{T}^{d+1})$ is closed. Let Q be the canonical projection: $X \longrightarrow X/N(T^d)$. Then the space $R(T^{d+1}) + N(T^d) = Q^{-1}\big(R(\widehat{T}^{d+1})\big)$ is closed. Hence we have $R(T^{d+1}) + N(T^d)$ and $R(T^{d+1}) \cap N(T^d)$ are closed and so by referring to Lemma 2.4, we infer that $R(T^{d+1})$ is closed. Thus $T \in Sq\phi(X)$. □

5. Some Classes of Linear Relations Defined by Means of Ascent and Descent.

In this section we study the connection between the class of strictly quasi-Fredholm linear relations and some classes of linear relations related to the notions of ascent, descent, essentially ascent and essentially descent.
Now we give the definitions of the classes $R_i(X)$ for $i \in \{1, 2, 3\}$ related to the notions of descent and essentially descent.

Definition 5.1. *Let X be a Banach space.*

$$\begin{array}{lcl} R_1(X) & = & \{T \in BCR(X) : T \in \phi_-(X) \textit{ and } \delta(T) < \infty\} \\ R_2(X) & = & \{T \in BCR(X) : \delta(T) < \infty \textit{ and } R(T^{\delta(T)}) \textit{ is closed }\} \\ R_3(X) & = & \{T \in BCR(X) : \delta_e(T) < \infty \textit{ and } R(T^{\delta_e(T)}) \textit{ is closed}\}. \end{array}$$

In the next proposition we give the dual versions of $R_i(X)$, $1 \leq i \leq 3$ according to $Sq\phi(X)$.

Proposition 5.1. *Let $T \in BCR(X)$, such that $\rho(T) \neq \emptyset$. Then we have:*

i) *$T \in R_1(X)$ if and only if $T \in Sq\phi(d)(X), c_0(T) < \infty$ and $c_d(T) = 0$, for some $d \in \mathbb{N}$.*

ii) *$T \in R_2(X)$ if and only if $T \in Sq\phi(d)(X)$ and $c_d(T) = 0$, for some $d \in \mathbb{N}$.*

iii) *$T \in R_3(X)$ if and only if $T \in Sq\phi(d)(X)$ and $c_d(T) < \infty$, for some $d \in \mathbb{N}$.*

Proof

i) Let $T \in Sq\phi(d)(X)$, such that $c_d(T) = 0$ and $c_0(T) < \infty$. Then, $\dim X/R(T) < \infty$ and $\delta(T) \leq d < \infty$. Using Proposition 3.1 in [8], we obtain that $R(T)$ is closed. It follows that $T \in R_1(X)$.
Conversely, let $T \in R_1(X)$. Then $c_0(T) = \dim X/R(T) < \infty$. Since $\big(c_n(T)\big)_n$ is decreasing then $c_n(T) < \infty$ for all $n \geq 0$. By using Proposition 3.2, we have $k_n(T) = c_n(T) - c_{n+1}(T) < \infty$ for all $n \geq 0$. As $R(T)$ is closed, then by Lemma 4.1, we have $R(T^n)$ is closed for all $n \in \mathbb{N}$. On the other hand, we have for all $n \geq \delta(T), c_n(T) = 0$.

This implies that $k_n(T) = 0$ for all $n \geq \delta(T)$. So, for $d := \delta(T)$, we have $T \in Sq\phi(d)(X)$ and $c_d(T) = 0$.
ii) Let $T \in Sq\phi(d)(X)$ with $c_d(T) = 0$. Then $\delta(T) \leq d < \infty$ and $R(T^{d+1})$ is closed. Since $c_{\delta(T)}(T) = 0$, then $c_n(T) = 0$ for all $n \geq \delta(T)$, and so $k_n(T) < \infty$ for all $n \geq \delta(T)$. As we have $R(T^{d+1})$ is closed with $d+1 \geq \delta(T)+1$, thus by Lemma 4.1, we have $R(T^n)$ is closed for all $n \geq \delta(T)$. In particular $R(T^{\delta(T)})$ is closed. Thus $T \in R_2(X)$.
Conversely, let $T \in R_2(X)$. Hence, $\delta(T) < \infty$ and $R(T^{\delta(T)})$ is closed. Further, since $c_{\delta(T)}(T) = 0$, then $R(T^{\delta(T)+1})$ is also closed. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. Clearly $d \leq \delta(T)$ and for all $n \geq d$, $k_n(T) = 0$. So, by Lemma 4.1, $R(T^{d+1})$ is closed, which implies that $T \in Sq\phi(d)(X)$. It remains to show that $c_d(T) = 0$. We have $\dim X/\Big(R(T) + N(T^{\delta(T)})\Big) < \infty$ and $k_n(T) = 0$ for all $n \geq d$. This implies, by Corollary 2.6 in [10], that $R(T)+N(T^d) = R(T)+N(T^n)$ for all $n \geq d$. In particular for $n := \delta(T)$. Thus $c_d(T) = \dim X/\Big(R(T)+N(T^d)\Big) < \infty$. Hence $k_d(T) = c_d(T) - c_{d+1}(T) = 0$ and so, $c_d(T) = 0$.
iii) Let $T \in Sq\phi(d)(X)$ with $c_d(T) < \infty$. It follows that $\delta_e(T) \leq d < \infty$ and $k_n(T) = c_n(T) - c_{n+1}(T) < \infty$ for all $n \geq \delta_e(T)$. Then, Lemma 4.1, leads to $R(T^n)$ is closed for all $n \geq \delta_e(T)$. So $T \in R_3(X)$.
Conversely, let $T \in R_3(X)$. Then $\delta_e(T) < \infty$. So $k_n(T) < \infty$ for all $n \geq \delta_e(T)$ and there exists $n_0 \in \mathbb{N}$ such that $k_{n_0}(T) = 0$. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. Thus by Lemma 4.1, we have $R(T^n)$ is closed for all $n \geq d$ and so $T \in Sq\phi(d)(X)$. It remains to prove that $c_d(T) < \infty$. We have $R(T^{\delta_e(T)})$ is closed and $c_{\delta_e(T)}(T) < \infty$. Then $\dim\Big(X\Big/R(T) + N(T^{\delta_e(T)})\Big) < \infty$. We have $\frac{N(T)\cap R(T^d)}{N(T)\cap R(T^{d+1})} \simeq \frac{N(T^{d+1})+R(T)}{N(T^d)+R(T)}$ then suppose that $d < \delta_e(T)$, we get $N(T^{\delta_e(T)}) + R(T) = N(T^{\delta_e(T)-1}) + R(T)$ and hence $c_{\delta_e(T)-1}(T) < \infty$ which is absurd. So $d \geq \delta_e(T)$ and hence $c_d(T) < \infty$.

□

Now we give the definitions of the classes $R_i(X)$ for $i \in \{4, 5, 6\}$ related to the notions of ascent and essentially ascent.

Definition 5.2. *Let X be a Banach space.*

$$\begin{array}{rcl} R_4(X) & = & \{T \in BCR(X) : \ T \in \phi_+(X) \text{ and } a(T) < \infty\} \\ R_5(X) & = & \{T \in BCR(X) : \ a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\} \\ R_6(X) & = & \{T \in BCR(X) : \ a_e(T) < \infty \text{ and } R(T^{a_e(T)+1}) \text{ is closed}\}. \end{array}$$

In the next proposition we give the dual versions of $R_4(X), R_5(X)$ and $R_6(X)$ according to the set $Sq\phi(X)$.

Proposition 5.2. *Let $T \in BCR(X)$, such that $\rho(T) \neq \emptyset$. Then*

i) *$T \in R_4(X)$ if and only if $T \in Sq\phi(d)(X), c'_0(T) < \infty$ and $c'_d(T) = 0$, for some $d \in \mathbb{N}$.*

ii) *$T \in R_5(X)$ if and only if $T \in Sq\phi(d)(X)$ and $c'_d(T) = 0$, for some $d \in \mathbb{N}$.*

iii) *$T \in R_6(X)$ if and only if $T \in Sq\phi(d)(X)$ and $c'_d(T) < \infty$, for some $d \in \mathbb{N}$.*

Proof
i) Let $T \in Sq\phi(d)(X)$ such that $c'_0(T) < \infty$ and $c'_d(T) = 0$. Then $\dim N(T) = c'_0(T) < \infty$ and $a(T) \leq d < \infty$. As $c'_n(T) < \infty$ for all $n \in \mathbb{N}$, then $k_n(T) = c'_n(T) - c'_{n+1}(T) < \infty$ for all $n \in \mathbb{N}$. So by Lemma 4.1, we have $R(T^n)$ is closed for all $n \geq 0$. Thus $R(T)$ is closed. So $T \in \phi_+(T)$ and $a(T) < \infty$.
Conversely, let $T \in R_4(X)$. Thus $c'_n(T) < \infty$ for all $n \in \mathbb{N}$ and so $k_n(T) = c'_n(T) - c'_{n+1}(T) < \infty$ for all $n \geq 0$. As $R(T)$ is closed, then by Lemma 4.1, $R(T^n)$ is closed for all $n \in \mathbb{N}$. On the other hand, since $\Big(c'_n(T)\Big)_n$ is decreasing, then there exists $n_0 \in \mathbb{N}$ such that $k_n(T) = 0$ for all $n \geq n_0$. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. It is clear that $d \leq a(T) < \infty, k_n(T) = 0$ for all $n \geq d$ and $c'_d(T) = c'_{a(T)}(T) = 0$. Thus $T \in Sq\phi(d)(X)$, $c'_d(T) = 0$ and $c'_0(T) < \infty$.
ii) Let $T \in Sq\phi(d)(X)$ such that $c'_d(T) = 0$. Then $a(T) \leq d$ and $c'_n(T) = 0$, for all $n \geq a(T)$. It follows that $k_n(T) = 0$ for all $n \geq a(T)$. This implies that $d \leq a(T)$. Using Proposition 4.2, it follows that $R(T^n)$ is closed for all $n \geq d$. Thus $R(T^{a(T)+1})$ is closed. So $T \in R_5(X)$.
Conversely, let $T \in R_5(X)$. Then $a(T) < \infty$ and so $c'_n(T) = 0$, for all $n \geq a(T)$. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$. Then $a(T) \geq d$, $k_n(T) = 0$ for all $n \geq d$ and $R(T^{a(T)+1})$ is closed. Hence, by Lemma 4.1, $R(T^n)$ is closed for all

$n \geq d$ and so $T \in Sq\phi(d)(X)$. It remains to prove that $c'_d(T) = 0$. We have $\alpha(T_{a(T)+1}) = c'_{a(T)+1}(T) = 0$, then $\dim\left(N(T) \cap R(T^{a(T)+1})\right) = 0$ and so $\dim\left(N(T) \cap R(T^m)\right) = 0$ for all $m \geq a(T)+1$. Then $\dim\left(N(T) \cap R(T^d)\right) = 0$ and so $c'_d(T) = 0$.
iii) Let $T \in Sq\phi(d)(X)$ such that $c'_d(T) < \infty$. Then $a_e(T) \leq d < \infty$ and so $k_n(T) = c'_n(T) - c'_{n+1}(T) < \infty$ for all $n \geq a_e(T)$. Since $R(T^{d+1})$ is closed with $d+1 \geq a_e(T)+1$, then by Lemma 4.1, $R(T^{a_e(T)+1})$ is also closed. Hence $T \in R_6(X)$.
Conversely, let $T \in R_6(X)$ then $a_e(T) < \infty$. So there exists $p \in \mathbb{N}$ such that $c'_p(T) < \infty$. Thus $\dim\left(R(T^p) \cap N(T)\right) < \infty$. The sequence $\left(R(T^p) \cap N(T)\right)_p$ is stationary for p large enough. Let $d = \inf\{n \in \mathbb{N}, k_n(T) = 0\}$ then $d < \infty$. On the other hand, $R(T^{a_e(T)+1})$ is closed leads to $R(T^{d+1})$ is closed, by Lemma 4.1. So $T \in Sq\phi(d)(X)$. It remains to prove that $c'_d(T) < \infty$. We have $k_n(T) = 0$, for all $n \geq d$, then $\dim\left(N(T) \cap R(T^d)\right) = \dim\left(N(T) \cap R(T^p)\right)$ for all $p \geq d$ so $\dim\left(N(T) \cap R(T^d)\right) = c'_d(T) < \infty$.

□

As a consequence of Propositions 5.1 and 5.2, we have the following results.

Corollary 5.1. *Let $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$. Then*

i) *$T \in R_2(X)$ if and only if $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and T_d is onto.*

ii) *$T \in R_3(X)$ if and only if $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and T_d is lower semi-Fredholm.*

iii) *$T \in R_5(X)$ if and only if $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and T_d is bounded below.*

iv) *$T \in R_6(X)$ if and only if $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and T_d is upper semi-Fredholm.*

Proof

i) Let $T \in R_2(X)$. Then by referring to Proposition 5.1 and Lemma 3.1, we infer that $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and $c_0(T_d) = c_d(T) = 0$. Thus T_d is onto.
Conversely, if T_d is onto for some $d \in \mathbb{N}$ then by Lemma 3.1, $c_d(T) = c_0(T_d) = 0$. Since by hypothesis $T \in Sq\phi(d)(X)$, then by Proposition 5.1, we have $T \in R_2(X)$.
ii) Let $T \in R_3(X)$. Then by Proposition 5.1, we have $T \in Sq\phi(d)(X)$ and $c_d(T) < \infty$ for some $d \in \mathbb{N}$. It follows, by Lemma 3.1, that $c_0(T_d) = c_d(T) < \infty$. Since $T \in Sq\phi(d)(X)$ then $R(T_d) = R(T^{d+1})$ is closed and so T_d is lower semi-Fredholm.
Conversely, if $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$ and T_d is lower semi-Fredholm, then $c_0(T_d) < \infty$. So by Lemma 3.1, $c_d(T) = c_0(T_d) < \infty$. Hence by Proposition 5.1, it follows that $T \in R_3(X)$.
The results of iii) and iv) are deduced from Proposition 5.2 and Lemma 3.1, by the same way. □

6. Perturbation of Semi-B-Fredholm Linear Relations.

The purpose of this section is to investigate the perturbation problem of semi-B-Fredholm linear relation under a finite rank operator and as consequences we deduce the stability of various essential spectra related to semi-B-Fredholm linear relations classes.

Definition 6.1. *([7], Definition 3.1) Let X be a Banach space. A linear relation $T \in LR(X)$ is called B-Fredholm if T is a range space relation and there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and the restriction $T_d = T_{/R(T^d)}$ is a Fredholm linear relation.*

Definition 6.2. *Let X be a Banach space. A linear relation $T \in LR(X)$ is called upper semi-B-Fredholm (resp. Lower semi-B-Fredholm) if T is a range space relation and there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and the restriction $T_d = T_{/R(T^d)}$ is an upper-semi Fredholm linear relation (resp. Lower-semi-Fredholm).*

Definition 6.3. *Let R be one of the classes R_i, with $1 \leq i \leq 6$. We define the associated B-class BR as the set:*

$$BR = \left\{T \in LR(X) : \text{ there exists } n \in \mathbb{N} \text{ such that } R(T^n) \text{ is closed and } T_n = T_{/R(T^n)} \in R\right\}.$$

The following propositions will be required in the proof of the main results of this section.

Proposition 6.1. *Let X be a Banach space and $T \in BCR(X)$, such that $\rho(T) \neq \emptyset$. Then the following statements are equivalent:*

i) $T \in R_6(X)$.

ii) *T is upper semi-B-Fredholm linear relation.*

iii) $T \in BR_6(X)$.

Proof

i) $\Rightarrow$ *ii)* If $T \in R_6(X)$ then by Corollary 5.1 and Proposition 4.2, $T \in Sq\phi(d)(X)$, $R(T^d)$ is closed and T_d is upper semi Fredholm for some $d \in \mathbb{N}$. Then T is upper semi-B-Fredholm.
ii) $\Rightarrow$ *iii)* First, we show that if T is upper semi-Fredholm then $T \in R_6(X)$. Indeed, if T is upper semi-Fredholm then $T \in Sq\phi(d)(X)$ for some $d \in \mathbb{N}$. As $c'_0(T) < \infty$ and $\left(c'_n(T)\right)_n$ is decreasing then $c'_d(T) < \infty$. Using Proposition 5.2, we get $T \in R_6(X)$. Now we have T is upper semi-B-Fredholm then there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and T_d is upper semi-Fredholm. Hence $T_d \in R_6$ and so $T \in BR_6(X)$.
iii) $\Rightarrow$ *i)* Now, let $T \in BR_6(X)$. There exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n \in R_6(X)$. It follows by Proposition 5.2, that $T_n \in Sq\phi(d)(R(T^n))$ and $c'_d(T_n) < \infty$ for some $d \in \mathbb{N}$. Hence we have $k_i(T_n) = 0$ for all $i \geq d$. So, by Lemma 3.1, we have $k_i(T) = 0$ for all $i \geq n + d$. As we have $R(T^{n+d+1}) = R((T_n)^{d+1})$ is closed then $T \in Sq\phi(n + d)(X)$ and $c'_{d+n}(T) = c'_d(T_n) < \infty$. Using Proposition 5.2 , we get $T \in R_6(X)$. □

Proposition 6.2. *Let X be a Banach space and $T \in BCR(X)$, such that $\rho(T) \neq \emptyset$. Then the following statements are equivalent:*

i) $T \in R_3(X)$.

ii) *T is lower semi-B-Fredholm linear relation.*

iii) $T \in BR_3(X)$.

The proof may be sketched in a similar way to the proof of Proposition 6.1, it suffices to replace Proposition 5.2 by Proposition 5.1.
We are now ready to express the main result of this section which gives a generalization of Proposition 2.7 in [3].

Theorem 6.1. *Let X be a Banach space and F be a bounded operator of finite rank. Let $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$. Suppose that $TF = FT$. If T is upper semi-B-Fredholm linear relation then $T + F$ is upper semi-B-Fredholm linear relation.*

Proof

Since T is upper semi-B-Fredholm, then there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and T_d is upper-semi Fredholm. By Proposition 6.1, it follows that $T \in R_6(X)$. Thus $a_e(T) < \infty$ and $R(T^{a_e(T)+1})$ is closed. We verify that $a_e(T + F) < \infty$ and $R(T + F)^{a_e(T)+1}$ is closed.
Step 1: We claim that $a_e(T + F) < \infty$. We first prove that for all non negative integer n

$$\dim\left[N(T^n)/N(T + F)^n \cap N(T^n)\right] < \infty. \tag{4}$$

Indeed, we have $TF = FT$. Then for all n, $T^nF = FT^n$. So, $T^n(0) = F(T^n(0))$. It follows that $T^n(0) \subset R(F)$. Let $x \in N(T^n)$. Then,

$$\begin{aligned}(T+F)^n x &\subset \sum_{i=0}^{n} C_n^i T^{n-i} F^i x \\ &\subset T^n x + F\Big(\sum_{i=1}^{n} C_n^i F^{i-1} T^{n-i} x\Big) \\ &\subset T^n(0) + R(F) \subset R(F).\end{aligned}$$

Let us introduce, the relation A given by:

$$A = (T+F)^n_{/N(T^n)} : \qquad \begin{aligned} N(T^n) &\longrightarrow R(F) \\ x &\longmapsto (T+F)^n x \end{aligned}$$

and $\widehat{A}$ induced by A by:

$$\widehat{A} : \qquad \begin{aligned} N(T^n)/N(A) &\longrightarrow R(F) \\ \overline{x} &\longmapsto Ax. \end{aligned}$$

Therefore, $N(\widehat{A}) = \{\overline{0}\}$ and according to Proposition I.6.4 in [5], we get:

$$\dim D(\widehat{A}) = \dim \Big[N(T^n)/N(T+F)^n \cap N(T^n)\Big] \leq \dim \Big(R(F)\Big) < +\infty.$$

Now, let $p = a_e(T)$. Given $n \geq p+1$, it follows by substituting $T+F$ for T in (4) that:

$$\dim \Big[N(T+F)^n/N(T^n) \cap N(T+F)^n\Big] < +\infty.$$

Since, $\dim \Big[N(T^n)/N(T^p)\Big] < +\infty$, we have

$$\dim \Big[N(T+F)^n/N(T^p) \cap N(T+F)^n\Big] < +\infty. \tag{5}$$

We shall now show that for all $n \geq p+1$, there exists a subspace v_n such that $\dim(v_n) < +\infty$ and

$$N(F) \cap N(T^p) \subset \Big[N(T+F)^n + v_n\Big] \cap N(T^p) \subset N(T^p).$$

Let $x \in N(F) \cap N(T^p)$. Then, $(T+F)^n x \subset R(F)$. Thus,

$$x \in (T+F)^{-n}(R(F)) = N((T+F)^n) + v_n.$$

Where $\dim(v_n) < +\infty$. Hence, $N(F) \cap N(T^p) \subset \Big[N(T+F)^n + v_n\Big] \cap N(T^p)$. Using now the hypothesis $\dim R(F) < +\infty$, it follows that $codim(N(F)) < +\infty$. Then

$$\dim \Big[N(T^p)/N(F) \cap N(T^p)\Big] < +\infty \tag{6}$$

and so,

$$\dim \Big[N(T^p)/\Big(N(T+F)^n + v_n\Big) \cap N(T^p)\Big] < +\infty. \tag{7}$$

We prove now that:

$$\dim \Big[\Big(N(T+F)^n + v_n\Big)/N(F) \cap N(T^p)\Big] < +\infty. \tag{8}$$

In fact, we have
$\dim \Big[\Big(N(T+F)^n + v_n\Big)/N(F) \cap N(T^p)\Big] \leq \dim \Big[\Big(N(T+F)^n + v_n\Big)/\Big(N(T+F)^n + v_n\Big) \cap N(T^p)\Big] + dim\Big[\Big(N(T+F)^n +$

$v_n\Big) \cap N(T^p)/N(F) \cap N(T^p)\Big]$.
Since, $N(T+F)^n \cap N(T^p) \subset \Big[\Big(N(T+F)^n + v_n\Big) \cap N(T^p)\Big]$, then using (5), we get:

$$\begin{aligned}&\dim\Big[\Big(N(T+F)^n + v_n\Big)/\Big(N(T+F)^n + v_n\Big) \cap N(T^p)\Big]\\ &\leq \dim\Big[\Big(N(T+F)^n + v_n\Big)/N(T+F)^n \cap N(T^p)\Big]\\ &\leq \dim\Big[N(T+F)^n/N(T+F)^n \cap N(T^p)\Big] + \dim(v_n) < +\infty\end{aligned}$$

and by (6), we get:

$$\dim\Big[\Big(N(T+F)^n + v_n\Big) \cap N(T^p)/N(F) \cap N(T^p)\Big] \leq \dim\Big[N(T^p)/N(F) \cap N(T^p)\Big] < +\infty.$$

Thus, proves (8). As, consequence of (8) we get:

$$\begin{aligned}&\dim\Big[N(T+F)^{n+1} + v_n + v_{n+1}/N(T+F)^n + v_n + v_{n+1}\Big]\\ &\leq \dim\Big[N(T+F)^{n+1} + v_n + v_{n+1}/N(F) \cap N(T^p)\Big]\\ &- \dim\Big[N(T+F)^n + v_n + v_{n+1}/N(F) \cap N(T^p)\Big] < +\infty.\end{aligned}$$

So, $\dim\Big[N(T+F)^{n+1}/N(T+F)^n\Big] < +\infty$. Thus $a_e(T+F) \leq p+1$.
Step2: We claim that $R(T+F)^{a_e(T+F)+1}$ is closed. Let $n \geq q+1$ with $q = \max(a_e(T), a_e(T+F))$. Denote T_0 and F_0, the restrictions of T and F to $R(T^q)$ respectively. Then T_0 is both closed and upper semi-Fredholm. Indeed, for $n \geq a_e(T)$, we have $c'_n(T) < +\infty$ and so, $k_n(T) < +\infty$. As $R(T^{a_e(T)+1})$ is closed using Lemma 4.1, we get $R(T^q)$ is closed. So, T_0 is a closed linear relation with $R(T_0) = R(T^{q+1})$ closed and by Lemma 3.1, $\alpha(T_0) = c'_q(T) < +\infty$. Then $T_0 \in \phi_+$. Moreover $T_0 + F_0$ is closed us a sum of a closed linear relation and bounded operator. So $T_0 + F_0$ is upper semi-Fredholm and by Proposition 24 in [1], we get $(T_0 + F_0)^n \in \phi_+(R(T^q))$. Hence $R(T_0 + F_0)^n$ is closed.
We claim now that

$$(T+F)^n T^q(X) =_e T^q(T+F)^n(X). \tag{9}$$

We prove (9) by induction on n. For $n = 1$, let $x \in X$ and $y \in T^q x$. Then, $T^q x = y + T^q(0)$. Hence:

$$\begin{aligned}(T+F)T^q x = (T+F)(y + T^q(0)) &= (T+F)y + (T+F)T^q(0)\\ &\subset Ty + Fy + (T+F)T^q(0)\\ &\subset TT^q x + FT^q x + (T+F)T^q(0)\\ &\subset T^q(T+F)x + (T+F)T^q(0)\\ &\subset T^q(T+F)x + V_1\end{aligned}$$

where V_1 is a finite dimensional subspace. Hence, $(T+F)T^q(X) \subset T^q(T+F)(X) + V_1$. On the other hand we have,

$$\begin{aligned}T^q(T+F)x &= T^{q+1}x + T^q F x\\ &= Ty + Fy + T^{q+1}(0) + FT^q(0)\\ &= (T+F)y + T^{q+1}(0) + FT^q(0)\\ &\subset (T+F)T^q x + W_1\end{aligned}$$

where W_1 is a finite dimensional subspace. Hence, $T^q(T+F)(X) \subset (T+F)T^q(X) + W_1$. Then, the result is proved for $n = 1$. Suppose now that $(T+F)^n T^q(X) \subset T^q(T+F)^n(X) + V_n$ and $T^q(T+F)^n(X) \subset (T+F)^n T^q(X) + W_n$. Let $x \in X$

$$\begin{aligned}(T+F)^{n+1}T^q x &= (T+F)(T+F)^n T^q x\\ &\subset (T+F)\Big[T^q(T+F)^n x + V_n\Big]\\ &\subset (T+F)T^q(T+F)^n x + (T+F)V_n.\end{aligned}$$

Let $z \in (T+F)^n x$, then $(T+F)^n x = z + (T+F)^n(0)$. So,

$$\begin{array}{rcl}(T+F)^{n+1}T^q x & \subset & (T+F)T^q z + (T+F)T^q(T+F)^n(0) + (T+F)V_n \\ & \subset & T^q(T+F)z + V_1 + (T+F)T^q(T+F)^n(0) + (T+F)V_n \\ & \subset & T^q(T+F)(T+F)^n x + V_1 + (T+F)T^q(T+F)^n(0) + (T+F)V_n.\end{array}$$

Then $(T+F)^{n+1}T^q(X) \subset T^q(T+F)^{n+1}(X) + V_{n+1}$. By the same way, we prove that $T^q(T+F)^{n+1}(X) \subset (T+F)^{n+1}T^q(X) + W_{n+1}$ and so, (9) is proved.
Then $R(T_0+F_0)^n = (T_0+F_0)^n(R(T^q)) = (T+F)^n T^q(X) =_e T^q(T+F)^n(X)$. It follows that $T^q(T+F)^n(X)$ is closed. So, $T^{-q}(T^q(T+F)^n(X)) = (T+F)^n(X) + N(T^q) = R(T+F)^n + N(T^q)$ is closed. It remains to prove that: $R(T+F)^n \cap N(T^q)$ is closed. By (7), we have

$$\dim\left[R(T+F)^n \cap N(T^p)/R(T+F)^n \cap \left[N(T+F)^n + v_n\right] \cap N(T^p) < +\infty.$$

Then it suffices to prove that $R(T+F)^n \cap \left(N(T+F)^n + v_n\right)$ is closed.
As $\dim\left[N(T+F)^n + v_n/N(T+F)^n\right] < +\infty$, then

$$\dim\left[R(T+F)^n \cap \left[N(T+F)^n + v_n\right]/R(T+F)^n \cap N(T+F)^n\right] < +\infty.$$

Then it suffices to prove that $R(T+F)^n \cap N(T+F)^n$ is closed. This is an immediate consequence of Lemmas 4.1, 4.2 and 4.4 in [15], and $a_e(T+F) < +\infty$. The result is now an immediate consequence of Lemma 2.4 and Lemma 4.1. In view of Proposition 6.1, since we have $T+F \in R_6(X)$, then $T+F$ is upper semi-B-Fredholm. □

Theorem 6.2. *Let X be a Banach space and F be a bounded operator with finite rank and $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$. Suppose that $TF = FT$. If T is lower semi-B-Fredholm linear relation then $T+F$ is lower semi-B-Fredholm linear relation.*

Proof

Since T is lower semi-B-Fredholm, then there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and T_d is lower-semi Fredholm. By Proposition 6.2, it follows that $T \in R_3(X)$. Then $d_e(T) < \infty$ and $R(T^{d_e(T)})$ is closed. We claim that $d_e(T+F) < \infty$ and $R(T+F)^{d_e(T+F)}$ is closed. Indeed,
Step 1: We prove that $d_e(T+F) < \infty$. First we show that, for all $n \geq 1$,

$$\dim\left[R(T^n)/R(T+F)^n \cap R(T^n)\right] \leq \dim R(F) < +\infty. \tag{10}$$

Let $y_1, ..., y_m \in R(T^n)$ such that $\left(\overline{y}_1, \overline{y}_2 ..., \overline{y}_m\right)$ is linearly independent in $R(T^n)/R(T+F)^n \cap R(T^n)$. Then there exist $x_1, x_2, ..., x_m \in X$ such that $y_i \in T^n x_i$ for all $1 \leq i \leq m$. Hence, $y_i \in R(T+F)^n + R(F)$. So, there exists z_i and t_i such that $y_i \in (T+F)^n z_i + Ft_i$. Suppose that $\dim\left(R(F)\right) < m$. Then there exist $\alpha_1, ..., \alpha_m$ not all zero such that $\alpha_1 Ft_1 + ... + \alpha_m Ft_m = 0$. It follows that, $\alpha_1 y_1 + ... + \alpha_m y_m \in (T+F)^n\left(\alpha_1 z_1 + ... + \alpha_m z_m\right)$. This leads to, $\alpha_1 \overline{y_1} + ... + \alpha_m \overline{y}_m = \overline{0}$. Which is a contradiction. Let $d = d_e(T) < +\infty$, then for all $n \geq d$, we have

$$\dim\left[R(T^d)/R(T^n)\right] < +\infty. \tag{11}$$

By combination of (10) and (11) we get, for all $n \geq d$

$$\dim\left[R(T^d)/R(T+F)^n \cap R(T^n)\right] < +\infty. \tag{12}$$

Thus,

$$\dim\left[R(T^d) + R(F)/R(T+F)^n \cap R(T^d)\right] < +\infty, \text{ for all } n \geq d+1. \tag{13}$$

Using (13) and substituting $T+F$ for T in (10) we get

$$\dim\left[R(T^d)+R(F)/R(T+F)^n\right] < +\infty, \text{ for all } n \geq d+1.$$

Therefore, $\dim\left[R(T+F)^n/R(T+F)^{n+1}\right] = \dim\left[R(T^d)+R(F)/R(T+F)^{n+1}\right] - \dim\left[R(T^d)+R(F)/R(T+F)^n\right] < +\infty$ for all $n \geq d$. Thus $d_e(T+F) < +\infty$.

Step 2: We show that $R(T+F)^{d_e(T+F)}$ is closed. Since $d_e(T) < \infty$ so $R(T^{d_e(T)}) =^e R(T^n)$ for all $n \geq d_e(T)$. Suppose that $d_e(T) \leq d_e(T+F)$ then $R(T^{d_e(T)}) =_e R(T^{d_e(T+F)})$. As F is of finite rank so we have $R(T+F)^{d_e(T+F)} =_e R(T^{d_e(T+F)})$. Indeed, by (10) we get

$$\dim\left[R(T^{d_e(T+F)})/R(T+F)^{d_e(T+F)} \cap R(T^{d_e(T+F)})\right] \leq \dim R(F) < \infty.$$

Hence, $R(T^{d_e(T+F)}) \subset_e R(T+F)^{d_e(T+F)}$. We replace T by $T+F$ and F by $-F$ in (10) we get

$$\dim\left[R(T+F)^{d_e(T+F)}/R(T)^{d_e(T+F)} \cap R(T+F)^{d_e(T+F)}\right] \leq \dim R(F) < \infty.$$

Hence, $R(T+F)^{d_e(T+F)} \subset_e R(T)^{d_e(T+F)}$. It follows that, $R(T+F)^{d_e(T+F)} =_e R(T^{d_e(T+F)})$. Since $R(T^{d_e(T+F)})$ is closed then $R(T+F)^{d_e(T+F)}$ is closed.

Now, if $d_e(T+F) \leq d_e(T)$ then $R(T+F)^{d_e(T+F)} =_e R(T+F)^{d_e(T)}$. As F is of finite rank then we have $R(T+F)^{d_e(T)} =_e R(T^{d_e(T)})$. Indeed, using (10) we get

$$\dim\left[R(T^{d_e(T)})/R(T+F)^{d_e(T)} \cap R(T^{d_e(T)})\right] \leq \dim R(F) < \infty.$$

Hence $R(T^{d_e(T)}) \subset_e R(T+F)^{d_e(T)}$. By interchanging T by $T+F$ and F by $-F$ in (10), we have $\dim\left[R(T+F)^{d_e(T)}/R(T^{d_e(T)}) \cap R(T+F)^{d_e(T)}\right] < \infty$. Hence $R(T+F)^{d_e(T)} \subset_e R(T^{d_e(T)})$. It follows that, $R(T+F)^{d_e(T)} =_e R(T^{d_e(T)})$. As $R(T^{d_e(T)})$ is closed then $R(T+F)^{d_e(T)}$ is closed which implies that, $R(T+F)^{d_e(T+F)}$ is closed. Thus we have $T+F \in R_3(X)$ so by Proposition 6.2, $T+F$ is lower semi-B-Fredholm. □

By means of B-Fredholm linear relation classes, we can define the following spectra :

$$\begin{array}{lcl}
\sigma_{BF}(T) & = & \left\{\lambda \in \mathbb{C} \text{ such that } T-\lambda I \text{ is not a B-Fredholm relation}\right\}, \\
\sigma_{USBF}(T) & = & \left\{\lambda \in \mathbb{C} \text{ such that } T-\lambda I \text{ is not an upper semi-B-Fredholm relation}\right\}, \\
\sigma_{LSBF}(T) & = & \left\{\lambda \in \mathbb{C} \text{ such that } T-\lambda I \text{ is not a lower semi-B-Fredholm relation}\right\}, \\
\sigma_{SBF}(T) & = & \left\{\lambda \in \mathbb{C} \text{ such that } T-\lambda I \text{ is not a semi-B-Fredholm relation}\right\},
\end{array}$$

the B-Fredholm spectrum, upper semi-B-Fredholm spectrum, lower semi-B-Fredholm spectrum and semi-B-Fredholm sepctrum, respectively.

As consequences of the above theorems we give the following corollary.

Corollary 6.1. *Let X be a Banach space and F be a bounded operator such that F is of finite rank and $T \in BCR(X)$ such that $\rho(T) \neq \emptyset$ and $\rho(T+F) \neq \emptyset$. Suppose that $TF = FT$, then*

i) $\sigma_{USBF}(T+F) = \sigma_{USBF}(T)$,

ii) $\sigma_{LSBF}(T+F) = \sigma_{LSBF}(T)$,

iii) $\sigma_{SBF}(T+F) = \sigma_{SBF}(T)$,

iv) $\sigma_{BF}(T+F) = \sigma_{BF}(T)$.

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