Filomat 28:3 (2014), 509–522
DOI (will be added later)

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete Convergence for Weighted Sums of a Class of Random Variables

Xin Deng[a], Meimei Ge[a], Xuejun Wang[a], Yanfang Liu[a], Yu Zhou[a]

[a] *School of Mathematical Sciences, Anhui University, Hefei 230601, P.R. China*

Abstract. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers and $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying the Rosenthal type inequality, which is stochastically dominated by a random variable X. Under mild conditions, we present some results on complete convergence for weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ of random variables satisfying the Rosenthal type inequality. The results obtained in the paper generalize some known ones in the literatures.

1. Introduction

In many stochastic models, the assumption that random variables are independent is not plausible. So it is of interest to extend the results from independent framework to dependent variables. The emphasis of the paper is to consider a class of random variables satisfying the Rosenthal type inequality.

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables. The Rosenthal type inequality is expressed as follows: for any $r \geq 2$ and every $n \geq 1$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Z_i - EZ_i)\right|^r\right) \leq C\left[\sum_{i=1}^{n} E|Z_i - EZ_i|^r + \left(\sum_{i=1}^{n} E(Z_i - EZ_i)^2\right)^{r/2}\right]. \tag{1.1}$$

The main purpose of the paper is to study the complete convergence for a class of random variables satisfying the Rosenthal type inequality. The concept of complete convergence was introduced by Hsu and Robbins [1] as follows.

Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant θ (write $X_n \to \theta$ completely) if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty.$$

2010 *Mathematics Subject Classification*. 60F15
Keywords. complete convergence; weighted sums; Rosenthal type inequality
Received: 18 April 2013; Accepted: 08 April 2014
Communicated by Svetlana Jankovic
Supported by the National Natural Science Foundation of China (11201001, 11171001, 11126176), the Natural Science Foundation of Anhui Province (1208085QA03, 1308085QA03, 1408085QA02), the Research Teaching Model Curriculum of Anhui University (xjyjkc1407) and the Students Innovative Training Project of Anhui University (201310357004, 201410357117).
Email address: wxjahdx2000@126.com (Xuejun Wang)

In view of the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdos [2] proved the converse. The result of Hsu-Robbins-Erdos is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz [3] for the strong law of large numbers. Due to Baum and Katz [3], one has:

Theorem 1.1. *Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n^\alpha\right) < \infty$ *for all* $\varepsilon > 0$.

Motivated by applications to sequential analysis of time series, Lai [4] extended this theorem from i.i.d. case to other dependent cases namely for some classes of φ-mixing and strong mixing sequences of random variables satisfying the following additional assumption: there exist some $\beta > 1$ and positive integer m such that as $x \to \infty$,

$$\sup_{i>m} P\left(|X_1| > x, |X_i| > x\right) = O\left(P^\beta(|X_1| > x)\right). \tag{1.2}$$

Peligrad [5] proved that the equivalence of (i) and (ii) in Theorem 1.1 holds for φ-mixing sequence without the additional assumption (1.2), but under the condition of strict stationarity. Wang [6] proved the equivalence of (i) and (ii) in Theorem 1.1 under the condition of stationarity, which is weaker than that of Peligrad [5]. Shao [7] generalized and improved the corresponding results of Wang [6] under the condition of identical distributions. Sung [8] obtained the weighted version of Baum and Katz type result for identically distributed ρ^*-mixing random variables. Wu [9] extended the well-known Baum and Katz [3] complete convergence theorem from the i.i.d. case to negatively dependent random variables and Wu [10] improved and extended this type theorem from the i.i.d. case to weighted sums of pairwise negative quadrant dependent random variables. For more details about the Baum and Katz type result, one can refer to Wang and Hu [11] and Shen [12]. The main purpose of the paper is to prove the Baum and Katz type result for weighted sums of a class of random variables satisfying the Rosenthal type inequality, which are stochastically dominated by a random variable X. The concept of stochastic domination is as follows.

Definition 1.2. *A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$.

Throughout the paper, let $I(A)$ be the indicator function of the set A. The symbol C denotes a positive constant which is not necessarily the same one in each appearance, $\lfloor x \rfloor$ denotes the integer part of x and $a \wedge b \doteq \min\{a, b\}$. $a_n = O(b_n)$ stands for $|a_n/b_n| \le C$.

2. Main Results

Our main results are as follows.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers satisfying $|a_{ni}| \le C$ for $1 \le i \le n$ and $n \ge 1$, where C is a positive constant. Suppose that (1.1) holds for any $r \ge 2$ and every $n \ge 1$, where $Z_i = a_{ni}X_iI(|X_i| \le n^\alpha)$ for $1 \le i \le n$. Then*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^\alpha\right) < \infty, \text{ for all } \varepsilon > 0. \tag{2.1}$$

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying $|a_{ni}| > 1$. Suppose that (1.1) holds for any $r \geq 2$ and every $n \geq 1$, where $Z_i = a_{ni}X_i I(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. If*

$$\sum_{i=1}^{n} |a_{ni}|^q \leq n \text{ for some } q > p, \tag{2.2}$$

Then (2.1) holds true.

Together with Theorems 2.1 and 2.2, we can get the following result.

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \text{ for some } q > p. \tag{2.3}$$

Suppose that for any $r \geq 2$ and every $n \geq 1$, (1.1) holds for $Z_i = a_{ni}X_i I(|X_i| \leq n^\alpha)$ and $Z_i = a_{ni}X_i I(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. Then (2.1) holds true.

If the condition stochastic domination is replaced by identical distribution, then we can get the following result on complete convergence for weighted sums of identically distributed random variables satisfying the Rosenthal type inequality.

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables with $EX_1 = 0$. Let $p > 1/\alpha$, $1/2 < \alpha \leq 1$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying (2.3). Suppose that for any $r \geq 2$ and every $n \geq 1$, (1.1) holds for $Z_i = a_{ni}X_i I(|X_i| \leq n^\alpha)$ and $Z_i = a_{ni}X_i I(|a_{ni}X_i| \leq n^\alpha)$ for $1 \leq i \leq n$. If $E|X_1|^p < \infty$, then (2.1) holds true.*

Conversely, suppose that for every $n \geq 1$,

$$E\left|\sum_{i=1}^{n}(Z_i - EZ_i)\right|^2 \leq C\sum_{i=1}^{n} E(Z_i - EZ_i)^2, \tag{2.4}$$

where $Z_i = I(|X_i| > n^\alpha)$ for $1 \leq i \leq n$. If (2.1) holds for any array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfying (2.3), then $E|X_1|^p < \infty$.

Remark 2.1. There are many sequences of random variables satisfying the Rosenthal type inequality (1.1). See for example, independent random variables, φ-mixing identically distributed random variables with $\sum_{n=}^{\infty} \varphi^{1/2}(2^n) < \infty$ (see Shao [7]), ρ-mixing identically distributed random variables with $\sum_{n=}^{\infty} \rho^{2/p}(2^n) < \infty$ (see Shao [13]), negatively associated random variables (see Shao [14]), $\tilde{\rho}$-mixing random variables (see Utev and Peligrad [15]), φ-mixing random variables with $\sum_{n=}^{\infty} \varphi^{1/2}(n) < \infty$ (see Wang et al. [16]), asymptotically almost negatively associated random variables (see Yuan and An [17]), negatively superadditive-dependent random variables (see Wang et al. [18]). Therefore, the main results of the paper also hold for these sequences of random variables.

Remark 2.2. Bai and Cheng [19] presented a necessary and sufficient condition for the almost sure convergence of weighted sums of independent and identically distributed random variables as follows (see Theorem 2.1 in Bai and Cheng [19]): let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with mean zero and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Denote $T_n = \sum_{i=1}^{n} a_{ni}X_i$. Suppose that $E|X|^\beta < \infty$ and

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n) \tag{2.5}$$

holds for some $1 < p < 2$ and $1/p = 1/\alpha + 1/\beta$. Then

$$T_n/n^{1/p} \to 0 \ \ a.s. \tag{2.6}$$

Conversely, if (2.6) is true for any coefficient arrays satisfying (2.5), then $E|X|^\beta < \infty$.

Bai and Cheng's assumptions reduce to requiring the existence of a moment of order $\beta > 1/\alpha$ and the assumption on the weights becomes $q > 1/\alpha$. Compared with Theorem 2.1 in Bai and Cheng [19], Theorem 2.4 in our paper presents a necessary and sufficient condition for the complete convergence of weighted sums of a class of random variables satisfying the Rosenthal type inequality, which is more general than almost sure convergence. It is easily seen that if $p\alpha \geq 2$, then (2.1) implies (2.6). Hence, the result of Theorem 2.4 in the paper generalizes and improves the corresponding one of Theorem 2.1 in Bai and Cheng [19].

3. Proof of the Main results

To prove the main results of the paper, we need the following useful lemma. For the proof, one can refer to Wu ([20], [21]), Shen ([22], [23]) or Shen and Wu [24].

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1 \left[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\right] \tag{3.1}$$

and

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b), \tag{3.2}$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

Proof of Theorem 2.1.

For $1 \leq i \leq n$ and $n \geq 1$, define $X'_{ni} = X_i I(|X_i| \leq n^\alpha)$. Since $EX_i = 0$ and $|a_{ni}| \leq C$, we have that

$$\begin{aligned}
n^{-\alpha} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} EX'_{ni} \right| &= n^{-\alpha} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} EX_i I(|X_i| > n^\alpha) \right| \\
&\leq n^{-\alpha} \max_{1\leq j\leq n} \sum_{i=1}^{j} |a_{ni} EX_i I(|X_i| > n^\alpha)| \\
&\leq n^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^\alpha) \\
&\leq Cn^{-\alpha} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > n^\alpha) \\
&\leq Cn^{1-\alpha} E|X| I(|X| > n^\alpha) \\
&\leq Cn^{1-p\alpha} E|X|^p \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Hence for all n large enough, we have

$$n^{-\alpha} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni} EX'_{ni} \right| < \frac{\varepsilon}{2}.$$

It follows that

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \\
\le\ & \sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{i=1}^{n} P(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X'_{ni}\right| > \varepsilon n^{\alpha}\right) \\
\le\ & C\sum_{n=1}^{\infty} n^{p\alpha-1} P(|X| > n^{\alpha}) + C\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon n^{\alpha}}{2}\right) \\
\doteq\ & CI + CJ. \qquad (3.3)
\end{aligned}
$$

It is easily seen that

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty} n^{p\alpha-1}\sum_{i=n}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) \\
&= \sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) \sum_{n=1}^{i} n^{p\alpha-1} \\
&\le C\sum_{i=1}^{\infty} P(i^{\alpha} < |X| \le (i+1)^{\alpha}) i^{p\alpha} \\
&\le CE|X_1|^p < \infty. \qquad (3.4)
\end{aligned}
$$

Next, we show $J < \infty$. We have by Markov inequality and (1.1) that for any $r \ge 2$,

$$
\begin{aligned}
J &\le \left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\right|^r\right) \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni} - EX'_{ni})^2\right)^{r/2} + \sum_{i=1}^{n} |a_{ni}|^r E|X'_{ni} - EX'_{ni}|^r\right\} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left\{\left(\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni})^2\right)^{r/2} + \sum_{i=1}^{n} |a_{ni}|^r E|X'_{ni}|^r\right\} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n} E|X_i|^2 I(|X_i| \le n^{\alpha})\right)^{r/2} + C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n} E|X_i|^r I(|X_i| \le n^{\alpha}) \\
&\doteq CJ_1 + CJ_2 \qquad (3.5)
\end{aligned}
$$

In the last inequality, we used the fact that $|a_{ni}| \le C$ for $1 \le i \le n$ and $n \ge 1$.

If $p \ge 2$, then we take $r > \max\{(p\alpha - 1)/(\alpha - 1/2), p\}$. Taking into account Lemma 3.1, we have that

$$
\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left[\sum_{i=1}^{n}(EX^2 I(|X| \le n^{\alpha}) + n^{2\alpha}P(|X| > n^{\alpha}))\right]^{r/2} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left[\sum_{i=1}^{n}(EX^2 I(|X| \le n^{\alpha}) + EX^2 I(|X| > n^{\alpha}))\right]^{r/2} \\
&\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty. \qquad (3.6)
\end{aligned}
$$

Since $r > p$, it follows by Lemma 3.1 again and (3.4) that

$$\begin{aligned}
J_2 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n}(E|X|^r I(|X|\leq n^{\alpha}) + n^{r\alpha}P(|X|>n^{\alpha}))\\
&= C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1}E|X|^r I(|X|\leq n^{\alpha}) + C\sum_{n=1}^{\infty} n^{p\alpha-1}P(|X|>n^{\alpha})\\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-1}\sum_{i=1}^{n}E|X|^r I((i-1)^{\alpha}<|X|\leq i^{\alpha}) + C\\
&= C\sum_{i=1}^{\infty} E|X|^r I((i-1)^{\alpha}<|X|\leq i^{\alpha})\sum_{n=i}^{\infty} n^{p\alpha-r\alpha-1} + C\\
&\leq C\sum_{i=1}^{\infty} E|X|^r I((i-1)^{\alpha}<|X|\leq i^{\alpha}) i^{p\alpha-r\alpha} + C\\
&\leq CE|X|^p + C < \infty. \qquad (3.7)
\end{aligned}$$

If $p < 2$, then we take $r = 2$. Since $r > p$, (3.7) still holds, and so $J_1 = J_2 < \infty$. This completes the proof of the theorem.

Proof of Theorem 2.2.

If $p < 2$, then we can take $\delta > 0$ such that $p < p + \delta < \min\{2, q\}$. Since $|a_{ni}| > 1$, we have that $\sum_{i=1}^{n} |a_{ni}|^{p+\delta} \leq \sum_{i=1}^{n} |a_{ni}|^q \leq n$. Thus we may assume that (2.2) holds for some $p < q < 2$ when $p < 2$.

Let $S'_{nj} = \sum_{i=1}^{j} a_{ni}X_i I(|a_{ni}X_i| \leq n^{\alpha})$ for $1 \leq j \leq n$ and $n \geq 1$. In view of $EX_i = 0$, we have by Lemma 3.1 that

$$\begin{aligned}
n^{-\alpha}\max_{1\leq j\leq n}\left|ES'_{nj}\right| &\leq n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X_i|I(|a_{ni}X_i|>n^{\alpha})\\
&\leq n^{-p\alpha}\sum_{i=1}^{n} E|a_{ni}X_i|^p\\
&\leq Cn^{-p\alpha}\sum_{i=1}^{n}|a_{ni}|^p E|X|^p\\
&\leq Cn^{-p\alpha}\left(\sum_{i=1}^{n}|a_{ni}|^q\right)^{p/q} n^{1-p/q}E|X|^p\\
&\leq Cn^{1-p\alpha}E|X|^p \to 0, \text{ as } n\to\infty.
\end{aligned}$$

Hence for all n large enough, we have that

$$n^{-\alpha}\max_{1\leq j\leq n}\left|ES'_{nj}\right| < \varepsilon/2.$$

It follows that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^{\alpha}\right) \\
\le & \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le i\le n}|a_{ni}X_i| > n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|S'_{nj}\right| > \varepsilon n^{\alpha}\right) \\
\le & C\sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|a_{ni}X| > n^{\alpha}) + C\sum_{n=1}^{\infty} P\left(\max_{1\le j\le n}\left|S'_{nj} - ES'_{nj}\right| > \frac{\varepsilon n^{\alpha}}{2}\right) \\
\doteq & CI + CJ
\end{aligned} \tag{3.8}$$

For $1 \le j \le n-1$ and $n \ge 2$, let

$$I_{nj} = \left\{1 \le i \le n : n^{1/q}(j+1)^{-1/q} < |a_{ni}| \le n^{1/q} j^{-1/q}\right\}. \tag{3.9}$$

Thus, $\{I_{nj}, 1 \le j \le n-1\}$ are disjoint and $\bigcup_{j=1}^{n-1} I_{nj} = \{1 \le i \le n : a_{ni} \ne 0\}$. It follows by (2.2) that

$$n \ge \sum_{\{1\le i\le n: a_{ni}\ne 0\}} |a_{ni}|^q = \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^q \ge n\sum_{j=1}^{k} \frac{1}{j+1}\#I_{nj} \ge \frac{n}{k+1}\sum_{j=1}^{k} \#I_{nj}, \tag{3.10}$$

which implies that $\sum_{j=1}^{k} \#I_{nj} \le k+1$ for $1 \le k \le n-1$.

Denote $t = 1/(\alpha - 1/q)$. It is easily checked that,

$$\begin{aligned}
I \le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1}\sum_{i\in I_{nj}} P(|a_{ni}X| > n^{\alpha}) \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1} P\left(|X|^t > nj^{t/q}\right)\#I_{nj} \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{j=1}^{n-1} \#I_{nj} \sum_{k\ge \lfloor nj^{t/q}\rfloor} P\left(k < |X|^t \le k+1\right) \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\infty} P\left(k < |X|^t \le k+1\right) \sum_{j=1}^{(n-1)\wedge\lfloor((k+1)/n)^{q/t}\rfloor} \#I_{nj} \\
\le\ & C\sum_{n=2}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\infty} P\left(k < |X|^t \le k+1\right)\left(n \wedge \left(\left\lfloor\left(\frac{k+1}{n}\right)^{q/t}\right\rfloor + 1\right)\right) \\
\le\ & C\sum_{n=1}^{\infty} n^{p\alpha-2}\sum_{k=n}^{\lceil n^{1+t/q}\rceil} P\left(k < |X|^t \le k+1\right)\left\lfloor\left(\frac{k+1}{n}\right)^{q/t} + 1\right\rfloor \\
& +C\sum_{n=1}^{\infty} n^{p\alpha-1}\sum_{k=\lfloor n^{1+t/q}\rfloor+1}^{\infty} P\left(k < |X|^t \le k+1\right) \\
=:\ & I_1 + I_2.
\end{aligned} \tag{3.11}$$

Since $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we obtain

$$\begin{aligned} I_1 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-q/t} \sum_{k=n}^{\lfloor n^{1+t/q}\rfloor} P\left(k < |X|^t \leq k+1\right) k^{q/t} \\ &\leq C\sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{q/t} \sum_{n=\lfloor k^{q/(q+t)}\rfloor}^{k} n^{p\alpha-2-q/t} \\ &\leq C\sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{q/t-q\alpha(q-p)/(q+t)} \\ &\leq CE|X|^p < \infty. \end{aligned} \tag{3.12}$$

We also obtain

$$\begin{aligned} I_2 &\leq C\sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) \sum_{n=1}^{\lfloor k^{q/(t+q)}\rfloor} n^{p\alpha-1} \\ &\leq C\sum_{k=1}^{\infty} P\left(k < |X|^t \leq k+1\right) k^{p(\alpha-1/q)} \\ &\leq CE|X|^p < \infty. \end{aligned} \tag{3.13}$$

From $I_1 < \infty$ and $I_2 < \infty$, we have $I < \infty$. Thus, it remains to show that $J < \infty$. We have by Markov inequality and (1.1) that for any $r \geq 2$,

$$\begin{aligned} J &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2} E\left(\max_{1\leq j\leq n}\left|S'_{nj} - ES'_{nj}\right|^r\right) \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \leq n^{\alpha})\right)^{r/2} \\ &\quad +C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n} E|a_{ni}X_i|^r I(|a_{ni}X_i| \leq n^{\alpha}) \\ &\doteq J_1 + J_2. \end{aligned} \tag{3.14}$$

Observe that for $r \geq q$ and $n > m$,

$$n \geq \sum_{j=1}^{n-1}\sum_{i\in I_{nj}} |a_{ni}|^q \geq n\sum_{j=1}^{n-1}\frac{1}{j+1}\#I_{nj} \geq n(m+1)^{r/q-1}\sum_{j=m}^{n-1}(j+1)^{-r/q}\#I_{nj},$$

which implies that $\sum_{j=m}^{n-1} j^{-r/q}\#I_{nj} \leq Cm^{-(r/q-1)}$ for $r \geq q$ and $n > m$. For J_1 and J_2, we consider the following two cases.

(i) If $p \geq 2$, then we take $r > \max\{(p\alpha-1)/(\alpha-1/2), q\}$. Taking into account Lemma 3.1, we have that

$$\begin{aligned} J_1 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n}|a_{ni}|^2 EX^2\right)^{r/2} \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\left(\sum_{i=1}^{n}|a_{ni}|^q\right)^{r/2} \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2+r/2} < \infty. \end{aligned} \tag{3.15}$$

The second inequality follows by the fact that $|a_{ni}| > 1$.

It follows by Lemma 3.1 and $I < \infty$ that

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{p\alpha-r\alpha-2}\sum_{i=1}^{n}(E|a_{ni}X|^r I(|a_{ni}X|\le n^{\alpha}) + n^{r\alpha}P(|a_{ni}X|>n^{\alpha}))\\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2}\sum_{j=1}^{n-1}\sum_{i\in I_{nj}} E|a_{ni}X|^r I(|a_{ni}X|\le n^{\alpha})\\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{j=1}^{n-1} j^{-r/q}\#I_{nj}E|X|^r I\left(|X|^t\le n(j+1)^{t/q}\right)\\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{j=1}^{n-1} j^{-r/q}\#I_{nj}\sum_{0\le k\le\lfloor n(j+1)^{t/q}\rfloor} E|X|^r I\left(k<|X|^t\le k+1\right)\\
&= C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{j=1}^{n-1} j^{-r/q}\#I_{nj}\sum_{k=0}^{2n} E|X|^r I\left(k<|X|^t\le k+1\right)\\
&\quad +C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{j=1}^{n-1} j^{-r/q}\#I_{nj}\sum_{k=2n+1}^{\lfloor n(j+1)^{t/q}\rfloor} E|X|^r I\left(k<|X|^t\le k+1\right)\\
&\doteq CJ_3 + CJ_4. \qquad (3.16)
\end{aligned}$$

Since $p\alpha - r\alpha - 2 + r/q < q\alpha - r\alpha - 2 + r/q = -(r-q)(\alpha - 1/q) - 1 < -1$ and $q > p$, we have that

$$\begin{aligned}
J_3 &= \sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{k=0}^{2n} E|X|^r I\left(k<|X|^t\le k+1\right)\sum_{j=1}^{n-1} j^{-r/q}\#I_{nj}\\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q}\sum_{k=0}^{2n} E|X|^r I\left(k<|X|^t\le k+1\right)\\
&\le C\sum_{k=1}^{\infty} E|X|^r I\left(k<|X|^t\le k+1\right)\sum_{n=\lfloor k/2\rfloor}^{\infty} n^{p\alpha-r\alpha-2+r/q}\\
&\le C\sum_{k=1}^{\infty} E|X|^r I\left(k<|X|^t\le k+1\right)k^{p\alpha-r\alpha-1+r/q}\\
&\le C\sum_{k=1}^{\infty} P\left(k<|X|^t\le k+1\right)k^{p\alpha-1}\\
&\le CE|X|^p < \infty. \qquad (3.17)
\end{aligned}$$

Since $1/t + 1/q - \alpha = 0$ and $p\alpha - 2 - q/t = -\alpha(q-p) - 1 < -1$, we also have that

$$\begin{aligned}
J_4 &\leq C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{k=2n+1}^{\lfloor n^{(q+t)/q}\rfloor} E|X|^r I\left(k < |X|^t \leq k+1\right) \sum_{j=\lfloor (k/n)^{q/t}\rfloor -1}^{n-1} j^{-r/q} \# I_{nj} \\
&\leq C\sum_{n=2}^{\infty} n^{p\alpha-r\alpha-2+r/q} \sum_{k=2n+1}^{\lfloor n^{(q+t)/q}\rfloor} E|X|^r I\left(k < |X|^t \leq k+1\right) \left(\left\lfloor \left(\frac{k}{n}\right)^{q/t} \right\rfloor - 1\right)^{-(r/q-1)} \\
&\leq C\sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1\right) k^{-(r-q)/t} \sum_{n=\lfloor k^{q/(q+t)}\rfloor}^{\lfloor k/2\rfloor} n^{p\alpha-2-q/t} \\
&\leq C\sum_{k=5}^{\infty} E|X|^r I\left(k < |X|^t \leq k+1\right) k^{-(r-q)/t-(\alpha-1/q)(q-p)} \\
&\leq CE|X|^p < \infty.
\end{aligned} \tag{3.18}$$

From $J_3 < \infty$ and $J_4 < \infty$, we have $J_2 < \infty$.

(ii) If $p < 2$, then we take $r = 2$. As noted above, we may assume that $p < q < 2$. Since $r > q$, as in the case $p \geq 2$, we have $J_1 = J_2 \leq CE|X_1|^p < \infty$. This completes the proof of the theorem.

Proof of Theorem 2.3.

The assumption (2.3) implies that $\sum_{i=1}^{n} |a_{ni}|^q \leq Cn$ for some $q > p$, where C is a positive constant. For $n \geq 1$, let

$$A_n = \{1 \leq i \leq n : |a_{ni}|/C^{1/q} \leq 1\}, B_n = \{1 \leq i \leq n : |a_{ni}|/C^{1/q} > 1\}, \tag{3.19}$$

and let $a'_{ni} = a_{ni}/C^{1/q}$ if $i \in A_n$, $a'_{ni} = 0$ otherwise, and $a''_{ni} = a_{ni}/C^{1/q}$ if $i \in B_n$, $a''_{ni} = 0$ otherwise. Then

$$\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} \frac{a_{ni}}{C^{1/q}} X_i\right| \leq \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a'_{ni} X_i\right| + \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a''_{ni} X_i\right|. \tag{3.20}$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) &\leq \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a'_{ni} X_i\right| > \frac{\varepsilon n^{\alpha}}{2C^{1/q}}\right) \\
&+ \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a''_{ni} X_i\right| > \frac{\varepsilon n^{\alpha}}{2C^{1/q}}\right) \\
&\doteq I + J.
\end{aligned} \tag{3.21}$$

By Theorem 2.1, we have $I < \infty$. By Theorem 2.2, we have $J < \infty$. Then (2.1) holds. This completes the proof of the theorem.

Proof of Theorem 2.4.

The sufficient part can be obtained by Theorem 2.3. So we only need to show necessity.

Choose, for each $n \geq 1, a_{n1} = \cdots = a_{nn} = 1$. Then $\{a_{ni}\}$ satisfies (2.3). By (2.1), we obtain that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0, \tag{3.22}$$

which implies that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}|X_j|>\varepsilon n^{\alpha}\right)\\
\le\ &\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right|>\frac{\varepsilon n^{\alpha}}{2}\right)+\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j-1} X_i\right|>\frac{\varepsilon n^{\alpha}}{2}\right)\\
<\ &\infty,\ \ \forall\varepsilon>0.
\end{aligned}\tag{3.23}$$

Observe that

$$\begin{aligned}
\infty &> \sum_{i=1}^{\infty}\sum_{n=2^{i-1}+1}^{2^i} n^{p\alpha-2} P\left(\max_{1\le j\le n}|X_j|>\varepsilon n^{\alpha}\right)\\
&\ge \begin{cases}\sum_{i=1}^{\infty}(2^{i-1})^{p\alpha-2}2^{i-1}P\left(\max\limits_{1\le j\le 2^{i-1}}|X_j|>\varepsilon(2^i)^{\alpha}\right), & \text{if } p\alpha\ge 2,\\ \sum_{i=1}^{\infty}(2^{i})^{p\alpha-2}2^{i-1}P\left(\max\limits_{1\le j\le 2^{i-1}}|X_j|>\varepsilon(2^i)^{\alpha}\right), & \text{if } 1<p\alpha<2,\end{cases}\\
&\ge \begin{cases}\sum_{i=1}^{\infty}P\left(\max\limits_{1\le j\le 2^{i-1}}|X_j|>\varepsilon(2^i)^{\alpha}\right), & \text{if } p\alpha\ge 2,\\ 2^{p\alpha-2}\sum_{i=1}^{\infty}P\left(\max\limits_{1\le j\le 2^{i-1}}|X_j|>\varepsilon(2^i)^{\alpha}\right), & \text{if } 1<p\alpha<2.\end{cases}
\end{aligned}$$

By the convergence of series, we have that for any $\varepsilon>0$,

$$P\left(\max_{1\le j\le 2^{i-1}}|X_j|>\varepsilon(2^i)^{\alpha}\right)\to 0\ \text{ as }\ i\to\infty.$$

Hence, for any $n\ge 1$, there exists a i such that $2^{i-2}\le n<2^{i-1}$, thus

$$\begin{aligned}
P\left(\max_{1\le j\le n}|X_j|>n^{\alpha}\right) &\le P\left(\max_{1\le j\le 2^{i-1}}|X_j|>(2^{i-2})^{\alpha}\right)\\
&= P\left(\max_{1\le j\le 2^{i-1}}|X_j|>2^{-2\alpha}(2^i)^{\alpha}\right)\to 0,\ \text{as } n\to\infty.
\end{aligned}\tag{3.24}$$

Since

$$P\left(\max_{1\le j\le n}|X_j|>n^{\alpha}\right)=\sum_{j=1}^{n}P\left(|X_j|>n^{\alpha},\max_{1\le i\le j-1}|X_i|\le n^{\alpha}\right),$$

it follows that

$$nP(|X_1|>n^{\alpha})=P\left(\max_{1\le j\le n}|X_j|>n^{\alpha}\right)+\sum_{j=1}^{n}P\left(|X_j|>n^{\alpha},\max_{1\le i\le j-1}|X_i|>n^{\alpha}\right).\tag{3.25}$$

It is easy to check that

$$\begin{aligned}
&\sum_{j=1}^{n} P\left(|X_j| > n^{\alpha}, \max_{1\le i\le j-1}|X_i| > n^{\alpha}\right)\\
= &\sum_{j=1}^{n} EI\left(|X_j| > n^{\alpha}\right) I\left(\max_{1\le i\le j-1}|X_i| > n^{\alpha}\right)\\
= &\sum_{j=1}^{n} E\left((I(|X_j| > n^{\alpha}) - P(|X_j| > n^{\alpha}))I\left(\max_{1\le i\le j-1}|X_i| > n^{\alpha}\right)\right)\\
&+\sum_{j=1}^{n} EP(|X_j| > n^{\alpha})I\left(\max_{1\le i\le j-1}|X_i| > n^{\alpha}\right)\\
\le &E\left(\sum_{j=1}^{n}(I(|X_j| > n^{\alpha}) - P(|X_1| > n^{\alpha}))I\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\right)\\
&+nP(|X_1| > n^{\alpha})P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\\
\doteq &I + II. \qquad (3.26)
\end{aligned}$$

According to the Cauchy-Schwarz inequality, we have by (2.4) that

$$\begin{aligned}
I^2 \le &\left(E\left|\sum_{j=1}^{n}(I(|X_j| > n^{\alpha}) - P(|X_1| > n^{\alpha}))I\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\right|\right)^2\\
\le &E\left(\sum_{j=1}^{n}(I(|X_j| > n^{\alpha}) - P(|X_1| > n^{\alpha}))\right)^2 P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\\
\le &C\left(\sum_{j=1}^{n} E(I(|X_j| > n^{\alpha}) - P(|X_1| > n^{\alpha}))^2\right)P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\\
= &Cn\mathrm{Var}(I(|X_1| > n^{\alpha}))P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\\
\le &CnP(|X_1| > n^{\alpha})P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\\
\le &\left[\frac{1}{4}nP(|X_1| > n^{\alpha}) + CP\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right)\right]^2. \qquad (3.27)
\end{aligned}$$

Now we return the estimate from the relation (3.27) into the relation (3.26) and then into relation (3.25) and get

$$\frac{3}{4}nP(|X_1| > n^{\alpha}) \quad\le\quad 2(1+C)P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right) + nP(|X_1| > n^{\alpha})P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right). \qquad (3.28)$$

By (3.24) and (3.28), for all n sufficiently large, we have

$$nP(|X_1| > n^{\alpha}) \le 4(1+C)P\left(\max_{1\le i\le n}|X_i| > n^{\alpha}\right). \qquad (3.29)$$

Relations (3.23) and (3.29) finally yield that

$$\sum_{n=1}^{\infty} n^{p\alpha-1}P(|X_1| > n^{\alpha}) < \infty.$$

Hence,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{p\alpha-1} P(|X_1| > n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{p\alpha-1} \sum_{i=n}^{\infty} P(i^{\alpha} < |X_1| \le (i+1)^{\alpha}) \\
&= \sum_{i=1}^{\infty} P(i^{\alpha} < |X_1| \le (i+1)^{\alpha}) \sum_{n=1}^{i} n^{p\alpha-1} \\
&\ge C \sum_{i=1}^{\infty} P(i^{\alpha} < |X_1| \le (i+1)^{\alpha}) i^{p\alpha} \\
&\ge C \sum_{i=1}^{\infty} E\left(I(i^{\alpha} < |X_1| \le (i+1)^{\alpha})(i+1)^{p\alpha}\right) \\
&\ge C \sum_{i=1}^{\infty} E\left(I(i^{\alpha} < |X_1| \le (i+1)^{\alpha})|X_1|^{p}\right) \\
&\ge CE|X_1|^p,
\end{aligned}$$

which implies that $E|X_1|^p < \infty$. This completes the proof of the theorem.

Acknowledgements. The authors are most grateful to the Section Editor Svetlana Jankovic and anonymous referee for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

References

[1] P.L. Hsu, H. Robbins, Complete convergence and the law of large numbers, Proceedings of the National Academy of Sciences of the United States of America 33 (1947) 25–31.
[2] P. Erdos, On a theorem of Hsu and Robbins, The Annals of Mathematical Statistics 20 (1949) 286–291.
[3] L.E. Baum, M. Katz, Convergence rates in the law of large numbers, Transactions of the American Mathematical Society 120 (1965) 108–123.
[4] T.L. Lai, Convergence rates and r-quick versions of the strong law for stationary mixing sequences, The Annals of Probability 5 (1977) 693–706.
[5] M. Peligrad, Convergence rates of the strong law for stationary mixing sequences, Z. Wahrsch. verw. Gebiete 70 (1985) 307–314.
[6] D.C. Wang, The complete convergence of the sums for the stationary sequence of φ-mixing random variables, Journal of System Science and Mathematical Science 7 (1987) 352-359.
[7] Q.M. Shao, A moment inequality and its applications, Acta Mathematica Sinica 31 (1988) 736–747.
[8] S.H. Sung, Complete convergence for weighted sums of ρ^*-mixing random variables, Discrete Dynamics in Nature and Society 2010 (2010) 13 pages.
[9] Q.Y. Wu, A complete convergence theorem for weighted sums of arrays of rowwise negatively dependent random variables, Journal of Inequalities and Applications 2012 (2012) 10 pages.
[10] Q.Y. Wu, Sufficient and necessary conditions of complete convergence for weighted sums of PNQD random variables, Journal of Applied Mathematics 2012 (2012) 10 pages.
[11] X.J. Wang, S.H. Hu, Some Baum-Katz type results for φ-mixing random variables with different distributions, RACSAM 106 (2012) 321–331.
[12] A.T. Shen, On the strong convergence rate for weighted sums of arrays of rowwise negatively orthant dependent random variables, RACSAM 107 (2013) 257–271.
[13] Q.M. Shao, Maximal inequalities for partial sums of ρ mixing sequences, The Annals of Probability 23 (1995) 948–965.
[14] Q.M. Shao, A comparison theorem on moment inequalities between negatively associated and independent random variables, Journal of Theoretical Probability 13 (2000) 343–356.
[15] S. Utev, M. Peligrad, Maximal inequalities and an invariance principle for a class of weakly dependent random variables, Journal of Theoretical Probability 16 (2003) 101–115.
[16] X.J. Wang, S.H. Hu, W.Z. Yang, Y. Shen, On complete convergence for weighted sums of φ mixing random variables, Journal of Inequalities and Applications 2010 (2010) 13 pages.
[17] D.M. Yuan, J. An, Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications, Science in China Series A: Mathematics 52 (2009) 1887–1904.

[18] X.J. Wang, X. Deng, L.L. Zheng, S.H. Hu, Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications, manuscript ID: 800066, DOI:10.1080/02331888.2013.800066, in press (2013).
[19] Z.D. Bai, P.E. Cheng, Marcinkiewciz strong laws for linear statistics, Statistics and Probability Letters 46 (2000) 105–112.
[20] Q.Y. Wu, Probability Limit Theory for Mixing Sequences, Science Press of China, Beijing (2006).
[21] Q.Y. Wu,A strong limit theorem for weighted sums of sequences of negatively dependent random variables, Journal of Inequalities and Applications 2010 (2010) 8 pages.
[22] A.T. Shen, On asymptotic approximation of inverse moments for a class of nonnegative random variables, Statistics, Article ID 801480, doi: 10.1080/02331888.2013.801480, in press (2013)
[23] A.T. Shen, Bernstein-type inequality for widely dependent sequence and its application to nonparametric regression models, Abstract and Applied Analysis 2013 (2013) 9 pages.
[24] A.T. Shen, R.C. Wu, Strong and weak convergence for asymptotically almost negatively associated random variables, Discrete Dynamics in Nature and Society 2013 (2013) 7 pages.