Filomat 31:14 (2017), 4441–4454
https://doi.org/10.2298/FIL1714441G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed Point Problems for α-ψ-Weakly Contractive Operators on KST Spaces

Liliana Guran[a], Amelia Bucur[b]

[a]*Department of Pharmaceutical Sciences, "Vasile Goldiş" Western University of Arad, Revoluţiei Avenue, no. 94-96, 310025, Arad, România.*
[b]*Department of Mathematics and Informatics, "Lucian Blaga" University of Sibiu, Dr.I.Raţiu Street, No.5-7, 550012, Sibiu, România.*

Abstract. Using the concept of w-distance we will investigate some existence, uniqueness and Ulam-Hyers stability results for fixed point problems concerning α-ψ-weakly contractive operators. Starting at the results of P. Amiri, S. Rezapour, N. Shahzad [1] the presented theorems extend and generalize several results concerning a w-distance.

1. Introduction and Preliminaries

In 2012 Samet, C. Vetro, P. Vetro [27] introduced the notion of α-ψ-contractive type operator and proved some fixed point results. They proved that several fixed point theorems, including the Banach contraction principle, can be derived from their main results. Many other results concerning this type of operators was obtained for different spaces, see [1, 3, 8, 15, 17].

In 1996, O. Kada, T. Suzuki and W. Takahashi [14] introduced the notion of w-distance. They elaborated, with the help of examples, that the concept of w-distance is general than that of metric on a nonempty set. They also proved a generalization of Caristi fixed point theorem (see [10]) employing the definition of w-distance on a complete metric space.

The first stability problem was raised by Ulam [30] during his talk at the University of Wisconsin in 1940, concerns the stability of group homomorphisms. The first affirmative partial answer to the question of Ulam was given, for Banach spaces, by Hyers [12] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability. Ulam-Hyers stability results in fixed point theory have been investigated by many authors, see e.g. Bota-Boriceanu and Petruşel [7], Hyers [12, 13] Lazăr [16], Petru, Petrusel, Yao [19], Rus [23, 25].

Recently, L. Guran [11] obtained some results concerning the Ulam-Hyers stability on a KST-space, a generalization of a usual metric space, which is a metric space endowed with a w-distance (in the sense of Kada-Suzuki-Takahashi).

Starting at the results of P. Amiri, S. Rezapour, N. Shahzad [1] we study on this paper the existence, uniqueness and generalized Ulam-Hyers w-stability a fixed point of α-ψ-contractive type operator on a KST-space.

Further, we recollect some essential definitions and fundamental results.

Let (X, d) be a metric space and $f : X \to X$ be a singlevalued operator. We will use the following notations:
$P(X)$ - *the set of all nonempty subsets of* X;
$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;
$P_{cp}(X)$ - *the set of all nonempty compact subsets of* X;
$Fix(f) := \{x \in X \mid x = f(x)\}$ - *the set of the fixed points of* F.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [14]) as follows.

2010 *Mathematics Subject Classification*. 46T99; 47H10; 54H25
Keywords. α-ψ-weakly contractive operator, Well-posedness, Fixed point, KST-space, Ulam-Hyers w-stability.
Received: 14 March 2016; Accepted: 22 September 2016
Communicated by Calogero Vetro
Research supported by "Lucian Blaga" University of Sibiu, research grants LBUS-IRG-2015-01.
Email addresses: lguran@uvvg.ro (Liliana Guran), amelia.bucur@ulbsibiu.ro (Amelia Bucur)

Definition 1.1. *Let (X,d) be a metric space. Then $w : X \times X \to \mathbb{R}_+$ is called a weak distance (briefly w-distance) on X if the following axioms are satisfied :*

1. *$w(x,z) \leq w(x,y) + w(y,z)$, for any $x,y,z \in X$;*
2. *for any $x \in X$, $w(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;*
3. *for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \leq \delta$ and $w(z,y) \leq \delta$ implies $d(x,y) \leq \varepsilon$.*

By definition, the triple (X,d,w) is a KST-space if X is a nonempty set, $d : X \times X \to \mathbb{R}_+$ is a metric on X and $w : X \times X \to \mathbb{R}_+$ is a w-distance on X.

Let (X,d,w) be a KST-space. We say that (X,d,w) is a complete KST-space if the metric space (X,d) is complete.

Some examples of w-distance can be find in [14].

For the proof of the main results we need the following crucial result for w-distance (see[29]).

Lemma 1.2. *Let (X,d) be a metric space and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0,+\infty[$ converging to zero and let $x,y,z \in X$. Then the following hold:*

1. *If $w(x_n,y) \leq \alpha_n$ and $w(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $w(x_n,y_n) \leq \alpha_n$ and $w(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $w(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
4. *If $w(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0,\infty)$. We denote by Φ, the class of the corporation function $\varphi : [0,\infty) \to [0,\infty)$. For more details and examples , see e.g. [6, 24].

Among them, we recall the following essential result.

Lemma 1.3. *(Berinde [6], Rus [24]) If $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;

(2) φ is continuous at 0;

(3) $\varphi(t) < t$, for any $t > 0$.

Next, we recall the definition of α-ψ-contractive and α-admissible mappings introduced by Samet et al. [27].

We denote with Ψ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. It is clear that if $\Phi \subset \Psi$ (see e.g. [18]) and hence, by Lemma 1.3 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

Definition 1.4. *(Samet et al. [27]) Let (X,d) be a metric space and $f : X \to X$ be a given mapping. We say that f is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)d(f(x),f(y)) \leq \psi(d(x,y)), \text{ for all } x,y \in X. \tag{1}$$

Definition 1.5. *(Samet et al. [27]) Let $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that f is α-admissible if*

$$x,y \in X, \quad \alpha(x,y) \geq 1 \Longrightarrow \alpha(f(x),f(y)) \geq 1.$$

For exemples see [27].

Next let us recall some important results concerning α-ψ-contractive mappings.

Theorem 1.6. *(Samet et al. [27]) Let (X,d) be a complete metric space and $f : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,f(x_0)) \geq 1$;

(iii) f is continuous.

Then, f has a fixed point, that is, there exists $x^* \in X$ such that $f(x^*) = x^*$.

Theorem 1.7. *(Samet et al. [27]) Let (X,d) be a complete metric space and $f : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,f(x_0)) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n.

Then, f has a fixed point, that is, there exists $x^* \in X$ such that $f(x^*) = x^*$.

2. Fixed Point Theorems for α-ψ-Weakly Contractive Operators

Let (X, d, w) be a complete KST-space and $f : X \to X$ be an operator. We say that f is a *α-ψ-weakly contractive operator of type I* if there exists two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that:

$$\alpha(x, y)w(f(x), f(y)) \leq \psi(M(x, y)) \quad \text{for all} \quad x, y \in X, \tag{2}$$

where $M(x, y) = \max\{w(x, y), w(x, f(x)), w(y, f(y))\}$.

Theorem 2.1. *Let (X, d, w) be a complete KST space and $f : X \to X$ be a α-ψ-weakly contractive operator of type I. Suppose that:*

i) f is α-admissible;

ii) there exists $x_0 \in X$, such that $\alpha(x_0, f(x_0)) \geq 1$ and $\alpha(x_0, f^2(x_0)) \geq 1$;

iii) f is continuous.

Then $Fix(f) \neq \emptyset$.

Proof. By hypothesis (ii) there exists a point $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ and $\alpha(x_0, f^2(x_0)) \geq 1$.

We define a sequence $(x_n)_{n\in\mathbb{N}} \in X$ by $x_{n+1} = f(x_n) = f^{n+1}(x_0)$ for all $n \geq 0$.

Suppose there exists $n_0 \geq n$ such that $x_{n_0} = x_{n_0+1}$ then the proof is complete for every $u = x_{n_0} = x_{n_0+1} = f(x_0) = f(u)$.

Since f is admissible we have that

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0)) \geq 1 \Rightarrow \alpha(f(x_0), f(x_1)) = \alpha(x_1, x_2)) \geq 1. \tag{3}$$

Thus result that

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{4}$$

for every $n \in \mathbb{N}$.

Using the same technique above, we get:

$$\alpha(x_0, x_2) = \alpha(x_0, f^2(x_0)) \geq 1 \Rightarrow \alpha(f(x_0), f(x_2)) = \alpha(x_1, x_3)) \geq 1. \tag{5}$$

Then we have

$$\alpha(x_n, x_{n+2}) \geq 1, \tag{6}$$

for all $n \in \mathbb{N}$.

First we prove that $w(x_n, x_{n+1}) \xrightarrow{d} 0$.

By (2) and (4) we obtain:

$$w(x_n, x_{n+1}) = w(f(x_{n-1}), f(x_n)) \leq \alpha(x_{n-1}, x_n)w(f(x_{n-1}), f(x_n)) \leq \tag{7}$$

$$\leq \psi(M(x_{n-1}, x_n)), \quad \text{for} \quad n \geq 1,$$

where

$$M(x_{n-1}, x_n) = \max\{w(x_{n-1}, x_n), w(x_{n-1}, f(x_{n-1})), w(x_n, f(x_n))\} = \tag{8}$$

$$= \max\{w(x_{n-1}, x_n), w(x_{n-1}, x_n), w(x_n, , x_{n+1})\} = \max\{w(x_{n-1}, x_n), w(x_n, x_{n+1})\}.$$

If for some $n \in \mathbb{N}, M(x_{n-1}, x_n) = w(x_n, x_{n+1})(\neq 0)$ then the inequality (7) turns into:

$$w(x_n, x_{n+1}) \leq \psi(M(x_{n-1}, x_n)) = \psi(w(x_n, x_{n+1})) < w(x_n, x_{n+1}), \text{ which is a contradiction.}$$

Hence $M(x_{n-1}, x_n) = w(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, and inequality (7) become:

$$w(x_n, x_{n+1}) \leq \psi(w(x_{n-1}, x_n)) < w(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{9}$$

Then we obtain: $w(x_n, x_{n+1}) < \psi^n(w(x_0, x_1))$. Using the properties of ψ we obtain that

$$w(x_n, x_{n+1}) \xrightarrow{d} 0. \tag{10}$$

Next we prove that $w(x_n, x_{n+2}) \xrightarrow{d} 0$.

$$w(x_n, x_{n+2}) := w(f(x_{n-1}), f(x_{n+1})) \leq \alpha(x_{n-1}, x_{n+1})w(f(x_{n-1}), f(x_{n+1})) \leq \psi(M(x_{n-1}, x_{n+1})), \tag{11}$$

for all $n \geq 1$, where

$$M(x_{n-1}, x_{n+1}) = \max\{w(x_{n-1}, x_{n+1}), w(x_{n-1}, f(x_{n-1})), w(x_{n+1}, f(x_{n+1}))\} =$$

$$= \max\{w(x_{n-1}, x_{n+1}), w(x_{n-1}, x_n), w(x_{n+1}, x_{n+2})\}.$$

By (9) we have

$$M(x_{n-1}, x_{n+1}) = \max\{w(x_{n-1}, x_{n+1}), w(x_{n-1}, x_n)\} \tag{12}$$

Take $a_n = w(x_n, x_{n+2})$ and $b_n = w(x_n, x_{n+1})$.
Thus from (11) we obtain

$$a_n := w(x_n, x_{n+2}) \leq \psi(M(x_{n-1}, x_{n+1})) = \psi(\max\{a_{n-1}, b_{n-1}\}) \text{ for all } n \in \mathbb{N}.$$

Again by (9) we obtain $b_n < b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}$. Therefore $\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}$ for all $n \in \mathbb{N}$.
Then we have a sequence $(\max\{a_n, b_n\})_{n\in\mathbb{N}}$ which is monotone nonincreasing. Then it is convergent to some $t > 0$.
Assume that $t > 0$. Then we have:

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\} = t.$$

Taking $n \to \infty$ we obtain:

$$t = \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \psi(\max\{a_{n-1}, b_{n-1}\}) \leq$$

$$\leq \psi(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}) = \psi(t) < t, \text{ which is a contradiction.}$$

Then $w(x_n, x_{n+2}) \xrightarrow{d} 0$.
Next we prove that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.
Suppose $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $\alpha(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$.
Consider now

$$w(x_n, x_{n+1}) = w(x_n, f(x_n)) = w(x_m, f(x_m)) = w(f(x_{m-1}), f(x_m)) \leq \tag{13}$$

$$\leq \alpha(x_{m-1}, x_m)w(f(x_{m-1}), f(x_m)) \leq \psi(M(x_{m-1}, x_m)),$$

where

$$M(x_{m-1}, x_m) = \max\{w(x_{m-1}, x_m), w(x_{m-1}, f(x_{m-1})), w(x_m, f(x_m))\} =$$

$$= \max\{w(x_{m-1}, x_m), w(x_{m-1}, x_m), w(x_m, x_{m+1})\} = \max\{w(x_{m-1}, x_m), w(x_m, x_{m+1})\}.$$

Case I. If $M(x_{m-1}, x_m) = w(x_{m-1}, x_m)$, then (13) become:

$$w(x_n, x_{n+1}) = w(x_n, f(x_n)) = w(x_m, f(x_m)) = w(x_m, x_{m+1}) \leq \tag{14}$$

$$\leq \alpha(x_m, x_{m+1})w(f(x_{m-1}), f(x_m)) \leq \psi(M(x_{m-1}, x_m)) =$$

$$= \psi(w(x_{m-1}, x_m)) \leq \psi^{m-n}(w(x_n, x_{n+1})).$$

Case II. If $M(x_{m-1}, x_m) = w(x_m, x_{m+1})$, then inequality (13) become:

$$w(x_n, x_{n+1}) = w(x_n, f(x_n)) = w(x_m, f(x_m)) = w(f(x_{m-1}), f(x_m)) \leq \tag{15}$$

$$\leq \alpha(x_{m-1}, x_m)w(f(x_{m-1}), f(x_m)) \leq \psi(M(x_{m-1}, x_m)) =$$

$$= \psi(w(x_m, x_{m+1})) \leq \psi^{m-n+1}(w(x_n, x_{n+1})).$$

By the properties of function Ψ, using Lemma 1.3, the inequalities (14) and (15) together yield $w(x_n, x_{n+1}) \leq \psi^{m-n}(w(x_n, x_{n+1})) < w(x_n, x_{n+1})$ and $w(x_n, x_{n+1}) \leq \psi^{m-n+1}(w(x_n, x_{n+1})) < w(x_n, x_{n+1})$. Contradiction. Then $x_m = x_n$ for all $n = m$.
Next we prove that $(x_n)_{n\in\mathbb{N}} \in X$ is a Cauchy sequence.
We must to prove that $w(x_n, x_{n+k}) \xrightarrow{d} 0$ for all $k \in \mathbb{N}$.

The case $k = 1$ and $k = 2$ are proved above. Following, we discuss the case for $k \geq 3$ arbitrary.
Case I. Suppose that $k = 2m + 1$ where $m \geq 1$. Then we have:

$$w(x_n, x_{n+k}) = w(x_n, x_{n+2m+1}) \leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{n+2m}, x_{n+2m+1}) \leq$$

$$\leq \sum_{p=n}^{n+2m} \psi^p(w(x_0, x_1)) \leq \sum_{p=n}^{\infty} \psi^p(w(x_0, x_1)) \xrightarrow{d} 0 \text{ as } n \to \infty. \tag{16}$$

Case II. Suppose that $k = 2m$ where $m \geq 2$. Then we have:

$$w(x_n, x_{n+k}) = w(x_n, x_{n+2m}) \leq w(x_n, x_{n+2}) + w(x_{n+2}, x_{n+3}) + ... + w(x_{n+2m-1}, x_{n+2m}) \leq$$

$$\leq w(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(w(x_0, x_1)) \leq w(x_0, x_{n+2}) + \sum_{p=n}^{\infty} \psi^p(w(x_0, x_1)) \xrightarrow{d} 0 \text{ as } n \to \infty. \tag{17}$$

Combining (16) with (17) we have that

$$w(x_n, x_{n+k}) \xrightarrow{d} 0 \text{ for all } k \geq 3. \tag{18}$$

Then, using Lemma 1.2(3) for every $m = n + k$ we obtain that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d, w) space. Since (X, d, w) is a complete KST space there exists $x^* \in X$ such that

$$\lim_{n\to\infty}(x_n, x^*) = 0. \tag{19}$$

By the continuity of f it follows that $x_{n+1} = f(x_n) \xrightarrow{d} f(x^*)$ as $n \to \infty$.
Then by $w(x_n, x_{n+1})) \xrightarrow{d} 0$ result that $w(x_n, f(x_n)) \xrightarrow{d} 0$ which imply

$$w(x_n, f(x^*)) \xrightarrow{d} 0. \tag{20}$$

Using Lemma 1.2(1), by (19) and (20), we obtain that $x^* = f(x^*)$. Then $Fix(f) \neq \emptyset$. □

Theorem 2.2. *Let (X, d, w) be a complete KST space and $f : X \to X$ be a α-ψ-weakly contractive operator of type I. Suppose that:*
i) f is α-admissible;
ii) there exists $x_0 \in X$, such that $\alpha(x_0, f(x_0)) \geq 1$ and $\alpha(x_0, f^2(x_0)) \geq 1$;
iii) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_{n_k})_{n\in\mathbb{N}}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$.
Then $Fix(f) \neq \emptyset$.

Proof. Follow the proof of Theorem 2.1 we know that the sequence $(x_n)_{n\in\mathbb{N}} \in X$ defined by $x_{n+1} = f(x_n)$ for all $n \geq 0$ is a Cauchy sequence and converge to a point $x^* \in X$.
In view of Lemma 1.2 we have:

$$\lim_{k\to\infty} w(x_{n_k}, x_{n_k+1}) \xrightarrow{d} 0, \text{ and } \lim_{k\to\infty} w(x_{n_k+1}, x^*) \xrightarrow{d} 0. \tag{21}$$

We must prove that $f(x^*) = x^*$.
Then we suppose that $f(x^*) \neq x^*$.
By (4) and hypothesis (iii) there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$.
By the definition of weakly α-ψ-contractive operator of type I we find that:

$$w(x_{n_k+1}, f(x^*)) \leq \alpha(x_{n_k}, x^*) w(f(x_{n_k}), f(x^*)) \leq \psi(M(x_{n_k}, x^*)), \tag{22}$$

where

$$M(x_{n_k}, x^*) = \max\{w(x_{n_k}, x^*), w(x_{n_k}, f(x_{n_k})), w(x^*, f(x^*))\} = \tag{23}$$

$$= \max\{w(x_{n_k}, x^*), w(x_{n_k}, x_{n_k+1}), w(x^*, f(x^*))\}.$$

By (10) and (21), letting $k \to \infty$, we obtain that: $\lim_{k\to\infty} M(x_{n_k}, x^*) = w(x^*, f(x^*))$.
Letting $k \to 0$ in (22) and using the lower semicontinuity of function ψ we get:

$$w(x^*, f(x^*)) \leq \psi(w(x^*, f(x^*)) < w(x^*, f(x^*)),$$

which is a contradiction. Hence we find that $x^* = f(x^*)$, then $Fix(f) \neq \emptyset$. □

Definition 2.3. *Let (X,d,w) be a KST space and $f : X \to X$ be a given operator. We say that f is a α-ψ-weakly contractive operator of type II if there exists two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that:*

$$\alpha(x,y)w(f(x),f(y)) \leq \psi(N(x,y)) \quad \text{for ony} \quad x,y \in X, \text{ where} \tag{24}$$

$$N(x,y) = \max\left\{w(x,y), \frac{w(x,f(x)) + w(y,f(y))}{2}\right\}.$$

By Theorem 2.1 is obvious that $N(x,y) \leq M(x,y)$. Then we have proved the following result.

Theorem 2.4. *Let (X,d,w) be a complete KST space and $f : X \to X$ be a weakly α-ψ-contractive operator of type II. Suppose that:*
i) f is α-admissible;
ii) there exists $x_0 \in X$, such that $\alpha(x_0, f(x_0)) \geq 1$ and $\alpha(x_0, f^2(x_0)) \geq 1$;
iii) f is continuous.
Then $Fix(f) \neq \emptyset$.

In the frame of Theorem 2.2 we get a similarly result for this new class of operators.

Theorem 2.5. *Let (X,d,w) be a complete KST space and $f : X \to X$ be a weakly α-ψ-contractive operator of type II. Suppose that:*
i) f is α-admissible;
ii) there exists $x_0 \in X$, such that $\alpha(x_0, f(x_0)) \geq 1$ and $\alpha(x_0, f^2(x_0)) \geq 1$;
iii) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{d} x \in X$ as $n \to \infty$, then there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $\alpha(x_{n_k}, x) \geq 1$, for all $k \in \mathbb{N}$.
Then $Fix(f) \neq \emptyset$.

Proof. By Theorem 2.2 (respectively Theorem 2.1) we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$ is a Cauchy sequence and converge to some $x^* \in X$.
In view of Lemma 1.2 we have that:

$$\lim_{k\to\infty} w(x_{n_k}, x^*) \xrightarrow{d} 0 \text{ and } \lim_{k\to\infty} w(x_{n_k}, x_{n_k+1}) \xrightarrow{d} 0. \tag{25}$$

We must prove that $f(x^*) = x^*$.
Suppose that $f(x^*) \neq x^*$.
By (10) and hypothesis (iii), there exists a subsequence $(x_{n_k})_{n_k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\alpha(x_{n_k+1}, x^*) \geq 1$ for all $k \in \mathbb{N}$.
By the definition of weakly (α,ψ)-contractive operator of type II we get:

$$w(x_{n_k+1}, f(x^*)) \leq \alpha(x_{n_k}, x^*)w(f(x_{n_k}), f(x^*)) \leq \psi(N(x_{n_k}, x^*)), \tag{26}$$

where

$$N(x_{n_k}, x^*) = \max\left\{w(x_{n_k}, x^*), \frac{w(x_{n_k}, f(x_{n_k})) + w(x^*, f(x^*))}{2}\right\}. \tag{27}$$

Letting $k \to \infty$ we obtain that:

$$\lim_{k\to\infty} N(x_{n_k}, x^*) = \frac{w(x^*, f(x^*))}{2}. \tag{28}$$

For k enough, we have $N(x_{n_k}, x^*) > 0$, which implies that

$$\psi(N(x_{n_k}, x^*)) < N(x_{n_k}, x^*).$$

By (25) and Lemma 1.2(2) result that $(x_{n_k+1}) \xrightarrow{d} x^*$ as $k \to \infty$.
Then, by (26) and (28), letting $k \to \infty$, we obtain:

$$w(x^*, f(x^*)) \leq \frac{w(x^*, f(x^*))}{2} \text{ which is a contradiction.}$$

Hence $f(x^*) = x^*$, which implies that $Fix(f) \neq \emptyset$.□

3. Uniqueness Results for Fixed Points of Weakly α-ψ-Contractive Operators Types

In view to obtain results concerning the uniqueness of fixed points of weakly α-ψ-contractive operators type I and II, let us present some additional conditions.

Conditions:

(U) For all $x, y \in Fix(f)$, we have $\alpha(x, y) \geq 1$.

(W) For each pair of (distinct) points u, v, there is a number $r_{u,v} > 0$ such that for every $z \in X : r_{u,v} < w(u, z) + w(z, v)$.

Let us remember that condition (W) was introduced on a metric space by Wilson [31] to replace the triangle inequality with the weakened condition. In our case, we translated this condition in terms of w-distance.

Theorem 3.1. *Adding condition (U) to the hypothesis of Theorem 2.1 (respectively Theorem 2.2) we obtain that $x^* \in X$ is a unique fixed point of f. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.*

Proof. Let u^* ne another fixed point of f, with $u^* \neq x^*$. By hypothesis (U) we have that $1 \leq \alpha(x^*, u^*) = \alpha(f(x^*), f(u^*))$.

Next we prove that for every $x^* \in X$ such that $x^* = f(x^*)$ we have that $w(x^*, x^*) = 0$.

For $x^* = f(x^*)$ we suppose that $w(x^*, x^*) \neq 0$.

Then we have:

$$w(x^*, x^*) = w(f(x^*), f(x^*)) \leq \alpha(x^*, x^*)w(f(x^*), f(x^*)) \leq \psi(M(x^*, x^*)) =$$

$$= \psi(\max\{w(x^*, x^*), w(x^*, f(x^*)), w(x^*, f(x^*))\}) = \psi(\max\{w(x^*, x^*), w(x^*, f(x^*))\}) =$$

$$= \psi(w(x^*, x^*)) < w(x^*, x^*) \text{ which is a contradiction.}$$

Then we get the conclusion

$$w(x^*, x^*) = 0 \text{ for } x^* = f(x^*). \tag{29}$$

Since $u^* = f(u^*)$ and $x^* = f(x^*)$ are fixed points results that $w(x^*, f(x^*)) = w(u^*, f(u^*)) = 0$.

Then we have the following inequality:

$$w(x^*, u^*) \leq \alpha(x^*, u^*)w(x^*, u^*) = \alpha(x^*, u^*)w(f(x^*), f(u^*)) \leq \psi(M(x^*, u^*)) =$$

$$= \psi(\max\{w(x^*, u^*), w(x^*, f(x^*)), w(u^*, f(u^*))\}) = \psi(w(x^*, u^*)) < w(x^*, u^*),$$

which is a contradiction. Hence $x^* = u^*$.□

A similarly result we can obtain and for the case of α-ψ-weakly contractive operators of type II.

Theorem 3.2. *Adding condition (U) to the hypothesis of Theorem 2.4 we obtain that $x^* \in X$ is the unique fixed point of f. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.*

Proof. As in the proof of Theorem 3.1 we suppose that v^* is another fixed point with $v^* \neq x^*$.

We have that $1 \leq \alpha(x^*, v^*) = \alpha(f(x^*), f(v^*))$.

For $x^* = f(x^*)$ we suppose that $w(x^*, x^*) \neq 0$.

Then using the properties of function ψ, we obtain:

$$w(x^*, x^*) = w(f(x^*), f(x^*)) \leq \alpha(x^*, x^*)w(f(x^*), f(x^*)) \leq \psi(N(x^*, x^*) =$$

$$= \psi\left(\max\left\{w(x^*, x^*), \frac{w(x^*, f(x^*)) + w(x^*, f(x^*))}{2}\right\}\right) = \psi(\max\{w(x^*, x^*), w(x^*, f(x^*))\}) =$$

$$= \psi(w(x^*, x^*)) < w(x^*, x^*), \tag{30}$$

which is a contradiction.

Then we get that

$$w(x^*, x^*) = 0 \text{ for } x^* = f(x^*). \tag{31}$$

Since $v^* = f(v^*)$ and $x^* = f(x^*)$ we get that $w(v^*, f(v^*)) = 0$ and $w(x^*, f(x^*)) = 0$.

Then we have the inequality:

$$w(x^*, v^*) \leq \alpha(x^*, v^*), w(f(x^*), f(v^*)) \leq \psi(N(u^*, v^*)) =$$

$$= \psi\left(\max\left\{w(x^*, v^*), \frac{w(x^*, f(x^*)) + w(v^*, f(v^*))}{2}\right\}\right).$$

Using (31) and $\psi(t) < t$ for every $t > 0$ we obtain that: $w(x^*, v^*) \leq \psi(w(x^*, v^*)) < w(x^*, v^*)$, which is a contradiction. Hence $x^* = v^*$.□

Next, let us consider some alternative conditions for uniqueness of fixed points of weakly α-ψ-contractive operators.

(H_1) For all $x, y \in Fix(f)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

(H_2) Let $x, y \in Fix(f)$. If there exists $\{z_n\}$ in X such that $\alpha(x, z_n) \geq 1$ and $\alpha(y, z_n) \geq 1$, then $w(z_n, z_{n+1}) \leq \inf\{w(x, z_n), w(y, z_n)\}$.

Next, let us present a new result of uniqueness of fixed point of weakly α-ψ-contractive operators of type I.

Theorem 3.3. *Adding conditions (H_1), (H_2) and (W) to the hypothesis of Theorem 2.1 (respectively Theorem 2.2) we find that $x^* \in X$ is a unique point. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.*

Proof. Suppose $u^* \in Fix(f)$ another fixed point such that $x^* \neq u^*$. Then $w(x^*, u^*) > 0$.
By (H_1) we get that there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1, \text{ and } \alpha(u^*, z) \geq 1. \tag{32}$$

Since f is admissible, from (32), we obtain: $\alpha(x^*, f^n(z)) \geq 1$ and $\alpha(u^*, f^n(z)) \geq 1$ for all $n \in \mathbb{N}$.
We define the sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ by $z_{n+1} = f(z_n)$ for all $n \geq 0$ and $z_0 = z$.
By (32), for all $n \in \mathbb{N}$, we get:

$$w(x^*, z_{n+1}) = w(f(x^*), f(z_n)) \leq \alpha(x^*, z_n)w(f(x^*), f(z_n)) \leq \psi(M(x^*, z_n)). \tag{33}$$

Since $w(x^*, f(x^*)) = w(x^*, x^*) = 0$ for $x^* = f(x^*)$ we have:

$$M(x^*, z_n) = \max\{w(x^*, z_n), w(x^*, f(x^*)), w(z_n, f(z_n))\} = \max\{w(x^*, z_n), w(z_n, z_{n+1})\}.$$

By (H_2) we obtain $M(x^*, z_n) = w(x^*, z_n)$ for all $n \in \mathbb{N}$.
Iteratively, using the inequality (33), we get:

$$w(x^*, z_{n+1}) \leq \psi^{n+1}(w(x^*, z_0)) \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality we obtain:

$$\lim_{n\to\infty} w(x^*, z_n) = 0. \tag{34}$$

Similarly we show that

$$\lim_{n\to\infty} w(u^*, z_n) = 0. \tag{35}$$

Using (W), there exists $r_{x^*,u^*} > 0$ such that, for all $n \in \mathbb{N}$,

$$r_{x^*,u^*} < w(x^*, z_n) + w(u^*, z_n).$$

Letting $n \to \infty$, in view of (34) and (35), it follows that $r_{x^*,u^*} = 0$ which is a contradiction. Hence $x^* = u^*$.□
Next we present another result of uniqueness concerning α-ψ-weakly contractive operators of type II.

Theorem 3.4. *Adding conditions (H_1),(H_2) and (W) to the hypothesis of Theorem 2.4 (respectively Theorem 2.5) we find that $x^* \in X$ is the unique fixed point of f. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.*

Proof. Suppose v^* is another fixed point of f and $x^* \neq v^*$. By (H_1) there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \text{ and } \alpha(v^*, z) \geq 1. \tag{36}$$

Since f is admissible, by (36) we have:

$$\alpha(x^*, f^n(z)) \geq 1 \text{ and } \alpha(v^*, f^n(z)) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{37}$$

We define the sequence $(z_n)_{n\in\mathbb{N}} \in X$ by $z_{n+1} = f(z_n)$ for all $n \geq 0$ and $z_0 = z$.
By (37) we have:

$$w(x^*, z_{n+1}) = w(f(x^*), f(z_n)) \leq \alpha(x^*, z_n)w(f(x^*), f(z_n)) \leq \psi(N(x^*, z_n)). \tag{38}$$

Since $w(x^*, f(x^*)) = w(x^*, x^*) = 0$ we have:

$$N(x^*, z_n) = \max\left\{w(x^*, z_n), \frac{w(x^*, f(x^*)) + w(z_n, f(z_n))}{2}\right\} =$$

$$= \max\left\{w(x^*, z_n), \frac{w(z_n, z_{n+1})}{2}\right\}. \tag{39}$$

By (H_2) we get

$$N(x^*, z_n) = w(x^*, z_n) \text{ for all } n \in \mathbb{N}.$$

Then, iteratively, using (38), we obtain

$$w(x^*, z_{n+1}) \leq \psi^{n+1}(w(x^*, z_0)) \text{ for all } n \in \mathbb{N}$$

Letting $n \to \infty$ we have:

$$\lim_{n\to\infty} w(x^*, z_n) = 0. \tag{40}$$

Similarly we prove that

$$\lim_{n\to\infty} w(v^*, z_n) = 0. \tag{41}$$

Regarding condition (W) for $r_{u,v} > 0$ we get:

$$r_{u,v} = w(x^*, z_n) + w(v^*, z_n).$$

Letting $n \to \infty$, by (40) and (41), result that $r_{u,v} = 0$ which is a contradiction.
Hence $x^* = v^*$. □

4. Consequences of Fixed Point Theorems for α-ψ-Weakly Contractive Type Operators

In this section let us present some results as consequences of fixed point theorems presented for α-ψ-weakly contractive operators of type I and II on KST spaces.

Corollary 4.1. *Let (X, d, w) be a complete KST space and $f : X \to X$ be a given continuous operator. Suppose that there exists a function $\psi \in \Psi$ such that*

$$w(f(x), f(y)) \leq \psi(M(x, y)) \textit{ for all } x \in X.$$

Then f has a unique fixed point. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.

Proof. If we take in Theorem 2.1 $\alpha(x, y) = 1$ we obtain immediately the conclusion. □

Corollary 4.2. *Let (X, d, w) be a complete KST space and $f : X \to X$ be a given continuous operator. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$w(f(x), f(y)) \leq \lambda M(x, y) \textit{ for all } x, y \in X.$$

Then f has a unique fixed point. Moreover, if $x^* = f(x^*)$, then $w(x^*, x^*) = 0$.

Proof. By taking $\psi(t) = \lambda t$ where $\lambda \in (0, 1)$ follow the conclusion. □

Remark 4.3. *Same results can be obtained for α-ψ-weakly contractive operators of type II on KST spaces.*

5. Applications of α-ψ-Weakly Contractive Operators of Type I and II

In this section we discuss the Ulam-Hyers stability, the well posedness and data dependence with respect to w-distance of fixed point problems for α-ψ-weakly contractive operators of type I and II.

First, let us recall the notions of Ulam-Hyers stability and weakly Picard operator with respect to a w-distance.

Definition 5.1. *Let (X, d, w) be a KST space and $f : X \to X$ be an given operator. By definition the fixed point equation:*

$$x = f(x) \tag{42}$$

is called generalized Ulam-Hyers w-stable if and only if the exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing, continuous at 0 and $\psi(0) = 0$ such that, for every $\varepsilon > 0$ and each $u^* \in X$ an ε-solution of the fixed point equation, is satisfied the following inequality: $w(u^*, f(u^*)) \leq \varepsilon$.

There exists a solution x^* of equation (38) such that

$$w(u^*, x^*) \leq \psi(\varepsilon).$$

In I. A. Rus [25], [26] we find the following definition.

Definition 5.2. *Let (X, d) be a metric space and $f : X \to X$ be an operator. f is a weakly Picard operator (WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from $x_0 \in X$ converge for all $x \in X$ and its limit is a fixed point for f.*

If f is a WPO then we consider the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$.

Notice that $f^\infty(x) = Fix(f)$.

Let us give our first result concerning α-ψ-weakly contractive operators of type I.

Theorem 5.3. *Let (X, d, w) be a complete KST space. Suppose that all the hypotheses of Theorem 2.1 (respectively Theorem 2.2) hold and additionally, that the function $\beta : [0, \infty) \to [0, \infty)$, $\beta(r) := r - \psi(r)$ is strictly increasing and onto, then the following hold:*

a) The fixed point equation (42) is generalized Ulam-Hyers w-stable if and only if for every $x^* \in X$ such that $x^* = f(x^*)$ we have $w(x^*, x^*) = 0$;

b) $Fix(f) = \{x^*\}$ and if $x_n \in X$, $n \in \mathbb{N}$ are such that $w(x_n, f(x_n)) \xrightarrow{d} 0$ as $n \to \infty$ then $x_n \xrightarrow{d} x^*$ as $n \to \infty$; that is, the fixed point equation (42) is well posed with respect to w-distance.

c) If $g : X \to X$ is such that there exists $\eta \in [0, \infty)$ with $w(g(x), f(x)) \leq \eta$ for $x \in X$. Then $y^* \in Fix(g)$ imply that $w(y^*, x^*) \leq \beta^{-1}(\eta)$.

Proof. a) Following the same steps as in the proof of Theorem 2.1 (respectively Theorem 2.2) we get that $f : X \to X$ is a WPO with respect to w-distance. Then $Fix(f) = \{x^*\}$.

Let $\varepsilon > 0$ and $u^* \in X$ be a solution of fixed point equation (42). Then $w(u^*, f(u^*)) < \varepsilon$.

Since f is α-ψ-weakly contractive operators of type I and since $x^* \in Fix(f)$, by the condition (U) there exists $u^* \in X$ such that $\alpha(u^*, x^*) \geq 1$.

We obtain:

$$w(u^*, x^*) = w(u^*, f(x^*)) \leq w(u^*, f(u^*)) + w(f(u^*), f(x^*)) \leq$$
$$\leq \varepsilon + \lambda(u^*, x^*)w(f(u^*), f(x^*)) \leq \varepsilon + \psi(M(u^*, x^*)),$$

where

$$M(u^*, x^*) = \max\{w(u^*, x^*), w(u^*, f(u^*)), w(x^*, f(x^*))\}.$$

Since $u^* = f(u^*)$ and $x^* = f(x^*)$ we have that $w(u^*, f(u^*)) = w(u^*, u^*) = 0$ and $w(x^*, f(x^*)) = w(x^*, x^*) = 0$.

Then $M(u^*, x^*) = w(u^*, x^*)$.

Result that $w(u^*, x^*) \leq \varepsilon + \psi(w(u^*, x^*))$.

Therefore

$$\beta(w(u^*, x^*)) := w(u^*, x^*) - \psi(w(u^*, x^*)) \leq$$
$$\leq \varepsilon + \psi(w(u^*, x^*)) - \psi(w(u^*, x^*)) \leq \varepsilon.$$

Then $w(u^*, x^*) \leq \beta^{-1}(\varepsilon)$.

Consequently, the fixed point equation (42) is generalized Ulam-Hyers w-stable.

b) Using the proof of Theorem 2.1 for $\alpha(x_{n-1}, x_n) \geq 1$ we obtain for $x_{n-1} = f(x_{n-2})$ and $x_n = f(x_{n-1})$ the following inequality:

$$w(x_{n-1}, x_n) \leq w(f(x_{n-2}), f(x_{n-1})) \leq \alpha(x_{n-2}, x_{n-1})w(f(x_{n-2}), f(x_{n-1})) \leq \tag{43}$$

$$\leq \psi(M(x_{n-2}, x_{n-1})) \text{ for } n \geq 1,$$

where

$$M(x_{n-2}, x_{n-1}) = \max\{w(x_{n-2}, x_{n-1}), w(x_{n-2}, f(x_{n-2})), w(x_{n-1}, f(x_{n-1}))\} =$$

$$= \max\{w(x_{n-2}, x_{n-1}), w(x_{n-2}, x_{n-1}), w(x_{n-1}, x_n)\} =$$

$$= \max\{w(x_{n-2}, x_{n-1}), w(x_{n-1}, x_n)\}.$$

Then, by (10), we have:

$$w(x_{n-1}, x_n) \leq \psi^{n-1}(w(x_0, x_1)) \xrightarrow{d} 0. \tag{44}$$

Since f is α-ψ-weakly contractive operator and since $x^* \in Fix(f)$ by hypothesis (H_2) there exists $x_n \in X$ such that $\alpha(x^*, x_n) \geq 1$.

By (18), for $n, k \in \mathbb{N}$, we have $w(x_n, x_{n+k}) \xrightarrow{d} 0$.

Since $w(x_n, \cdot) : X \times X \to \mathbb{R}_+$ is l.s.c. and by the properties of function ψ we obtain:

$$w(x_{n-1}, x^*) \leq \lim_{k\to\infty} \inf(x_{n-1}, x_{n+k}) \leq \lim_{k\to\infty} \sum_{k=n}^{\infty} \psi^{k-1}(w(x_0, x_1)) \xrightarrow{d} 0. \tag{45}$$

Then, by (44) and (45), using Lemma 1.2(1), we obtain that $x_n \xrightarrow{d} x^*$.

So, the fixed point equation (42) is well posed with respect to w-distance.

c) Since f is α-ψ-weakly contractive operators of type I and since $x^* \in Fix(f)$, by (H_1) there exists $x \in X$ such that $\alpha(x, x^*) \geq 1$.

Using the triangle inequality we have:

$$w(x, x^*) \leq w(x, f(x)) + w(f(x), x^*) = w(x, f(x)) + w(f(x), f(x^*)) \leq \tag{46}$$

$$\leq w(x, f(x)) + \alpha(x, x^*)w(f(x), f(x^*)) = w(x, f(x)) + \psi(M(x, x^*)),$$

where $M(x, x^*) = max\{w(x, x^*), w(x, f(x)), w(x^*, f(x^*))\}$.

By (29) we have that $w(x^*, f(x^*)) = w(x^*, x^*) = 0$ for $x^* = f(x^*), x^* \in X$.

Then we have next two cases:

Case I. If $M(x, x^*) = w(x, x^*)$ we have

$$w(x, x^*) \leq w(x, f(x)) + \psi(w(x, x^*)).$$

Therefore, using (46) we have:

$$\beta(w(x, x^*)) := w(x, x^*) - \psi(w(x, x^*)) \leq$$

$$\leq w(x, f(x)) + \psi(w(x, x^*)) - \psi(w(x, x^*)) \leq$$

$$\leq w(x, f(x)).$$

Then $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

Case II. If $M(x, x^*) = w(x, f(x))$ we have:

$$w(x, x^*) \leq w(x, f(x)) + \psi(w(x, f(x))) < w(x, f(x)) + w(x, f(x)) = 2w(x, f(x)).$$

Since $w(x_n, \cdot) : X \times X \to \mathbb{R}_+$ is l.s.c. and $x^* \in Fix(f)$, then, for every $x \in X$, we have $w(x, f(x)) \xrightarrow{d} w(x, x^*)$. Then, we conclude that $w(x, x^*) \leq w(x, f(x))$.

Therefore, using Lemma 1.3 concerning the properties of ψ, we obtain:

$$\beta(w(x, x^*)) := w(x, x^*) - \psi(w(x, x^*)) \leq 2w(x, f(x)) - \psi(w(x, x^*)) \leq$$

$$\leq 2w(x, f(x)) - \psi(w(x, f(x))) < 2w(x, f(x)) - w(x, f(x)) = w(x, f(x)).$$

Then $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

Thus, from both cases, we obtain that $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

For any operator $g : X \to X$ with $y^* \in Fix(g)$ if we denote $x := y^*$, there exists $\eta \in [0, \infty)$ with $w(g(y), f(x)) \leq \eta$ such that

$$w(x, x^*) = w(y^*, x^*) \leq \beta^{-1}(w(y^*, f(x^*))) \leq \beta^{-1}(w(g(y^*), f(x^*))) \leq \beta^{-1}(\eta). \square$$

Similarly result can be obtained and for α-ψ-weakly contractive operators of type II.

Theorem 5.4. *Let (X, d, w) be a complete KST space. Suppose that all the hypotheses of Theorem 2.4 (respectively Theorem 2.5) hold and additionally, that the function $\beta : [0, \infty) \to [0, \infty)$, $\beta(r) := r - \psi(r)$ is strictly increasing and onto, then the following hold:*

a) The fixed point equation (42) is generalized Ulam-Hyers w-stable if and only if for every $x^* \in X$ such that $x^* = f(x^*)$ we have $w(x^*, x^*) = 0$;

b) $Fix(f) = \{x^*\}$ and if $x_n \in X$, $n \in \mathbb{N}$ are such that $w(x_n, f(x_n)) \xrightarrow{d} 0$ as $n \to \infty$ then $x_n \xrightarrow{d} x^*$ as $n \to \infty$; that is, the fixed point equation (42) is well posed with respect to w-distance.

c) If $g : X \to X$ is such that there exists $\eta \in [0, \infty)$ with $w(g(x), f(x)) \leq \eta$ for $x \in X$. Then $y^* \in Fix(g)$ imply that $w(y^*, x^*) \leq \beta^{-1}(\eta)$.

Proof. a) Following the same steps as in the proof of Theorem 2.4 (respectively Theorem 2.5) we get that $f : X \to X$ is a WPO with respect to w-distance. Then $Fix(f) = \{x^*\}$.

Let $\varepsilon > 0$ and $u^* \in X$ be a solution of fixed point equation (42).

Then $w(u^*, f(u^*)) < \varepsilon$.

Since f is α-ψ-weakly contractive operators of type II and since $x^* \in Fix(f)$ from condition (U) there exists $u^* \in X$ such that $\alpha(u^*, x^*) \geq 1$.

We obtain:

$$w(u^*, x^*) = w(u^*, f(x^*)) \leq w(u^*, f(u^*)) + w(f(u^*), f(x^*)) \leq$$
$$\leq \varepsilon + \lambda(u^*, x^*) w(f(u^*), f(x^*)) \leq \varepsilon + \psi(N(u^*, x^*)),$$

where

$$N(u^*, x^*) = max\{w(u^*, x^*), \frac{w(u^*, f(u^*)) + w(x^*, f(x^*))}{2}\}.$$

Since $u^* = f(u^*)$ and $x^* = f(x^*)$ we have that $w(u^*, f(u^*)) = w(u^*, u^*) = 0$ and $w(x^*, f(x^*)) = w(x^*, x^*) = 0$.

Then $N(u^*, x^*) = w(u^*, x^*)$.

Result that $w(u^*, x^*) \leq \varepsilon + \psi(w(u^*, x^*))$.

Therefore

$$\beta(w(u^*, x^*)) := w(u^*, x^*) - \psi(w(u^*, x^*)) \leq$$
$$\leq \varepsilon + \psi(w(u^*, x^*)) - \psi(w(u^*, x^*)) \leq \varepsilon.$$

Then $w(u^*, x^*) \leq \beta^{-1}(\varepsilon)$.

Thus the fixed point equation (42) is generalized Ulam-Hyers w-stable.

b) In the same way as in the proof of Theorem 2.1 for $\alpha(x_{n-1}, x_n) \geq 1$ we obtain, for $x_{n-1} = f(x_{n-2})$ and $x_n = f(x_{n-1})$, the following inequality:

$$w(x_{n-1}, x_n) \leq w(f(x_{n-2}), f(x_{n-1})) \leq \alpha(x_{n-2}, x_{n-1}) w(f(x_{n-2}), f(x_{n-1})) \leq \tag{47}$$
$$\leq \psi(N(x_{n-2}, x_{n-1})) \text{ for } n \geq 1,$$

where

$$N(x_{n-2}, x_{n-1}) = \max\{w(x_{n-2}, x_{n-1}), \frac{w(x_{n-2}, f(x_{n-2})) + w(x_{n-1}, f(x_{n-1}))}{2}\}$$

Case I. For $N(x_{n-2}, x_{n-1}) = w(x_{n-2}, x_{n-1})$ we have $w(x_{n-1}, x_n) \leq \psi(w(x_{n-2}, x_{n-1}))$.

Since ψ is increasing result that $w(x_{n-1}, x_n) \leq \psi^{n-1}(w(x_0, x_1)) \xrightarrow{d} 0$.

Case II. For $N(x_{n-2}, x_{n-1}) = \frac{w(x_{n-2}, f(x_{n-2})) + w(x_{n-1}, f(x_{n-1}))}{2}$ and using the properties of ψ we get

$$w(x_{n-1}, x_n) \leq \psi(\frac{w(x_{n-2}, f(x_{n-2})) + w(x_{n-1}, f(x_{n-1}))}{2}) = \psi(\frac{w(x_{n-2}, x_{n-1}) + w(x_{n-1}, x_n)}{2}) <$$
$$< \frac{w(x_{n-2}, x_{n-1}) + w(x_{n-1}, x_n)}{2}.$$

Then $2w(x_{n-1}, x_n) \leq w(x_{n-2}, x_{n-1}) + w(x_{n-1}, x_n)$, which imply $w(x_{n-1}, x_n) \leq w(x_{n-2}, x_{n-1})$.

Since ψ is increasing we obtain that:

$$w(x_{n-1}, x_n) \leq \psi^{n-2}(w(x_0, x_1)) \xrightarrow{d} 0. \tag{48}$$

Then, by both cases we get that

$$w(x_{n-1}, x_n) \xrightarrow{d} 0. \tag{49}$$

Since f is α-ψ-weakly contractive operator of type II and since $x^* \in Fix(f)$ by hypothesis (H_2) there exists $x_n \in X$ such that $\alpha(x^*, x_n) \geq 1$.

Then, for $n, m \in \mathbb{N}$, with $m > n$, we have

$$w(x_n, x_m) \leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) ... + w(x_{m-1}, x_m) \leq$$

$$\leq \psi^n(w(x_0, x_1)) + \psi^{n+1}(w(x_0, x_1)) + ... + \psi^{m-1}(w(x_0, x_1)) \leq$$

$$\leq \sum_{p=n}^{m-1} \psi^p(w(x_0, x_1)) < \sum_{p=n}^{\infty} \psi^p(w(x_0, x_1)) \xrightarrow{d} 0.$$

Since $w(x_n, \cdot) : X \times X \to \mathbb{R}_+$ is l.s.c. and by the properties of function ψ we obtain:

$$w(x_{n-1}, x^*) \leq \lim_{m\to\infty} \inf(x_{n-1}, x_m) \leq \lim_{m\to\infty} \sum_{k=n}^{\infty} \psi^{p-1}(w(x_0, x_1)) \xrightarrow{d} 0.$$

Then

$$w(x_{n-1}, x^*) \xrightarrow{d} 0. \tag{50}$$

Then, by (49) and (50), using Lemma 1.2(1) we obtain that $x_n \xrightarrow{d} x^*$.

So, the fixed point equation (42) is well posed with respect to w-distance.

c) Since f is α-ψ-weakly contractive operators of type I and since $x^* \in Fix(f)$ from (H_1) there exists $x \in X$ such that $\alpha(x, x^*) \geq 1$.

Using the triangle inequality we have:

$$w(x, x^*) \leq w(x, f(x)) + w(f(x), x^*) = w(x, f(x)) + w(f(x), f(x^*)) \leq \tag{51}$$

$$\leq w(x, f(x)) + \alpha(x, x^*)w(f(x), f(x^*)) = w(x, f(x)) + \psi(N(x, x^*)),$$

where

$N(x, x^*) = max\{w(x, x^*), \frac{w(x, f(x)) + w(x^*, f(x^*))}{2}\}$.

Since $x^* \in Fix(f)$ we have that $w(x^*, f(x^*)) = w(x^*, x^*) = 0$ for $x^* = f(x^*), x^* \in X$.

Then $N(x, x^*) = max\{w(x, x^*), \frac{w(x, f(x))}{2}\}$.

We have two cases.

Case I. For $N(x, x^*) = w(x, x^*)$ we have $w(x, x^*) \leq w(x, f(x)) + \psi(w(x, x^*))$.

Therefore we have:

$$\beta(w(x, x^*)) := w(x, x^*) - \psi(w(x, x^*)) \leq w(x, f(x)) + \psi(w(x, x^*)) - \psi(w(x, x^*)) \leq w(x, f(x)).$$

Then $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

Case II. If $N(x, x^*) = \frac{w(x, f(x))}{2}$ we have:

$$w(x, x^*) \leq w(x, f(x)) + \psi(\frac{w(x, f(x))}{2}) < w(x, f(x)) + \frac{w(x, f(x))}{2} = \frac{3}{2}w(x, f(x)) < 2w(x, f(x)).$$

Since $w(x_n, \cdot) : X \times X \to \mathbb{R}_+$ is l.s.c. and $x^* \in Fix(f)$, we get that $w(x, f(x)) \xrightarrow{d} w(x, x^*)$ for every $x \in X$. This imply that $w(x, x^*) \leq w(x, f(x))$.

Then, using Lemma 1.3 concerning the properties of ψ we obtain:

$$\beta(w(x, x^*)) := w(x, x^*) - \psi(w(x, x^*)) \leq 2w(x, f(x)) - \psi(w(x, x^*))$$

$$\leq 2w(x, f(x)) - \psi(w(x, f(x))) < 2w(x, f(x)) - w(x, f(x)) = w(x, f(x)).$$

Then $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

Thus, from both cases we obtain that $w(x, x^*) \leq \beta^{-1}(w(x, f(x)))$.

For any operator $g : X \to X$ with $y^* \in Fix(g)$ if we denote $x := y^*$ there exists $\eta \in [0, \infty)$ with $w(g(y), f(x)) \leq \eta$ such that $w(x, x^*) = w(y^*, x^*) \leq \beta^{-1}(w(y^*, f(x^*))) \leq \beta^{-1}(w(g(y^*), f(x^*))) \leq \beta^{-1}(\eta)$. □

References

[1] P. Amiri, Sh. Rezapour, N. Shahzad, Fixed points of generalized α-ϕ-contractions, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, Vol. 108, Issue 2(2014), 519-526. DOI 10.1007/s13398-013-0123-9.

[2] H. Aydi, M-F. Bota, E. Karapınar and S. Moradi, A common fixed point for weak ϕ-contractions on b-metric spaces, Fixed Point Theory, 13(2012), No 2, 337-346.

[3] H. Aydi, E. Karapınar, B. Samet, Fixed points for generalized (α,*phi*)-contractions on generalized metric spaces, Journal of Ineq. and Appl. 2014, 2014:229, DOI: 10.1186/1029-242X-2014-229.

[4] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped. Inst. 30(1989), 26-37.

[5] V. Berinde, Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory, Preprint no. 3(1993), 3-9.

[6] V. Berinde, Contracţii generalizate şi aplicaţii, Editura Club Press 22, Baia Mare, 1997.

[7] M. F. Bota-Boriceanu, A. Petruşel, Ulam-Hyers stability for operatorial equations, Analele Univ. Al. I. Cuza, Iaşi, 57(2011), 65-74.

[8] M.-F. Bota, E. Karapınar, O. Mleşnite, Ulam-Hyers Stability Results for Fixed Point Problems via α-ψ-Contractive Mapping in (b)-Metric Space,Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2013, 6 pages, Article ID 825293, http://dx.doi.org/10.1155/2013/825293.

[9] N. Bourbaki, Topologie Générale, Herman, Paris, 1974.

[10] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc., 215(1976), 241-251.

[11] L. Guran, Ulam-Hyers stability of fixed point equations for singlevalued operators on KST spaces, Creat. Math. Inform., No. 1, 21(2012), 41 - 47.

[12] D. H. Hyers, On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America, vol. 27, no. 4, pp. 222-224, 1941.

[13] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, Proc. Am. Math. Soc., No.2, 126(1998), 425-430.

[14] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japonica, 44(1996), 381-391.

[15] E. Karapınar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal. 2012, Article ID 793486(2012).

[16] V. L. Lazăr, Ulam-Hyers stability for partial differential inclusions, Electronic Journal of Qualitative Theory of Differential Equations, 21 (2012), 1-19.

[17] B. Mohammadi, Sh. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ćirić generalized multifunctions. Fixed Point Theory Appl. 2013, Article ID 24(2013).

[18] N. Hussain, Z. Kadelburg, S. Radenović, and F.Al-Solamy, Comparison Functions and Fixed Point Results in Partial Metric Spaces, Abstract and Applied Analysis, vol. 2012, Article ID 605781, 15 pages, 2012.

[19] T. P. Petru, A. Petruşel and J.-C. Yao, Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, Taiwanese Journal of Mathematics, Vol. **15**, No. 5, pp. 2195-2212, October 2011.

[20] A. Petruşel, Multivalued weakly Picard operators and applications, Scientiae Mathematicae Japonicae, 1(2004), 1-34.

[21] A. Petruşel, I. A. Rus, Multivalued Picard and weakly Picard operators, Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia (Spain), July 2003, 207-226;

[22] A. Petruşel, I. A. Rus, A. Sântămărian, Data dependence of the fixed point set of multivalued weakly Picard operators, Nonlinear Analysis, 52(2003), no. 8, 1947-1959.

[23] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, Fixed Point Theory, 9(2008), No. 2, 541-559.

[24] I. A. Rus, Generalized contractions and applications, Cluj University Press, Cluj-Napoca, 2001.

[25] I. A. Rus, Remarks on Ulam stability of the operatorial equations, Fixed Point Theory, 10(2009), No. 2, 305-320.

[26] I.A. Rus, Ulam stability of ordinary differentioal equations, Studia Univ. Babeş-Bolyai Math., 54(2009), No. 4, 125-133.

[27] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Analysis 75 (2012), 2154-2165.

[28] N. Shioji, T. Suzuki, W. Takahashi, Contractive mappings, Kannan mappings and metric completeness, Proc. Amer. Math. Soc., 126(1998), 3117-3124.

[29] T. Suzuki, W. Takahashi, Fixed points theorems and characterizations of metric completeness, Topological Methods in Nonlinear Analysis, Journal of Juliusz Schauder Center, 8(1996), 371-382.

[30] S. M. Ulam, Problems in Modern Mathematics, John Wiley and Sons, New York, NY, USA, 1964.

[31] W.A. Wilson, On semimetric spaces, Am. J. Math. 53(2), 361-373 (1931).