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Some Classes of Operators Related to the Space of Convergent Series cs

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Abstract. Sequence space of convergent series can also be seen as a matrix domain of triangle. By using the theory of matrix domains of triangle, as well as the fact that cs is an AK space we can give the representation of some general bounded linear operators related to the cs sequence space. We will also give the conditions for compactness by using the Hausdorff measure of noncompactness.

1. Basic Notations

The set ω will denote all complex sequences $x = (x_k)_{k=0}^{\infty}$ and ℓ_∞, c, c_0 and ϕ will denote the sets of all bounded, convergent, null and finite sequences. As usual, let e and $e^{(n)}$,$(n = 0, 1, ...)$ represent the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a Schauder basis if for every $x \in X$ exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$. A subspace X of ω is called an FK space if it is a complete linear metric space with continuous coordinates $P_n : X \to C$, $(n = 0, 1, ...)$ where $P_n(x) = x_n$. An FK space $X \supset \phi$ is said to have AK if his Schauder basis is $(e^{(n)})_{n=0}^{\infty}$, and a normed FK space is BK space. The spaces c_0, c, l_∞ are all the BK spaces and among them only the c_0 has AK. The space ℓ_∞ has no Schauder base.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, and X and Y be subsets of ω. We denote with $A_n = (a_{nk})_{k=0}^{\infty}$ the sequences in the n-th row of A, $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^{\infty}$ (provided all the series $A_n x$ converge). Matrix domain of A in X is $X_A = \{x \in \omega | Ax \in X\}$ and (X, Y) is the class of all matrices A such that $X \subset Y_A$. If $(X, \|\cdot\|)$ is a normed space we write S_X for unit sphere and $\overline{B_X}$ for the closed unit ball in X. For $X \supset \phi$ a BK space and $a = (a_k) \in \omega$ we define

$$\|a\|_X^* = \sup_{x \in S_X} |\sum_{k=0}^{\infty} a_k x_k|$$

provided the right side exists and is finite. β-dual of X, X^β, is the set defined with $X^\beta = \{a = (a_k) \in \omega | \sum_{k=0}^{\infty} a_k x_k \ \textit{converges}, \forall x \in X\}$. It is clear that β-duals play important role since $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$. We write $B(X, Y)$ for the set of all bounded linear operators.

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$, $(n = 0, 1, ...)$. It is well-known that every triangle T has a unique inverse $S = (S_{nk})_{n,k=0}^{\infty}$ which also is a triangle, and $x = T(S(x)) = S(T(x))$ for all $x \in \omega$.

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2. The Space of Convergent Series

The space of all convergent series and the space of bounded series are both known and studied spaces. Here we are going to give their definitions and their most important properties, but, as the title of the paper suggests, our mainly preoccupation will be the *cs* space.

$$cs = \{\, x \in \omega | \sum_{k=0}^{\infty} x_k \text{ converges } \},$$

$$bs = \{\, x \in \omega | (\sum_{k=0}^{n} x_k)_{n=0}^{\infty} \in \ell_\infty \}.$$

They are *BK* spaces, *cs* is a closed subspace of *bs* so their norms are the same and are given with $\|x\|_{cs} = \|x\|_{bs} = \sup_n | \sum_{k=0}^{n} x_k|$. It is also known that *cs* is an *AK* space.

However, there exists another way to define these spaces - as matrix domains of triangle:

$$cs = (c)_\Sigma \ , \ bs = (\ell_\infty)_\Sigma \ ,$$

where the triangle Σ is given with $\Sigma = (\sigma_{nk})_{n,k=1}^{\infty}$ with $\sigma_{nk} = 1$ for $1 \leq k \leq n$ and $\sigma_{nk} = 0$ for $k > n$, $(n = 1, 2, ...)$. Now we can see that *bs* has no Schauder base.[5, Remark 24.]

Thanks to this definition, we can use the known results about matrix domains [8] to characterize the following classes of matrix transformations:

$$((c)_\Sigma, (c)_\Sigma), ((c)_\Sigma, (\ell_\infty)_\Sigma), ((\ell_\infty)_\Sigma, (\ell_\infty)_\Sigma), ((\ell_\infty)_\Sigma, (c)_\Sigma).$$

But we are going to use the results already obtained by applying the theory of functional analysis given by A.Wilansky [13] : (cs, cs) , (cs, bs) , (bs, cs) , (bs, bs) in the same order [13, Examples 8.4.6B, 8.4.6B, 8.5.9, 8.4.6C] or in [12]. We will also need the following characterizations when the final space is one of the classic sequence spaces: (cs, c_0) , (cs, c) , (cs, ℓ_∞) which are all given in [13, Example 8.4.5B] and they are:

$A \in (bs, bs)$ if and only if

$$\lim_k a_{nk} = 0 \text{ for all } n \tag{1}$$

and

$$\sup_m \sum_k | \sum_{n=0}^{m} (a_{nk} - a_{n,k-1})| < \infty. \tag{2}$$

$A \in (cs, bs)$ if and only if (2) holds and

$$\sup_m \left| \lim_k \sum_{n=0}^{m} a_{nk} \right| < \infty. \tag{3}$$

$A \in (bs, cs)$ if and only if (1) holds and

$$\lim_m \sum_k | \sum_{n=0}^{m} (a_{nk} - a_{n,k-1})| = \sum_k | \sum_n (a_{nk} - a_{n,k-1})|. \tag{4}$$

$A \in (cs, cs)$ if and only if

$$\sup_m \sum_k | \sum_{n=0}^{m} (a_{nk} - a_{n,k-1})| < \infty \tag{5}$$

and

$$\sum_n a_{nk} \text{ converges for all } k. \tag{6}$$

$A \in (cs, \ell_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk} - a_{n,k-1}| < \infty. \tag{7}$$

$A \in (cs, c)$ if and only if (7) holds and

$$\lim_n a_{nk} \text{ exists for all } k. \tag{8}$$

$A \in (cs, c_0)$ if and only if (7) holds and

$$\lim_n a_{nk} = 0. \tag{9}$$

When we are talking about operators and matrix transformations concerning the (cs, Y) class, where Y is some sequence space, especially important property of the cs space is that it is an AK space. Therefor, the next theorem is giving us the opportunity for creating some more new and useful results.

Theorem 2.1. *[13, Theorem 4.2.8] Let $X \supset \phi$ and Y be BK spaces. Then we have the following:*
a) $(X, Y) \subset B(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$ where $L_A(x) = Ax$ for all $x \in X$.
b) If X has AK then we have $B(X, Y) \subset (X, Y)$, that is, every operator $L \in B(X, Y)$ is given by a matrix $A \in (X, Y)$ where $Ax = L(x)$ for all $x \in X$.

We known that cs is AK space, so for every $L \in B(cs, Y)$, where Y is arbitrary sequence space, there exists an infinite matrix $A \in (cs, Y)$ such that $L(x) = Ax$ for every $x \in cs$. Hence, we can at the same time talk about general bounded linear operator on cs as well as the appropriate matrix operator L_A associated with matrix $A \in (cs, Y)$.

Before we proceed finding the conditions for compactness of certain operators, we need to mention one more interesting property of cs space i.e. a property of its dual space.

Theorem 2.2. *([13, Theorem 7.2.9], [7, Theorem 1.34]) Let X be a BK space, $\phi \subset X$. Then there is a linear one-to-one map $T : X^\beta \longrightarrow X^*$. If X has AK, then T is onto.*

Applying the results [13, Theorem 4.4.3], we obtain the representation for the functional f from cs^*:

$$f \in cs^* \iff f(x) = \mu \cdot \lim_{n\to\infty} \sum_{k=0}^{n} x_k + \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} x_k, \text{ where } (a_n)_{n=0}^\infty \in \ell_1.$$

On the other side, if we put $T = \Sigma$, $S = \Delta$, $R = S^t$ and apply result [8, Remark 3.3], related to $\beta-$dual of the space c_T to the class $A \in (cs, Y)$, for arbitrary sequence space Y, we obtain the following:

$$\eta = \lim_{n\to\infty} \sum_{k=0}^{n} x_k;$$

$$A_n \in cs^\beta = bv$$

and for $x \in cs$

$$A_n x = \sum_{k=0}^{\infty} \hat{a}_{nk} \sum_{j=0}^{k} x_j - \eta\alpha, \text{ where } \hat{a}_{nk} = a_{nk} - a_{n,k+1}.$$

Taking into account all mentioned, we can say that every bounded linear operator $L \in B(cs, Y)$, for arbitrary sequence space Y, can be represented by a matrix $A \in (cs, Y)$ such that $L(x) = Ax$ for each $x \in cs$.

3. Compactness for Some Classes of Operators

In this section, we will find under what conditions some classes of operators related to *cs* space are compact. First the definition and the most important properties of the Hausdorff measure of noncompactness will be listed, which is, due to the fundamental result by Goldenštein, Gohberg and Markus, the most effective way to characterize compact operators. It has been used in many research papers [2, 5, 7] recently.

Let (X, d) be a metric space, Q be a bounded subset of X and $K(x, r) = \{y \in X \mid d(x, y) < r\}$. Then the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\epsilon > 0 \mid Q \subset \bigcup_{i=1}^{n} K(x_i, r_i), x_i \in X,\ r_i < \epsilon\ (i = 1, \ldots, n),\ n \in N_0\}.$$

The following results and more properties of the measure of noncompactness can be found in [9] and [7].

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then we have

$\chi(Q) = 0$ if and only if Q is a totally bounded set,

$\chi(Q) = \chi(\overline{Q})$,

$Q_1 \subset Q_2$ implies $\chi(Q_1) \leq \chi(Q_2)$,

$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$

and

$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}$.

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have

$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)$,

$\chi(Q + x) = \chi(Q)\ (x \in X)$

and

$\chi(\lambda Q) = |\lambda|\chi(Q)$ for all $\lambda \in C$.

The Hausdorff measure of noncompactness of an operator $L \in B(X, Y)$, denoted by $\|L\|_\chi$, is defined as $\|L\|_\chi = \chi(L(S_X))$. We have that $\|L\|_\chi \leq \|L\|$ and L is a compact if and only if $\|L\|_\chi = 0$ [7, Corollary 2.26]

Theorem 3.1 (Goldenštein, Gohberg, Markus). *[7, Theorem 2.23] Let X be a Banach space with Schauder basis $\{e_1, e_2, ...\}$, Q be a bounded subset of X, and $P_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, ..., e_n\}$. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty}(\sup_{x\in Q} \|(I - P_n)x\|) \leq \chi(Q) \leq \limsup_{n\to\infty}(\sup_{x\in Q} \|(I - P_n)x\|),$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Specially, if $X = c$, then $a = 2$ in the previous theorem.

Theorem 3.2. *[10, Theorem 2.8.] Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, \ldots, x_n, 0, 0 \ldots)$ for $(x_0, x_1, \ldots, x_n, 0, 0 \ldots) \in X$, then*

$$\chi(Q) = \lim_{n\to\infty}(\sup_{x\in Q} \|(I - P_n)x\|).$$

Now, we can find under which conditions will a general bounded linear operator from a sequence space cs to one of the spaces c, c_0 and ℓ_∞ be compact.

Theorem 3.3. *Let $L \in B(cs, Y)$ where Y is one of the spaces c, c_0 or ℓ_∞. Then, L is given with a matrix $A = (a_{nk})_{n,k=0}^{\infty}$ such that $L(x) = Ax$ for every $x \in cs$, and we have:*

(i) if $Y = c$

$$\frac{1}{2} \cdot \lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) - \hat{\alpha}_k \mid + \mid \beta + \lim_{m\to\infty} a_{n,m+1} - \sum_{k=0}^{\infty} \hat{\alpha}_k \mid)) \leq \| L \|_\chi \leq$$

$$\leq \lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) - \hat{\alpha}_k \mid + \mid \beta + \lim_{m\to\infty} a_{n,m+1} - \sum_{k=0}^{\infty} \hat{\alpha}_k \mid))$$

where

$$\hat{\alpha}_k = \lim_{n\to\infty}(a_{nk} - a_{n,k+1}) \ for \ k = 0, 1, ...$$

and

$$\beta = \lim_{n\to\infty}(\sum_{k=0}^{\infty}(a_{nk} - a_{n,k+1}) + \lim_{m\to\infty} a_{n,m+1}) \ .$$

(ii) if $Y = c_0$ then

$$\| L \|_\chi = \lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) \mid + \mid \lim_{m\to\infty} a_{n,m+1} \mid))$$

(iii) if $Y = \ell_\infty$ then

$$0 \leq \| L \|_\chi \leq \lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) \mid + \mid \lim_{m\to\infty} a_{n,m+1} \mid)) \ .$$

Proof. Cases (i) and (ii) are simple consequences of theorem [2, Theorem 3.7]. The notations are the same and as usual, $\hat{a}_{nk}$ are the members of the matrix $\hat{A} \in (c, Y)$ associated with matrix $A \in (cs, Y)$, but knowing the matrices Σ, S and R, in our case we calculated

$$\hat{a}_{nk} = a_{nk} - a_{n,k+1} \text{ and } \gamma_n = -\lim_{m\to\infty} a_{n,m+1} \ .$$

In (iii) we start with projector $P_r : \ell_\infty \to \ell_\infty$ define as $P_r(x) = (x_0, x_1, ..., x_r, 0, 0, ...)$ and then, using the properties of measure χ, knowing the norm in ℓ_∞ and applying [2, Theorem 2.9 a)] we get:

$$0 \leq \chi(L(\overline{B}_{cs})) \leq \chi((I - P_r)(L(\overline{B}_{cs}))) \leq \sup_{x\in\overline{B}_{cs}} \| (I - P_r)(L(x)) \| \ \text{ for every } r$$

which leads to

$$0 \leq \| L \|_\chi \leq \lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid \hat{a}_{nk} \mid + \mid \gamma_n \mid)) \ .$$

□

Corollary 3.4. *1) $L \in B(cs, c)$ is compact if and only if*

$$\lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) - \hat{\alpha}_k \mid + \mid \beta + \lim_{m\to\infty} a_{n,m+1} - \sum_{k=0}^{\infty} \hat{\alpha}_k \mid)) = 0 \ . \tag{10}$$

2) $L \in B(cs, c_0)$ is compact if and only if

$$\lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) \mid + \mid \lim_{m\to\infty} a_{n,m+1} \mid)) = 0 \tag{11}$$

3) $L \in B(cs, \ell_\infty)$ is compact if

$$\lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid (a_{nk} - a_{n,k+1}) \mid + \mid \lim_{m\to\infty} a_{n,m+1} \mid)) = 0 \ . \tag{12}$$

Remark 3.5. *In the third case we have only "if condition" and, unlike some other cases in which this can be improved by using the work of Sargent [11], here it cannot be done because $cs = c_\Sigma$.*

The next theorem provide us with a very helpful result which will be used further on.

Theorem 3.6. *[8, Corollary 4.3] If $A \in (X, Y_T)$ then*
(i) $\| L_A \|_\chi = \| L_T \circ L_A \|_\chi$
and
(ii) $A \in (X, Y_T)$ *is compact if and only if* $TA \in (X, Y)$ *is compact* .

Before observing the compactness of some related operators, one more fact will be stated.

If with C we denote the matrix $C = \Sigma A = (c_{nk})_{n,k=0}^{\infty}$ we can calculate the following:

$$c_{nk} = \sum_{j=0}^{n} a_{jk},$$

$$\hat{c}_{nk} = R_k(C_n) = \sum_{j=0}^{n}(a_{jk} - a_{j,k+1}),$$

$$\gamma_n^{(C)} = \sum_{j=0}^{n} \gamma_j,$$

$$\hat{\alpha}_k(C) = \lim_{n\to\infty} \hat{c}_{nk}$$

and

$$\beta(C) = \lim_{n\to\infty}(\sum_{k=0}^{\infty} \hat{c}_{nk} - \gamma_n(C)).$$

Now, we can define the following result.

Theorem 3.7. *If $L \in B(cs, cs)$ then L is compact if and only if*

$$\lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid \hat{c}_{nk} - \hat{\alpha}_k(C) \mid + \mid \beta(C) - \gamma_n(C) - \sum_{k=0}^{\infty} \hat{\alpha}_k(C) \mid)) = 0 \ .$$

Proof. If we use Theorem 3.3, compactness of the class $B(cs, cs)$ where we use $cs = c_\Sigma$, will be reduced to that of the class $B(cs, c)$. □

Theorem 3.8. *If a matrix $A = (a_{nk})_{n,k=0}^{\infty}$ is from (bs, cs) then L_A is compact if and only if*

$$\lim_{r\to\infty}(\sup_{n\geq r}(\sum_{k=0}^{\infty} \mid \hat{c}_{nk} \mid + \mid \gamma_n(C) \mid)) = 0 \ .$$

Proof. Since $bs = (\ell_\infty)_\Sigma$, applying the result from [3] and Theorem 3.6, we obtain the result. □

Remark 3.9. *bs space does not have the AK property, so this theorem only "works" with a matrix operator.*

Theorem 3.10. *$L \in B(cs, bs)$ is compact if and only if for matrix $D = (d_{nk})_{n,k=0}^{\infty}$ where $d_{nk} = \sum_{j=0}^{k}(a_{jn} - a_{j,n-1})$ the following conditions hold:*

$$\sup_j \| (d_{nj})_{n=0}^{\infty} \|_\infty < \infty$$

and

$$\lim_{r\to\infty} \sup_j \| (d_{nj})_{n=r}^{\infty} \|_\infty = 0.$$

Proof. The space cs is AK space, so L is represented with a matrix $A \in (cs, bs)$ which is according to Theorem 3.6 the condition equivalent with $\Sigma A \in (cs, \ell_\infty)$. We know that $cs^\beta = bv$ [13, Theorem 7.3,5.], where $bv = (\ell_1)_\Delta$, $\ell_\infty = \ell_1^\beta$ and ℓ_1 is AK space. Now, if we use a result [13, Theorem 8.3.9] we get:

$$A \in (cs, bs) \iff \Sigma A \in (cs, \ell_\infty) \iff (\Sigma A)^t \in (\ell_1, bv) \iff \Delta(\Sigma A)^t \in (\ell_1, \ell_1)\ .$$

Let D be defined as $D = \Delta(\Sigma A)^t$. Applying the result [7, Theorem 2.28] for the compactness of operator $L \in B(\ell_1, \ell_1)$ to the operator associated with matrix D with entries d_{nk} defined as above, we obtain our result. □

Using the similar approach we can also improve the result in Corollary 3.4.3.

Theorem 3.11. *Let $L \in B(cs, \ell_\infty)$. Then L is compact if and only if*

$$\sup_{j\geq 0} \sum_{k=0}^{\infty} |a_{jk} - a_{j,k-1}| < \infty \ \text{ and } \ \lim_{n\to\infty} \sup_{j\geq 0} \sum_{k=0}^{\infty} |(a_{jk} - a_{j,k-1}) - (a_{jn} - a_{j,n-1})| = 0\ .$$

Proof. Since cs is AK space, for a bounded linear operator $L \in B(cs, \ell_\infty)$ exists a matrix $A \in (cs, \ell_\infty)$ such that $L(x) = Ax$ for every $x \in cs$. We know that $cs^\beta = bv$, $bv = (\ell_1)_\Delta$ $\ell_\infty = \ell_1^\beta$ i ℓ_1 is AK space, so we have

$$A \in (cs, \ell_\infty) \text{ is compact } \iff A^t \in (\ell_1, bv) \text{ is compact } \iff \Delta A^t \in (\ell_1, \ell_1) \text{ is compact}$$

For the last class if and only if conditions are given in [1, VI.B], and by applying them on matrix ΔA^t we obtain the result. □

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