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Simpson Type Integral Inequalities for Convex Functions via Riemann-Liouville Integrals

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Abstract. In this paper some new inequalities of Simpson-type are established for the classes of functions whose derivatives of absolute values are convex functions via Riemann-Liouville integrals. Also, by special selections of n,we give some reduced results.

1. Introduction

We will start with the following inequality that is well-known in the literature as Simpson's inequality and has several utilization in different fields of mathematics:

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $[a, b]$ and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in[a,b]} \left|f^{(4)}(x)\right| < \infty$. Then the folllowing inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4 .$$

Several researchers make effort to obtain new inequalities related to Simpson inequality. To consult some of them, one can take glance to the papers [5]-[10].

We will keep on our overview with the definition of convexity which is a significant concept for the inequality theory:

The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Let us call to mind the following notations and definitions:

Definition 1.1. *A real valued function $f(t)$, $t \ge 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exist a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty])$.*

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Definition 1.2. *A function* $f(t)$, $t \geq 0$ *is said to be in the space* C_μ^n, $n \in \mathbb{R}$, *if* $f^{(n)} \in C_\mu$.

Definition 1.3. *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *for a function* $f \in C_\mu$, $(\mu \geq -1)$ *is defined as*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha > 0, t > 0,$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

For further properties of this operator see the papers [1]-[4], [11] and [12].

The main purpose of the present paper is to give a new integral identity for the Riemann-Liouville fractional integrals and to prove several new integral inequalities that include generalizations for convex functions. In the conclusion part, we would like to call attention of the readers to appropriate selections of n give us refinements of previous studies.

2. The New Results

To prove our results, we obtain a new integral identity as following:

Lemma 2.1. $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b) *with* $a < b$. *If* $f' \in L[a,b]$, $n \geq 0$ *and* $\alpha > 0$, *then the following equality holds:*

$$\begin{aligned} & I(a,b;n,\alpha) = \frac{1}{6}\left[f(a) + f(b) + 2f\left(\frac{a+nb}{n+1}\right) + 2f\left(\frac{na+b}{n+1}\right)\right] \\ & -\frac{\Gamma(\alpha+1)(n+1)^\alpha}{6(b-a)^\alpha}\left[J_{a^+}^\alpha f\left(\frac{na+b}{n+1}\right) + J_{b^-}^\alpha f\left(\frac{a+nb}{n+1}\right)\right] \\ & -\frac{\Gamma(\alpha+1)(n+1)^\alpha}{3(b-a)^\alpha}\left[J_{\frac{a+nb}{n+1}^+}^\alpha f(b) + J_{\frac{na+b}{n+1}^-}^\alpha f(a)\right] \\ = \; & \frac{b-a}{2(n+1)}\left(\int_0^1 \left[\frac{2(1-t)^\alpha - t^\alpha}{3}\right] f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) dt\right. \\ & \left. + \int_0^1 \left[\frac{t^\alpha - 2(1-t)^\alpha}{3}\right] f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) dt\right) \end{aligned}$$

for all $x \in [a,b]$ *and where* $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Proof. Integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \left[\frac{2(1-t)^\alpha - t^\alpha}{3}\right] f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) dt \\ &= \frac{n+1}{3(b-a)}\left[f(a) + 2f\left(\frac{na+b}{n+1}\right)\right] - \frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}} \int_a^{\frac{na+b}{n+1}} f(x)\left(\frac{na+b}{n+1} - x\right)^{\alpha-1} dx \\ & \quad -\frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}} \int_a^{\frac{na+b}{n+1}} f(x)(x-a)^{\alpha-1} dx \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \left[\frac{t^\alpha - 2(1-t)^\alpha}{3}\right] f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right) dt \\ &= \frac{n+1}{3(b-a)}\left[f(b) + 2f\left(\frac{a+nb}{n+1}\right)\right] - \frac{\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}} \int_{\frac{a+nb}{n+1}}^b f(x)\left(x - \frac{a+nb}{n+1}\right)^{\alpha-1} dx \\ &\quad - \frac{2\alpha(n+1)^{\alpha+1}}{3(b-a)^{\alpha+1}} \int_{\frac{a+nb}{n+1}}^b f(x)(b-x)^{\alpha-1} dx. \end{aligned}$$

By adding I_1 and I_2 and multiplying the both sides $\frac{b-a}{2(n+1)}$, we can write

$$\begin{aligned} I_1 + I_2 &= \frac{1}{6}\left[f(a) + f(b) + 2f\left(\frac{a+nb}{n+1}\right) + 2f\left(\frac{na+b}{n+1}\right)\right] - \frac{\alpha(n+1)^\alpha}{6(b-a)^\alpha} \int_a^{\frac{na+b}{n+1}} f(x)\left(\frac{na+b}{n+1} - x\right)^{\alpha-1} dx \\ &\quad - \frac{\alpha(n+1)^\alpha}{3(b-a)^\alpha} \int_a^{\frac{na+b}{n+1}} f(x)(x-a)^{\alpha-1} dx - \frac{\alpha(n+1)^\alpha}{6(b-a)^\alpha} \int_{\frac{a+nb}{n+1}}^b f(x)\left(x - \frac{a+nb}{n+1}\right)^{\alpha-1} dx \\ &\quad - \frac{\alpha(n+1)^\alpha}{3(b-a)^\alpha} \int_{\frac{a+nb}{n+1}}^b f(x)(b-x)^{\alpha-1} dx. \end{aligned}$$

From the facts that,

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^{\frac{na+b}{n+1}} f(x)(x-a)^{\alpha-1} dx &= J^\alpha_{\frac{na+b}{n+1}^-} f(a) \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+nb}{n+1}}^b f(x)(b-x)^{\alpha-1} dx &= J^\alpha_{\frac{a+nb}{n+1}^+} f(b) \\ \frac{1}{\Gamma(\alpha)} \int_a^{\frac{na+b}{n+1}} f(x)\left(\frac{na+b}{n+1} - x\right)^{\alpha-1} dx &= J^\alpha_{a^+} f\left(\frac{na+b}{n+1}\right) \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+nb}{n+1}}^b f(x)\left(x - \frac{a+nb}{n+1}\right)^{\alpha-1} dx &= J^\alpha_{b^-} f\left(\frac{a+nb}{n+1}\right), \end{aligned}$$

we get the result. □

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$. If $f' \in L[a,b]$ and $\left|f'(x)\right|$ is convex function, then the following inequality holds for fractional integrals with $\alpha > 0$;*

$$|I(a,b;n,\alpha)| \leq \frac{b-a}{2(n+1)}\left[\frac{3 - 2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1} - 4\left(1 - \frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)}\right]\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

where $\Gamma(\alpha)$ is Euler Gamma function.

Proof. From the integral identity given in Lemma 1 and by using the properties of modulus, we have

$$\begin{aligned} |I(a,b;n,\alpha)| &\leq \frac{b-a}{2(n+1)}\left(\int_0^1 \left|\frac{2(1-t)^\alpha - t^\alpha}{3}\right| \left|f'\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right)\right| dt\right. \\ &\quad \left. + \int_0^1 \left|\frac{t^\alpha - 2(1-t)^\alpha}{3}\right| \left|f'\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right| dt\right). \end{aligned}$$

Since $\left|f'(x)\right|$ is convex function, we can write

$$\begin{aligned}
|I(a,b;n,\alpha)| &\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt\right.\\
&\quad\left.+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt\right)\\
&= \frac{b-a}{2(n+1)}\left(\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha-t^\alpha}{3}\right)\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt\right.\\
&\quad+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha-2(1-t)^\alpha}{3}\right)\left(\frac{n+t}{n+1}\left|f'(a)\right|+\frac{1-t}{n+1}\left|f'(b)\right|\right)dt\\
&\quad+\int_0^{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}\left(\frac{2(1-t)^\alpha-t^\alpha}{3}\right)\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt\\
&\quad\left.+\int_{\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}}^1\left(\frac{t^\alpha-2(1-t)^\alpha}{3}\right)\left(\frac{1-t}{n+1}\left|f'(a)\right|+\frac{n+t}{n+1}\left|f'(b)\right|\right)dt\right).
\end{aligned}$$

By a simple computation, we obtain desired result. □

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$. If $f'\in L[a,b]$ and $\left|f'(x)\right|^q$ is convex function, then the following inequality holds for fractional integrals with $\alpha>0$, $q>1$ and $p^{-1}+q^{-1}=1$;*

$$\begin{aligned}
|I(a,b;n,\alpha)| &\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q+\frac{1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{1}{2(n+1)}\left|f'(a)\right|^q+\frac{2n+1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

where $\Gamma(\alpha)$ is Euler Gamma function.

Proof. By using Lemma 1 and Hölder integral inequality, we can write

$$\begin{aligned}
&|I(a,b;n,\alpha)|\\
&\leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right)\\
&\leq \frac{b-a}{2(n+1)}\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Since $\left|f'(x)\right|^q$ is convex function, we can write

$$
\begin{aligned}
&|I(a,b;n,\alpha)| \\
\leq\ & \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|dt+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|dt\right) \\
\leq\ & \frac{b-a}{2(n+1)}\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{n+t}{n+1}\left|f'(a)\right|^q+\frac{1-t}{n+1}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1-t}{n+1}\left|f'(a)\right|^q+\frac{n+t}{n+1}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

By taking into account,

$$
\begin{aligned}
\int_0^1\left(\frac{n+t}{n+1}\left|f'(a)\right|^q+\frac{1-t}{n+1}\left|f'(b)\right|^q\right)dt &= \frac{2n+1}{2(n+1)}\left|f'(a)\right|^q+\frac{1}{2(n+1)}\left|f'(b)\right|^q \\
\int_0^1\left(\frac{1-t}{n+1}\left|f'(a)\right|^q+\frac{n+t}{n+1}\left|f'(b)\right|^q\right)dt &= \frac{1}{2(n+1)}\left|f'(a)\right|^q+\frac{2n+1}{2(n+1)}\left|f'(b)\right|^q,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|I(a,b;n,\alpha)| \leq\ & \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{2n+1}{2(n+1)}\left|f'(a)\right|^q+\frac{1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2(n+1)}\left|f'(a)\right|^q+\frac{2n+1}{2(n+1)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Which completes the proof. □

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$. If $f'\in L[a,b]$ and $\left|f'(x)\right|^q$ is convex function, then the following inequality holds for fractional integrals with $\alpha>0$ and $q\geq 1$;*

$$
\begin{aligned}
|I(a,b;n,\alpha)| \leq\ & \frac{b-a}{2(n+1)}\left(\frac{3-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left(K_1(\alpha,n)\left|f'(a)\right|^q+K_2(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(K_2(\alpha,n)\left|f'(a)\right|^q+K_1(\alpha,n)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

where $\Gamma(\alpha)$ is Euler Gamma function and

$$
\begin{aligned}
K_1(\alpha,n) &= \frac{(-4n-4)\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}+3n+2-2n\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)}+\frac{4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}-1}{3(\alpha+2)(n+1)} \\
K_2(\alpha,n) &= \frac{1-2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+1}}{3(\alpha+1)(n+1)}+\frac{1-4\left(1-\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}+2\left(\frac{2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}+1}\right)^{\alpha+2}}{3(\alpha+2)(n+1)}.
\end{aligned}
$$

Proof. By Lemma 1 and Power-Mean integral inequality, we can write

$$|I(a,b;n,\alpha)| \leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left|f'\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left|f'\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}.$$

By taking into account convexity of $|f'(x)|^q$, we get

$$|I(a,b;n,\alpha)| \leq \frac{b-a}{2(n+1)}\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\left(\int_0^1\left|\frac{2(1-t)^\alpha-t^\alpha}{3}\right|\left(\frac{n+t}{n+1}|f'(a)|^q+\frac{1-t}{n+1}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{t^\alpha-2(1-t)^\alpha}{3}\right|\left(\frac{1-t}{n+1}|f'(a)|^q+\frac{n+t}{n+1}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right).$$

Computing the above integrals, we get the result. □

3. Conclusion

In this section, we would like to point out some results that are special cases of our main results.

Remark 3.1. *In Lemma 1, if we choose $\alpha = 1$, we have Lemma 1.1 of [10].*

Remark 3.2. *In Theorem 1-2-3, if we choose $\alpha = 1$, we obtain Theorem 2.1., Teorem 2.2. and Theorem 2.3. in [10], respectively.*

Remark 3.3. *In Theorem 1, if we choose $\alpha = 1$ and $n = 1$, we have*

$$\left|\frac{1}{6}\left[f(a)+f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right| \leq \frac{5(b-a)}{72}\left(|f'(a)|+|f'(b)|\right).$$

which is the Corollary 1 of [9].

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