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Time-like Loxodromes on Helicoidal Surfaces in Minkowski 3-Space

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Abstract. Loxodromes in Euclidean 3-space are often used in navigation. We study time-like loxodromes which cut all meridians on helicoidal surfaces at a constant Lorentzian angle in Minkowski 3-space.

1. Introduction

A curve which cuts all meridians on a rotational surface (or a helicoidal surface) at a constant angle is called as a loxodrome. The equations of the loxodromes on the rotational surfaces in Euclidean 3-space were obtained by Noble [8]. Babaarslan and Munteanu [1] studied time-like loxodromes on the rotational surfaces in Minkowski 3-space. Also the equations of space-like loxodromes on the rotational surfaces in same space were obtained by Babaarslan and Yayli [2]. A natural generalization of the rotational surfaces is helicoidal surfaces. Loxodromes on helicoidal surfaces in Euclidean 3-space were studied by Babaarslan and Yayli [3]. Also they gave some important applications of them. Differential equations of the space-like loxodromes on helicoidal surfaces in Minkowski 3-space were found by Babaarslan and Kayacik [4].

In this paper, by using similar differential geometry methods, we obtain the equations of time-like loxodromes which cut all meridians on helicoidal surfaces at a constant Lorentzian angle in Minkowski 3-space. Also we give some examples of time-like loxodromes via Mathematica computer program.

2. Preliminaries

Let $\mathbb{E}_1^3$ be Minkowski 3-space. For two arbitrary vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$, the Lorentzian scalar product is given by

$$< u, v >= u_1v_1 + u_2v_2 - u_3v_3. \tag{1}$$

Also the pseudo-norm of the vector $u \in \mathbb{E}_1^3$ is defined by

$$\|u\| = \sqrt{|\langle u, u\rangle|}. \tag{2}$$

In $\mathbb{E}_1^3$, an arbitrary vector u has one of the following causal characters;

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i. it is space-like if $\langle u, u\rangle > 0$ or $u = 0$,

ii. it is time-like if $\langle u, u\rangle < 0$,

iii. it is light-like if $\langle u, u\rangle = 0$ and $u \neq 0$.

Let $\alpha : I \to \mathbb{E}^3_1$ be a regular curve in $\mathbb{E}^3_1$, where $I \subset \mathbb{R}$ is an open interval. The regular curve α is called;

i. space-like if $\langle \dot{\alpha}, \dot{\alpha}\rangle > 0$,

ii. time-like if $\langle \dot{\alpha}, \dot{\alpha}\rangle < 0$,

iii. light-like if $\langle \dot{\alpha}, \dot{\alpha}\rangle = 0$ (see [7]).

Let $S : U \to \mathbb{E}^3_1$ be a smooth immersed surface in $\mathbb{E}^3_1$, where $U \subset \mathbb{R}^2$ is an open set. S is non-degenerate if the induced metric on its tangent plane (its first fundamental form) is non-degenerate. The non-degenerate surface S is called;

i. space-like if its first fundamental form is a Riemannian metric,

ii. time-like if its first fundamental form is a Lorentzian metric (see [9]).

A helicoidal surface H in $\mathbb{E}^3_1$ is defined as the orbit of a plane curve (profile curve) under a Lorentzian screw motion (Lorentzian rotation about an axis together with a translation in the direction of the axis) [6]. By using the Lorentzian screw motions, three different types of helicoidal surfaces can be obtained in $\mathbb{E}^3_1$ as follows:

Case i. Taking profile curve $\beta = \beta(u) = (f(u), 0, g(u))$, $u \in I \subset \mathbb{R}$, we can obtain the following helicoidal surface whose rotation axis is space-like;

$$H(u, v) = (f(u) + \lambda v, g(u)\sinh v, g(u)\cosh v), \tag{3}$$

where $g(u) \neq 0$ and $\lambda \in \mathbb{R}^+$.

Case ii. Taking profile curve $\beta = \beta(u) = (0, f(u), g(u))$, $u \in I \subset \mathbb{R}$, we can obtain the following helicoidal surface whose rotation axis is time-like;

$$H(u, v) = (-f(u)\sin v, f(u)\cos v, g(u) + \lambda v), \tag{4}$$

where $f(u) \neq 0$ and $\lambda \in \mathbb{R}^+$.

Case iii. Taking profile curve $\beta = \beta(u) = (0, f(u), g(u))$, we can obtain the following helicoidal surface whose rotation axis is light-like;

$$H(u, v) = \Big((f(u) - g(u))v, (g(u) - f(u))\frac{v^2}{2} + f(u) + \lambda v, (g(u) - f(u))\frac{v^2}{2} + g(u) + \lambda v\Big), \tag{5}$$

where $f(u) \neq g(u)$ and $\lambda \in \mathbb{R}^+$.

If we take $\lambda = 0$ in the equations (3)-(5), then we have the rotational surfaces in $\mathbb{E}^3_1$ (see [4], [5]).

A basis of the tangent plane at each point of helicoidal surface H can be given by $\{H_u, H_v\}$. Thus the first fundamental form of H is

$$I = ds^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{6}$$

where $E = \langle H_u, H_u\rangle$, $F = \langle H_u, H_v\rangle$ and $G = \langle H_v, H_v\rangle$ are the coefficients of first fundamental form of H.

By using these coefficients, one can give the causal characters of the non-degenerate surfaces. For example; H is a space-like or time-like surface if and only if $\det(I) = EG - F^2 > 0$ or $\det(I) = EG - F^2 < 0$, respectively (see [7], [11]).

Also the arc-length of any curve on the helicoidal surface H between u_1 and u_2 can be defined by

$$s = \int_{u_1}^{u_2} \sqrt{\left|E + 2F\frac{dv}{du} + G(\frac{dv}{du})^2\right|}du \tag{7}$$

(see [4]).

3. Time-like Loxodromes on the Helicoidal Surfaces Having Space-like Meridians

In this section, we obtain the equations of time-like loxodromes on the helicoidal surfaces having space-like meridians with space-like, time-like and light-like axis, respectively. Firstly, we give the following definition:

Definition 3.1. *If u is a space-like vector and v is a time-like vector in $\mathbb{E}^3_1$. Then*

$$|\langle u, v\rangle| = \|u\|\|v\| \sinh\varphi,$$

where $\varphi \in \mathbb{R}^+ \cup \{0\}$ is the Lorentzian time-like angle between u and v [10].

3.1. Time-like loxodromes on the helicoidal surfaces having space-like meridians with space-like axis

Let us consider the helicoidal surface H which is given by (3). Also assume that $f'^2(u) - g'^2(u) = 1$ for all $u \in J \subset \mathbb{R}$. The meridian curve ($v$ = constant) is given by

$$H(u) = (f(u) + \lambda v, g(u)\sinh v, g(u)\cosh v).$$

If we derivative with respect to u, then we have

$$H_u(u) = (f'(u), g'(u)\sinh v, g'(u)\cosh v).$$

Since the meridian curve is space-like, we have

$$\langle H_u(u), H_u(u)\rangle = f'^2(u) - g'^2(u) = 1$$

for all $u \in J \subset \mathbb{R}$.

The coefficients of first fundamental form of helicoidal surface H are

$$E = \langle H_u, H_u\rangle = 1,\ F = \langle H_u, H_v\rangle = \lambda f'(u) \text{ and } G = \langle H_v, H_v\rangle = g^2(u) + \lambda^2. \tag{8}$$

By using (6) and (8), the first fundamental form of H is given by

$$ds^2 = du^2 + 2\lambda f'(u)dudv + (g^2(u) + \lambda^2)dv^2.$$

The helicoidal surface H is time-like if and only if $EG - F^2 = g^2(u) - \lambda^2 g'^2(u) < 0$ for all $u \in J \subset \mathbb{R}$.

Let us assume that the time-like loxodrome $\alpha(t)$ is the image of a curve $(u(t), v(t))$ lying on (uv)-plane under H. At the point $H(u, v)$ where the time-like loxodrome cuts the space-like meridians at a constant Lorentzian time-like angle φ, we have

$$\begin{aligned}\varepsilon\sinh\varphi &= \frac{Edu + Fdv}{\sqrt{-E^2du^2 - 2EFdudv - EGdv^2}}\\ &= \frac{du + \lambda f'(u)dv}{\sqrt{-du^2 - 2\lambda f'(u)dudv - (g^2(u) + \lambda^2)dv^2}}.\end{aligned} \tag{9}$$

From (9), we obtain the following differential equation of the time-like loxodrome:

$$\Big(\sinh^2\varphi(g^2(u) + \lambda^2) + \lambda^2 f'^2(u)\Big)\big(\frac{dv}{du}\big)^2 + 2\lambda\cosh^2\varphi f'(u)\frac{dv}{du} = -\cosh^2\varphi. \tag{10}$$

The general solution of (10) is

$$v = \int_{u_0}^{u} \frac{-2\lambda\cosh^2\varphi f'(u) + \varepsilon\sqrt{\sinh^2 2\varphi\,(-g^2(u) + \lambda^2(f'^2(u) - 1))}}{2\sinh^2\varphi\,(g^2(u) + \lambda^2) + 2\lambda^2 f'^2(u)}du, \tag{11}$$

where $\varepsilon = \pm 1$.

Now we give an example.

Example 3.2. Taking $f(u) = 2u$, $g(u) = \sqrt{3}u$, $\lambda = 2$, $\varepsilon = 1$, $\varphi = 2$, $u \in (-1, 1)$ and $u_0 = 0$, we have $v \in (-0.139041, 0.139041)$. Thus the arc-length of the time-like loxodrome is equal to 0.845363. Also we can draw the time-like helicoidal surface, the space-like meridian ($v = 0$) and the time-like loxodrome in Figure 1.

Figure 1: Time-like loxodrome (blue), space-like meridian (green)

3.2. Time-like loxodromes on the helicoidal surfaces having space-like meridians with time-like axis

Let us consider the helicoidal surface H which is given by (4). Thus the meridian curve is

$$H(u) = (-f(u)\sin v, f(u)\cos v, g(u) + \lambda v).$$

Differentiating with respect to u yields

$$H_u(u) = (-f'(u)\sin v, f'(u)\cos v, g'(u)).$$

The meridian curve $H(u)$ and the profile curve $\beta(u)$ have same causal character, because

$$\langle H_u(u), H_u(u)\rangle = f'^2(u) - g'^2(u) = 1$$

for all $u \in J \subset \mathbb{R}$.

The coefficients of first fundamental form of H are

$$E = 1,\ F = -\lambda g'(u) \text{ and } G = f^2(u) - \lambda^2. \tag{12}$$

Thus we have

$$ds^2 = du^2 - 2\lambda g'(u)dudv + (f^2(u) - \lambda^2)dv^2.$$

The helicoidal surface H is time-like if and only if $f^2(u) - \lambda^2 f'^2(u) < 0$ for all $u \in J \subset \mathbb{R}$.

The Lorentzian time-like angle φ between the time-like loxodrome $\alpha(t)$ and the space-like meridian $H(u)$ is defined by the angle φ between their tangent vectors at the point $H(u, v)$ and it is given by

$$\varepsilon \sinh\varphi = \frac{du - \lambda g'(u)dv}{\sqrt{-du^2 + 2\lambda g'(u)dudv - (f^2(u) - \lambda^2)dv^2}}. \tag{13}$$

If we arrange this equation, then we obtain the following differential equation

$$\Big(\sinh^2\varphi(f^2(u)-\lambda^2)+\lambda^2 g'^2(u)\Big)\Big(\frac{dv}{du}\Big)^2-2\lambda\cosh^2\varphi g'(u)\frac{dv}{du}=-\cosh^2\varphi. \tag{14}$$

Thus the general solution of this differential equation is given by

$$v=\int_{u_0}^{u}\frac{2\lambda\cosh^2\varphi g'(u)+\varepsilon\sqrt{\sinh^2 2\varphi\,(-f^2(u)+\lambda^2(g'^2(u)+1))}}{2\sinh^2\varphi\,(f^2(u)-\lambda^2)+2\lambda^2 g'^2(u)}du, \tag{15}$$

where $\varepsilon=\pm 1$.

Also the following example can be given.

Example 3.3. Taking $f(u)=u$, $g(u)=2$, $\lambda=2$, $\varepsilon=1$, $\varphi=1$, $u\in(-2,2)$ and $u_0=0$, we have $v\in(-2.06251, 2.06251)$. Also the arc-length of the time-like loxodrome is equal to 0.70184. We can draw the time-like helicoidal surface, the space-like meridian ($v=0$) and the time-like loxodrome in Figure 2.

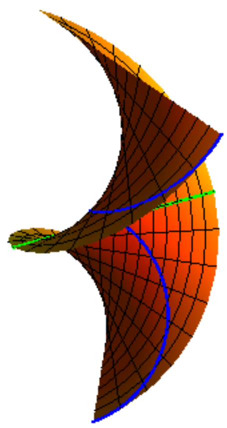

Figure 2: Time-like loxodrome (blue), space-like meridian (green)

3.3. Time-like loxodromes on the helicoidal surfaces having space-like meridians with light-like axis

Let us consider the helicoidal surface H which is given by (5). The meridian curve is

$$H(u)=\Big((f(u)-g(u))v,(g(u)-f(u))\frac{v^2}{2}+f(u)+\lambda v,(g(u)-f(u))\frac{v^2}{2}+g(u)+\lambda v\Big),$$

and it is space-like if and only if $f'^2(u)-g'^2(u)=1$ for all $u\in J\subset\mathbb{R}$.

The coefficients of first fundamental form of H are given by

$$E=1,\ F=\lambda(f'(u)-g'(u)) \text{ and } G=(f(u)-g(u))^2. \tag{16}$$

Substituting these equations into (6), the first fundamental form of H is found as

$$ds^2=du^2+2\lambda(f'(u)-g'(u))dudv+(f(u)-g(u))^2dv^2.$$

The helicoidal surface H is time-like if and only if

$$(f(u)-g(u))^2-\lambda^2(f'(u)-g'(u))^2<0$$

for all $u \in J \subset \mathbb{R}$.

The Lorentzian time-like angle φ between time-like loxodrome and space-like meridian is given by the following equation

$$\varepsilon \sinh \varphi = \frac{du + \lambda(f'(u) - g'(u))dv}{\sqrt{-du^2 - 2\lambda(f'(u) - g'(u))dudv - (f(u) - g(u))^2 dv^2}}. \tag{17}$$

Thus the differential equation of the time-like loxodrome is

$$\left(\sinh^2 \varphi (f(u) - g(u))^2 + \lambda^2 (f'(u) - g'(u))^2\right)\left(\frac{dv}{du}\right)^2 + 2\lambda \cosh^2 \varphi (f'(u) - g'(u))\frac{dv}{du} = -\cosh^2 \varphi, \tag{18}$$

and its general solution is

$$v = \int_{u_0}^{u} \frac{-2\lambda \cosh^2 \varphi (f'(u) - g'(u)) + \varepsilon \sqrt{\sinh^2 2\varphi \left(-(f(u) - g(u))^2 + \lambda^2 (f'(u) - g'(u))^2\right)}}{2\sinh^2 \varphi (f(u) - g(u))^2 + 2\lambda^2 (f'(u) - g'(u))^2} du, \tag{19}$$

where $\varepsilon = \pm 1$.

Example 3.4. Taking $f(u) = \sinh u$, $g(u) = \cosh u$, $\lambda = 2$, $\varepsilon = 1$, $\varphi = 1$, $u \in (-1, 1)$ and $u_0 = 0$, we get $v \in (-0.192045, 0.505886)$. Thus the arc-length of the time-like loxodrome is equal to 0.338181. We can draw the time-like helicoidal surface, the space-like meridian ($v = 0.2$) and the time-like loxodrome in Figure 3.

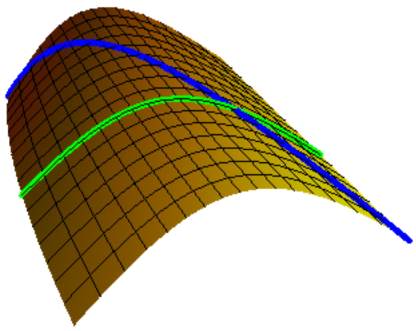

Figure 3: Time-like loxodrome (blue), space-like meridian (green)

4. Time-like Loxodromes on the Helicoidal Surfaces Having Time-like Meridians

In this section, we study time-like loxodromes on the helicoidal surfaces having time-like meridians with space-like, time-like and light-like axis, respectively. Firstly, we give the following definition:

Definition 4.1. *If u and v are positive (negative) time-like vectors in $\mathbb{E}^3_1$. Then*

$$\langle u, v \rangle = -\|u\|\|v\| \cosh \theta,$$

where $\theta \in \mathbb{R}^+$ is the Lorentzian time-like angle between u and v [10].

4.1. Time-like loxodromes on the helicoidal surfaces having time-like meridians with space-like axis

Let us consider the helicoidal surface H which is given by (3). Since the profile curve $\beta(u)$ is parametrized by arc-length, we have

$$f'^2(u) - g'^2(u) = -1$$

for all $u \in J \subset \mathbb{R}$. The meridian curve is given by

$$H(u) = (f(u) + \lambda v, g(u)\sinh v, g(u)\cosh v).$$

Differentiating with respect to u, we have

$$H_u(u) = (f'(u), g'(u)\sinh v, g'(u)\cosh v).$$

The meridian curve $H(u)$ and the profile curve $\beta(u)$ have same causal character, because

$$\langle H_u(u), H_u(u)\rangle = f'^2(u) - g'^2(u) = -1$$

for all $u \in J \subset \mathbb{R}$.

The coefficients of first fundamental form of H are given by

$$E = -1,\ F = \lambda f'(u) \text{ and } G = g^2(u) + \lambda^2. \tag{20}$$

Thus we get

$$ds^2 = -du^2 + 2\lambda f'(u)dudv + (g^2(u) + \lambda^2)dv^2.$$

The helicoidal surface H is time-like, because $EG - F^2 = -g^2(u) - \lambda^2 g'^2(u) < 0$ for all $u \in J \subset \mathbb{R}$.

The Lorentzian time-like angle θ between time-like loxodrome $\alpha(t)$ and time-like meridian $H(u)$ is given by the following formulation of differential geometry

$$-\cosh\theta = \frac{-du + \lambda f'(u)dv}{\sqrt{du^2 - 2\lambda f'(u)dudv - (g^2(u) + \lambda^2)dv^2}}. \tag{21}$$

From this equation, we obtain

$$\left(\cosh^2\theta(g^2(u) + \lambda^2) + \lambda^2 f'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2\lambda\sinh^2\theta f'(u)\frac{dv}{du} = \sinh^2\theta \tag{22}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{-2\lambda\sinh^2\theta f'(u) + \varepsilon\sqrt{\sinh^2 2\theta\,(g^2(u) + \lambda^2(f'^2(u) + 1))}}{2\cosh^2\theta\,(g^2(u) + \lambda^2) + 2\lambda^2 f'^2(u)}du, \tag{23}$$

where $\varepsilon = \pm 1$.

Let us give the following example.

Example 4.2. Taking $f(u) = 1$, $g(u) = u$, $\lambda = 1$, $\varepsilon = 1$, $\theta = 1/2$, $u \in (0, 1)$ and $u_0 = 0$, we have $v \in (0, 0.407298)$. Thus the arc-length of the time-like loxodrome is equal to 0.886819. Also we can draw the time-like helicoidal surface, the time-like meridian ($v = 0.25$) and the time-like loxodrome in Figure 4.

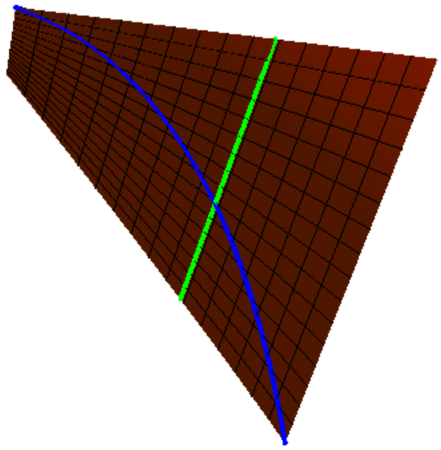

Figure 4: Time-like loxodrome (blue), time-like meridian (green)

4.2. Time-like loxodromes on the helicoidal surfaces having time-like meridians with time-like axis

Let us consider the helicoidal surface H which is given by (4). The meridian curve $H(u)$ is

$$H(u) = (-f(u)\sin v, f(u)\cos v, g(u) + \lambda v).$$

$H(u)$ is time-like if and only if

$$\langle H_u(u), H_u(u)\rangle = f'^2(u) - g'^2(u) = -1$$

for all $u \in J \subset \mathbb{R}$.

The coefficients of first fundamental form of H are

$$E = -1,\ \ F = -\lambda g'(u) \text{ and } G = f^2(u) - \lambda^2. \tag{24}$$

Substituting these equations into (6), the first fundamental form of H is

$$ds^2 = -du^2 - 2\lambda g'(u)dudv + (f^2(u) - \lambda^2)dv^2.$$

The helicoidal surface H is time-like, because $EG - F^2 = -f^2(u) - \lambda^2 f'^2(u) < 0$ for all $u \in J \subset \mathbb{R}$.

The Lorentzian time-like angle θ between the time-like loxodrome $\alpha(t)$ and the time-like meridian $H(u)$ is given by

$$-\cosh\theta = \frac{-du - \lambda g'(u)dv}{\sqrt{du^2 + 2\lambda g'(u)dudv - (f^2(u) - \lambda^2)dv^2}}. \tag{25}$$

From (25), the differential equation of the time-like loxodrome is

$$\left(\cosh^2\theta(f^2(u) - \lambda^2) + \lambda^2 g'^2(u)\right)(\frac{dv}{du})^2 - 2\lambda\sinh^2\theta g'(u)\frac{dv}{du} = \sinh^2\theta, \tag{26}$$

and its general solution is

$$v = \int_{u_0}^{u} \frac{2\lambda\sinh^2\theta g'(u) + \varepsilon\sqrt{\sinh^2 2\theta\,(f^2(u) + \lambda^2(g'^2(u) - 1))}}{2\cosh^2\theta\,(f^2(u) - \lambda^2) + 2\lambda^2 g'^2(u)}du, \tag{27}$$

where $\varepsilon = \pm 1$.

Also the following example can be given.

Example 4.3. Taking $f(u) = u$, $g(u) = \sqrt{2}u$, $\lambda = 1$, $\varepsilon = -1$, $\theta = 1/4$, $u \in (0, 1/4)$ and $u_0 = 0$, we have $v \in (-0.07159, 0)$. Also the arc-length of the time-like loxodrome is equal to 0.129974. We can draw the time-like helicoidal surface, the time-like meridian ($v = -0.04$) and the time-like loxodrome in Figure 5.

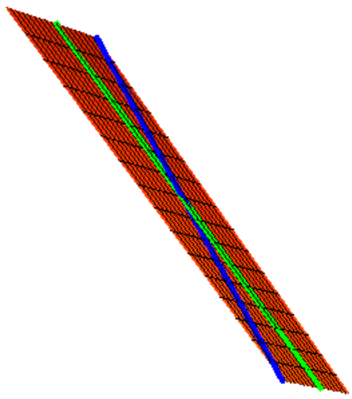

Figure 5: Time-like loxodrome (blue), time-like meridian (green)

4.3. Time-like loxodromes on the helicoidal surfaces having time-like meridians with light-like axis

Let us consider the helicoidal surface H which is given by (5). The meridian curve is given by

$$H(u) = \Big((f(u) - g(u))v, (g(u) - f(u))\frac{v^2}{2} + f(u) + \lambda v, (g(u) - f(u))\frac{v^2}{2} + g(u) + \lambda v\Big),$$

and it is time-like if and only if $f'^2(u) - g'^2(u) = -1$ for all $u \in J \subset \mathbb{R}$.

The coefficients of first fundamental form of H are

$$E = -1,\ F = \lambda(f'(u) - g'(u)) \text{ and } G = (f(u) - g(u))^2. \tag{28}$$

Substituting the equations in (28) into (6), we find

$$ds^2 = -du^2 + 2\lambda(f'(u) - g'(u))dudv + (f(u) - g(u))^2 dv^2.$$

The helicoidal surface H is time-like, because

$$EG - F^2 = -(f(u) - g(u))^2 - \lambda^2(f'(u) - g'(u))^2 < 0$$

for all $u \in J \subset \mathbb{R}$.

As it was mentioned earlier, at the intersection point $H(u, v)$, we get

$$-\cosh\theta = \frac{-du + \lambda(f'(u) - g'(u))dv}{\sqrt{du^2 - 2\lambda(f'(u) - g'(u))dudv - (f(u) - g(u))^2 dv^2}}. \tag{29}$$

From this equation, the differential equation of the time-like loxodrome is

$$\Big(\cosh^2\theta(f(u) - g(u))^2 + \lambda^2(f'(u) - g'(u))^2\Big)(\frac{dv}{du})^2 + 2\lambda\sinh^2\theta(f'(u) - g'(u))\frac{dv}{du} = \sinh^2\theta. \tag{30}$$

The general solution of (30) is

$$v = \int_{u_0}^{u} \frac{-2\lambda \sinh^2\theta(f'(u) - g'(u)) + \varepsilon\sqrt{\sinh^2 2\theta\left((f(u) - g(u))^2 + \lambda^2(f'(u) - g'(u))^2\right)}}{2\cosh^2\theta(f(u) - g(u))^2 + 2\lambda^2(f'(u) - g'(u))^2}du, \tag{31}$$

where $\varepsilon = \pm 1$.

Finally, we give the following example.

Example 4.4. Taking $f(u) = \cosh u$, $g(u) = \sinh u$, $\lambda = 1$, $\varepsilon = 1$, $\theta = 1$, $u \in (1, 2)$ and $u_0 = 0$, we get $v \in (0.601447, 2.23635)$. Thus the arc-length of the time-like loxodrome is equal to 1.256. We can draw the time-like helicoidal surface, the time-like meridian ($v = 1.5$) and the time-like loxodrome in Figure 6.

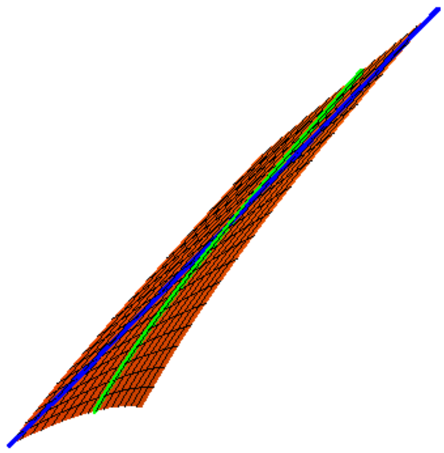

Figure 6: Time-like loxodrome (blue), time-like meridian (green)

Remark 4.5. *If we take $\lambda = 0$ in the equations (23), (27) and (31), respectively, then we find the equations of the time-like loxodromes on the rotational surfaces having time-like meridians in Minkowski 3-space. In other words, these equations coincide with the equations in [1].*

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