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The Lower and Upper Solution Method for Three-Point Boundary Value Problems with Integral Boundary Conditions on a Half-Line

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Abstract. This paper deal with the following second-order three-point boundary value problem with integral boundary condition on a half-line

$$u''(x) + q(x)f(x, u(x), u'(x)) = 0, \quad x \in (0, +\infty),$$
$$u(0) = \lambda \int_0^\eta u(s)ds, \quad u'(+\infty) = C,$$

where $\lambda > 0$, $0 < \lambda\eta < 1$ and $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo's condition which plays an important role in the nonlinear term depend on the first-order derivative explicitly. By using Schauder's fixed point theorem, the upper and lower solution method and topological degree theory, first we give sufficient conditions for the existence of at least one solution and next at least three solutions of the above problem. Moreover, an example is included to demonstrate the efficiency of the main results.

1. Introduction

In this paper, we shall examine an existence theory for second-order ordinary differential equations together with integral boundary conditions on a half-line

$$\begin{aligned} &u''(x) + q(x)f(x, u(x), u'(x)) = 0, \quad x \in (0, +\infty), \\ &u(0) = \lambda \int_0^\eta u(s)ds, \quad \lim_{x\to+\infty} u'(x) = u'(+\infty) = C, \end{aligned} \tag{1}$$

where $\lambda > 0$, $0 < \lambda\eta < 1$, $q : (0, +\infty) \to (0, +\infty)$, $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and $C \geq 0$. By applying the upper and lower solutions method, we give easily verifiable sufficient conditions for the existence of solutions of BVP (1). These solutions may be unbounded in this paper.

Multi-point boundary value problems for second-order differential equations in a finite interval and on an infinite interval included the large amount of priori work and many excellent results are obtained by using Avery-Peterson fixed point theorem, shooting method, lower and upper solution method, Leray-Schauder continuation theorem and so on, see for instance [1-13,15]. Meanwhile, BVPs with integral

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boundary conditions for ordinary differential equations have been extensively examined by many authors, for example see [11-16]. But, there is a little work related to boundary value problems with integral boundary conditions on an infinite interval.

In [12], Akcan and Hamal considered the boundary value problem (BVP):

$$\begin{aligned}&u''(x) + f(x, u(x), u'(x)) = 0, \quad x \in (0, 1),\\ &u(0) = u(1) = \alpha \int_0^\eta u(s)ds,\end{aligned}$$

where $f : (0, 1) \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is continuous and $\alpha, \eta \in (0, 1)$. In that study, the proof was based upon Avery and Peterson fixed point theorem.

In [5], Lian and Geng examined Sturm-Liouville boundary value problem on a half-line:

$$\begin{aligned}&u''(t) + \phi(t)f(t, u(t), u'(t)) = 0, \ \ t \in (0, +\infty),\\ &u(0) - au'(0) = B, \ \ u'(+\infty) = C,\end{aligned} \tag{2}$$

where $\phi : (0, +\infty) \to (0, +\infty)$, $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $a > 0, B, C \in \mathbb{R}$. By assuming the existence of two pairs of unbounded upper and lower solutions, they showed that the problem (2) has at least three solutions on a half-line.

Motivated and inspired by the above works, we present existence theory of solutions for the BVP (1). The plan of our paper is as follows: In Section 2, we give some definitions and lemmas which we need to prove the main results. This includes the construction of Green's function for a second-order boundary value problem with integral boundary conditions, properties of Green's function, definitions of upper and lower solutions of (1) and Nagumo's condition. In Section 3, we present two main results. In our first result, we use Schauder's fixed point theorem to establish the existence of at least one solution of (1) which lies between the assumed pair of upper and lower solutions. In our second result, we assume the existence of two pairs of upper and lower solutions and employ the degree theory to prove the existence of at least three solutions of (1). Finally, we demonstrate the importance of our results through one example.

2. Preliminaries

For the convenience of the reader, in this section we provide some necessary definitions and preparatory results which will be needed to prove the the existence of solutions of (1). We begin with constructing Green's function for the linear boundary value problem

$$\begin{aligned}&u''(x) + v(x) = 0, \ \ x \in (0, +\infty),\\ &u(0) = \lambda \int_0^\eta u(s)ds, \ \ u'(+\infty) = C.\end{aligned} \tag{3}$$

Lemma 2.1. *Let $v \in C[0, +\infty)$ and $\int_0^\infty v(s)ds < +\infty$. Then the solution $u \in C^1[0, +\infty) \cap C^2(0, +\infty)$ of the problem (3) can be expressed as*

$$u(x) = \frac{C\lambda\eta^2}{2(1 - \lambda\eta)} + Cx + \int_0^\infty G(x, s)v(s)ds,$$

where

$$G(x, s) = \frac{1}{2(1 - \lambda\eta)} \begin{cases} 2s - \lambda s^2, & 0 \le s \le \min\{\eta, x\} < +\infty;\\ 2\lambda\eta(s - x) + 2x - \lambda s^2, & 0 \le x \le s \le \eta < +\infty;\\ \lambda\eta^2 + 2s - 2\lambda\eta s, & 0 \le \eta \le s \le x < +\infty;\\ \lambda\eta^2 + 2x - 2\lambda\eta x, & 0 \le \max\{\eta, x\} \le s < +\infty.\end{cases} \tag{4}$$

Proof. Since $v \in C[0,+\infty)$ and $\int_0^\infty v(s)ds < +\infty$, we can integrate (3) from x to $+\infty$, and use $u'(+\infty) = C$, to get

$$u'(x) = C + \int_x^\infty v(s)ds.$$

Integrating the above equation on [0,x] and applying Fubini's theorem, we obtain

$$u(x) = u(0) + Cx + \int_0^x sv(s)ds + \int_x^\infty xv(s)ds. \tag{5}$$

Integrating (5) from 0 to η, we have

$$\int_0^\eta u(s)ds = u(0)\eta + C\frac{\eta^2}{2} + \int_0^\eta (\eta - s)sv(s)ds + \int_0^\eta \frac{s^2}{2}v(s)ds + \int_\eta^\infty \frac{\eta^2}{2}v(s)ds,$$

and from $u(0) = \lambda \int_0^\eta u(s)ds$, we have

$$u(0) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + \frac{\lambda}{1-\lambda\eta}\int_0^\eta (\eta - s)sv(s)ds + \frac{\lambda}{1-\lambda\eta}\int_0^\eta \frac{s^2}{2}v(s)ds + \frac{\lambda}{1-\lambda\eta}\int_\eta^\infty \frac{\eta^2}{2}v(s)ds.$$

Hence from (5), we have

$$u(x) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + Cx$$
$$+\begin{cases} \displaystyle\int_0^x \frac{2s-\lambda s^2}{2(1-\lambda\eta)}v(s)ds + \int_x^\eta \frac{2\lambda\eta(s-x)+2x-\lambda s^2}{2(1-\lambda\eta)}v(s)ds + \int_\eta^\infty \frac{\lambda\eta^2+2x-2\lambda\eta x}{2(1-\lambda\eta)}v(s)ds, & x \le \eta; \\ \displaystyle\int_0^\eta \frac{2s-\lambda s^2}{2(1-\lambda\eta)}v(s)ds + \int_\eta^x \frac{\lambda\eta^2+2s-2\lambda\eta s}{2(1-\lambda\eta)}v(s)ds + \int_x^\infty \frac{\lambda\eta^2+2x-2\lambda\eta x}{2(1-\lambda\eta)}v(s)ds, & \eta \le x; \end{cases}$$

which is the same as

$$u(x) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + Cx + \int_0^\infty G(x,s)v(s)ds, \ \forall x \in [0,+\infty).$$

This completes the proof of the lemma. □

Lemma 2.2. *Let the Green function $G(x,s)$ be as in (4). Then for all $x, s \in [0,+\infty)$, $\lambda > 0$ and $0 < \lambda\eta < 1$, $G(x,s)$ is continuous and $G(x,s) \ge 0$.*

Proof. The continuity of $G(x,s)$ with respect to $(x,s) \in [0,+\infty)\times[0,+\infty)$ is clear. Let define

$$g_1(x,s) = 2s - \lambda s^2 \text{ for } s \in [0,\min\{x,\eta\}], \ \ g_2(x,s) = 2\lambda\eta(s-x) + 2x - \lambda s^2 \text{ for } s \in [x,\eta],$$

$$g_3(x,s) = \lambda\eta^2 + 2s - 2\lambda\eta s \text{ for } s \in [\eta, x] \text{ and } g_4(x,s) = \lambda\eta^2 + 2x - 2\lambda\eta x \ \text{ for } s \in [\max\{x,\eta\},+\infty).$$

We solely need to prove that $g_1(x,s) \ge 0$ for $0 \le s \le \min\{x,\eta\} < +\infty$, because the proofs of others are similar. From the definition of $g_1(x,s)$, we have

$$g_1(x,s) = 2s - \lambda s^2 = s(2-\lambda s) \ge s(2-\lambda\eta) > s \ge 0,$$

for $0 \le s \le \min\{x,\eta\} < +\infty$, which completes the proof. □

Lemma 2.3. *For any $s \in [0,+\infty)$, $G(x,s)$ is nondecreasing with respect to x, that is for any $s \in [0,+\infty)$, $\frac{\partial G(x,s)}{\partial x} \ge 0$, $x \in [0+\infty)$. Moreover, $G(0,s) \le G(x,s) \le G(s,s)$.*

Proof. From (4) it is easy to see that $\dfrac{\partial G(x,s)}{\partial x} \geq 0$, for $s, x \in [0+\infty)$; this means $G(x,s)$ is nondecreasing with respect to x. Because of this and $0 \leq x$, we obtain $G(0,s) \leq G(x,s)$ where

$$G(x,s) \geq G(0,s) = \frac{1}{2(1-\lambda\eta)} \begin{cases} 2\lambda\eta s - \lambda s^2, & s \leq \eta; \\ \lambda\eta^2, & \eta \leq s. \end{cases}$$

By using nondecreasing of G with respect to x, we have

$$\begin{aligned} \frac{G(x,s)}{G(s,s)} &= \begin{cases} 1, & 0 \leq s \leq \min\{\eta, x\} < +\infty; \\ \dfrac{2\lambda\eta(s-x) + 2x - \lambda s^2}{2s - \lambda s^2}, & 0 \leq x \leq s \leq \eta < +\infty; \\ 1, & 0 \leq \eta \leq s \leq x < +\infty; \\ \dfrac{\lambda\eta^2 + 2x(1-\lambda\eta)}{\lambda\eta^2 + 2s(1-\lambda\eta)}, & 0 \leq \max\{\eta, x\} \leq s < +\infty; \end{cases} \\ &\leq 1, \end{aligned}$$

which implies $G(x,s) \leq G(s,s)$ for $s, x \in [0+\infty)$. This completes the proof of the lemma. □

Let

$$X = \left\{ u \in C^1[0,+\infty) : \lim_{x\to+\infty} \frac{u(x)}{1+x} \text{ and } \lim_{x\to+\infty} u'(x) \text{ exist} \right\}$$

with the norm $\|u\| = \max\{\|u\|_1, \|u\|_\infty\}$, where

$$\|u\|_1 = \sup_{x\in[0,+\infty)} \frac{|u(x)|}{1+x}, \quad \|u\|_\infty = \sup_{x\in[0,+\infty)} |u'(x)|.$$

Then by the standard arguments, it follows that $(X, \|.\|)$ is a Banach space. In what follows, we shall need the following modified version of the Arzela-Ascoli lemma [16].

Lemma 2.4. *Let $M \subset X$. Then M is relatively compact if the following conditions hold:*

1. *all functions from M are uniformly bounded in X;*
2. *the functions in $\{y : y = \frac{u}{1+x}, u \in M\}$ and $\{z : z = u'(x), u \in M\}$ are locally equi-continuous on $[0,+\infty)$;*
3. *the functions in $\{y : y = \frac{u}{1+x}, u \in M\}$ and $\{z : z = u'(x), u \in M\}$ are equi-convergent at $+\infty$, that is, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that*

   $$|y(x) - y(+\infty)| < \epsilon, \quad |z(x) - z(+\infty)| < \epsilon,$$

   for all $x > \delta$, and $u \in M$.

Definition 2.5. *A function $\alpha \in X \cap C^2(0,+\infty)$ is called a lower solution of (1) if*

$$\alpha''(x) + q(x)f(x, \alpha(x), \alpha'(x)) > 0, \quad x \in (0,+\infty), \tag{6}$$

$$\alpha(0) \leq \lambda \int_0^\eta \alpha(s)ds, \quad \alpha'(+\infty) \leq C. \tag{7}$$

Similarly, a function $\beta \in X \cap C^2(0,+\infty)$ is called an upper solution of (1) if

$$\beta''(x) + q(x)f(x, \beta(x), \beta'(x)) < 0, \quad x \in (0,+\infty), \tag{8}$$

$$\beta(0) \geq \lambda \int_0^\eta \beta(s)ds, \quad \beta'(+\infty) \geq C. \tag{9}$$

Definition 2.6. *We say $\alpha(\beta)$ is a strict lower solution (strict upper solution) for problem (1) if the above inequality (7) (or(9)) is strict for $x \in (0,+\infty)$.*

Definition 2.7. *Let $\alpha, \beta \in X \cap C^2(0, +\infty)$ be a pair of lower and upper solutions of (1) satisfying $\alpha(x) \leq \beta(x)$, $x \in [0, +\infty)$. A continuous function $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is said to satisfy the Nagumo's condition with respect to the pair of functions α, β, if there exist a nonnegative function $\phi \in C[0, +\infty)$ and a positive function $h \in C[0, +\infty)$ such that*

$$|f(x, u, v)| \leq \phi(x)h(|v|) \tag{10}$$

for all $x \in [0, +\infty)$, $\alpha(x) \leq u \leq \beta(x)$, $v \in \mathbb{R}$ and

$$\int_0^\infty \frac{s}{h(s)} ds = +\infty. \tag{11}$$

3. Main Results

The following result guarantees the existence of at least one solution of the problem (1).

Theorem 3.1. *Assume that α, β are the lower and upper solutions of (1) satisfying $\alpha(x) \leq \beta(x)$, and suppose that $f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous satisfying the Nagumo's condition with respect to the pair of functions α, β. If*

$$\int_0^\infty \max\{s, 1\}q(s)ds < +\infty, \int_0^\infty \max\{s, 1\}q(s)\phi(s)ds < +\infty \tag{12}$$

and there exists a constant $\gamma > 1$ such that

$$m = \sup_{x \in [0, +\infty)} (1 + x)^\gamma q(x)\phi(x) < +\infty \tag{13}$$

where $\phi(x)$ is the function in Nagumo's condition of f, then (1) has at least one solution $u \in X \cap C^2(0, +\infty)$ satisfying

$$\alpha(x) \leq u(x) \leq \beta(x), \ |u'(x)| < N \text{ for all } x \in [0, +\infty);$$

here, N is a constant depending on α, β, h and C.

Proof. We can choose an r such that

$$r \geq \max\left\{ \sup_{x \in [0, +\infty)} \beta'(x), \sup_{x \in [0, +\infty)} \alpha'(x), C \right\}, \tag{14}$$

and an $N > r$ such that

$$\int_r^N \frac{s}{h(s)} ds > m\left(\sup_{x \in [0, +\infty)} \frac{\beta(x)}{(1 + x)^\gamma} - \inf_{x \in [0, +\infty)} \frac{\alpha(x)}{(1 + x)^\gamma} + \|\beta\|_1 \frac{\gamma}{\gamma - 1} \right). \tag{15}$$

We define the following auxiliary functions

$$f_1(x, u, v) = \begin{cases} f(x, \beta, v), & u > \beta(x); \\ f(x, u, v), & \alpha(x) \leq u \leq \beta(x); \\ f(x, \alpha, v), & u < \alpha(x), \end{cases}$$

and

$$f^*(x, u, v) = \begin{cases} f_1(x, u, N), & v > N; \\ f_1(x, u, v), & |v| \leq N; \\ f_1(x, u, -N), & v < -N. \end{cases} \tag{16}$$

Now we consider the modified problem

$$\begin{aligned} &u''(x) + q(x)f^*(x, u(x), u'(x)) = 0, \quad x \in (0, +\infty), \\ &u(0) = \lambda \int_0^\eta u(s)ds, \ u'(+\infty) = C. \end{aligned} \tag{17}$$

As an application of Schauder's fixed point theorem we will prove that (17) has at least one solution u such that $\alpha(x) \leq u(x) \leq \beta(x)$ and $|u'(x)| < N$, $x \in [0, +\infty)$. To show this, for $u \in X$, we define two operators as follows

$$(T_1 u)(x) = \int_0^\infty G(x,s)q(s)f^*(s,u(s),u'(s))ds,\ x \in [0,+\infty), \tag{18}$$

and

$$(Tu)(x) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + Cx + (T_1u)(x),\ x \in [0,+\infty). \tag{19}$$

Now we divide the proof into three steps.

Step 1. From the following three parts we shall conclude that $T : X \to X$ is completely continuous.
(1) $T : X \to X$ is well defined: For each $u \in X$, in view of (10), (12) and (16), we have

$$\begin{aligned} \left|\int_0^\infty q(s)f^*(s,u(s),u'(s))ds\right| &\leq \int_0^\infty H_0 q(s)\phi(s)ds \\ &\leq \int_0^\infty \max\{s,1\}H_0q(s)\phi(s)ds < +\infty, \end{aligned} \tag{20}$$

where $H_0 = \max\limits_{0 \leq x \leq \|u\|_\infty} h(x)$. For $u \in X$, we find from (20) that

$$\int_1^\infty sH_0q(s)\phi(s)ds \leq \int_0^\infty \max\{s,1\}H_0q(s)\phi(s)ds < +\infty. \tag{21}$$

Since

$$\int_x^\infty H_0q(s)\phi(s)ds \leq \int_x^\infty sH_0q(s)\phi(s)ds < +\infty,\ x \geq 1, \tag{22}$$

it follows that

$$\lim_{x\to+\infty}\int_x^\infty H_0q(s)\phi(s)ds = 0. \tag{23}$$

Thus by Lemma 2.2, Lemma 2.3 and (21) , we have

$$\begin{aligned} \left|\lim_{x\to+\infty}\frac{(T_1u)(x)}{1+x}\right| &\leq \lim_{x\to+\infty}\int_0^\infty \frac{G(x,s)}{1+x}H_0q(s)\phi(s)ds \\ &\leq \lim_{x\to+\infty}\frac{1}{1+x}\int_0^\infty G(s,s)H_0q(s)\phi(s)ds \\ &= \lim_{x\to+\infty}\frac{1}{1+x}\int_0^\eta \frac{(2s-\lambda s^2)}{2(1-\lambda\eta)}H_0q(s)\phi(s)ds + \lim_{x\to+\infty}\frac{1}{1+x}\int_\eta^\infty \frac{(\lambda\eta^2+2s-2\lambda\eta s)}{2(1-\lambda\eta)}H_0q(s)\phi(s)ds \\ &= 0, \end{aligned}$$

which implies $\lim\limits_{x\to+\infty}\dfrac{(T_1u)(x)}{1+x} = 0$. Therefore, it follows that

$$\lim_{x\to+\infty}\frac{(Tu)(x)}{1+x} = \lim_{x\to+\infty}\frac{\frac{C\lambda\eta^2}{2(1-\lambda\eta)}+Cx}{1+x} + \lim_{x\to+\infty}\frac{(T_1u)(x)}{1+x} = C.$$

Now from (23), we have

$$\left|\lim_{x\to+\infty}\int_x^\infty q(s)f^*(s,u(s),u'(s))ds\right| \leq \lim_{x\to+\infty}\int_x^\infty H_0q(s)\phi(s)ds = 0,$$

and hence

$$\lim_{x\to+\infty}(Tu)'(x) = \lim_{x\to+\infty} C + \int_x^{\infty} q(s)f^*(s, u(s), u'(s))ds = C.$$

Consequently, it follows that $Tu \in X$.

(2) $T : X \to X$ is continuous. For any convergent sequence $u_n \to u$ in X, we have

$$u_n(x) \to u(x),\ \ u'_n(x) \to u'(x),\ \ n \to +\infty,\ x \in [0, +\infty).$$

Thus the continuity of f^* implies that

$$|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))| \to 0,\ n \to +\infty,\ s \in [0, +\infty).$$

Since $u'_n(x) \to u'(x)$, we have $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < +\infty$. Let $H_1 = \max_{0\le x\le \max\{\|u\|_\infty, \sup_{n\in\mathbb{N}}\|u_n\|_\infty\}} h(x)$. Then we obtain

$$\begin{aligned}
&\int_0^{\infty} sq(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\le 2\int_0^{\infty} sH_1 q(s)\phi(s)ds < +\infty.
\end{aligned} \tag{24}$$

Therefore from the Lebesgue dominated convergence theorem and (24) it follows that

$$\begin{aligned}
\|Tu_n - Tu\|_1 &= \sup_{x\in[0,+\infty)} \frac{|Tu_n(x) - Tu(x)|}{1+x} = \sup_{x\in[0,+\infty)} \frac{|T_1u_n(x) - T_1u(x)|}{1+x} \\
&= \sup_{x\in[0,+\infty)} \left|\int_0^{\infty} \frac{G(x,s)}{1+x} q(s)(f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s)))ds\right| \\
&\le \sup_{x\in[0,+\infty)} \int_0^{\infty} G(x,s)q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\le \int_0^{\infty} G(s,s)q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&= \int_0^{\eta} \frac{(2s-\lambda s^2)}{2(1-\lambda\eta)} q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\quad + \int_\eta^{\infty} \frac{(\lambda\eta^2 + 2s - 2\lambda\eta s)}{2(1-\lambda\eta)} q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\le \int_0^{\eta} \frac{s(2-\lambda s)}{2(1-\lambda\eta)} q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\quad + \int_\eta^{\infty} \frac{s(\lambda\eta + 2 - 2\lambda\eta)}{2(1-\lambda\eta)} q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds \\
&\le \frac{1}{(1-\lambda\eta)} \int_0^{\infty} sq(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds,
\end{aligned}$$

which approaches zero as $n \to \infty$. Lastly, we have

$$\begin{aligned}
\|Tu_n - Tu\|_\infty &= \sup_{x\in[0,+\infty)} |(Tu_n)'(x) - (Tu)'(x)| = \sup_{x\in[0,+\infty)} |(T_1u_n)'(x) - (T_1u)'(x)| \\
&= \sup_{x\in[0,+\infty)} \left|\int_x^{\infty} q(s)(f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s)))ds\right| \\
&\le \int_0^{\infty} q(s)|f^*(s, u_n(s), u'_n(s)) - f^*(s, u(s), u'(s))|ds,
\end{aligned}$$

which approaches zero as $n \to \infty$. As a result $\|Tu_n - Tu\| \to 0$, as $n \to +\infty$; so $T : X \to X$ is continuous.
(3) We will next show that $T : X \to X$ is relatively compact. Let A be any bounded subset of X, then for $u \in A$, let $H_2 = \max\limits_{0 \le x \le \|u\|_\infty, u \in A} h(x) < +\infty$, similar to the above proof, we get

$$\begin{aligned}
\|Tu\|_1 &= \sup_{x\in[0,+\infty)} \frac{|Tu(x)|}{1+x} \\
&= \sup_{x\in[0,+\infty)} \left| \frac{C\lambda\eta^2}{2(1-\lambda\eta)(1+x)} + \frac{Cx}{1+x} + \int_0^\infty \frac{G(x,s)}{1+x} q(s) f^*(s,u(s),u'(s))ds \right| \\
&\le \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + C + \frac{1}{(1-\lambda\eta)} \int_0^\infty sq(s)|f^*(s,u(s),u'(s))|ds \\
&\le \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + C + \frac{1}{(1-\lambda\eta)} \int_0^\infty sH_2 q(s)\phi(s)ds < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\|Tu\|_\infty &= \sup_{x\in[0,+\infty)} |(Tu)'(x)| \\
&= \sup_{x\in[0,+\infty)} \left| C + \int_x^\infty q(s) f^*(s,u(s),u'(s))ds \right| \\
&\le C + \int_0^\infty H_2 q(s)\phi(s)ds < +\infty,
\end{aligned}$$

which implies that $\|Tu\| < +\infty$. Thus TA is uniformly bounded. Meanwhile, for any $k > 0$, if $x_1, x_2 \in [0,k]$, we have

$$\begin{aligned}
\left| \frac{(Tu)(x_1)}{1+x_1} - \frac{(Tu)(x_2)}{1+x_2} \right| &\le \left| \frac{C\lambda\eta^2}{2(1-\lambda\eta)(1+x_1)} - \frac{C\lambda\eta^2}{2(1-\lambda\eta)(1+x_2)} \right| + \left| \frac{Cx_1}{1+x_1} - \frac{Cx_2}{1+x_2} \right| \\
&\quad + \int_0^\infty \left| \frac{G(x_1,s)}{1+x_1} - \frac{G(x_2,s)}{1+x_2} \right| q(s)|f^*(s,u(s),u'(s))|ds \\
&\le \left| \frac{C\lambda\eta^2}{2(1-\lambda\eta)(1+x_1)} - \frac{C\lambda\eta^2}{2(1-\lambda\eta)(1+x_2)} \right| + \left| \frac{Cx_1}{1+x_1} - \frac{Cx_2}{1+x_2} \right| \\
&\quad + \int_0^\infty \left| \frac{G(x_1,s)}{1+x_1} - \frac{G(x_2,s)}{1+x_2} \right| H_2 q(s)\phi(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
|(Tu)'(x_1) - (Tu)'(x_2)| &= \left| \int_{x_1}^\infty q(s) f^*(s,u(s),u'(s))ds - \int_{x_2}^\infty q(s) f^*(s,u(s),u'(s))ds \right| \\
&\le \int_{x_1}^{x_2} H_2 q(s)\phi(s)ds.
\end{aligned}$$

Then, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{(Tu)(x_1)}{1+x_1} - \frac{(Tu)(x_2)}{1+x_2} \right| < \epsilon,\ |(Tu)'(x_1) - (Tu)'(x_2)| < \epsilon$$

if $|x_1 - x_2| < \delta,\ x_1, x_2 \in [0,k]$.
Since k is arbitrary, we know the functions belonging to $\left\{ \frac{TA}{1+x} \right\}$ and the functions belonging to $\{(TA)'\}$ are locally equi-continuous on $[0,+\infty)$. Now for $u \in A$ one has

$$\left| \frac{(Tu)(x)}{1+x} - \lim_{x\to+\infty} \frac{(Tu)(x)}{1+x} \right| = \left| \frac{(Tu)(x)}{1+x} - C \right| \to 0, \text{ as } x \to +\infty,$$

and

$$|(Tu)'(x) - \lim_{x\to+\infty}(Tu)'(x)| = |(Tu)'(x) - C| = \left|\int_x^{\infty} f^*(s, u(s), u'(s))ds\right| \to 0,$$

as $x \to +\infty$ which yield that the functions from $\left\{\frac{TA}{1+x}\right\}$ and the functions from $\{(TA)'\}$ are equi-convergent at $+\infty$.

Consequently, the conditions of Lemma 2.4 hold and so TX is relatively compact. Therefore $T : X \to X$ is completely continuous. Schauder's fixed point theorem guarantees that T has at least one fixed point $u \in X$ which is a solution of (17).

Step 2. If u is a solution of (17), then it holds $\alpha(x) \leq u(x) \leq \beta(x)$, $x \in [0, +\infty)$. We solely need to show $\alpha(x) \leq u(x)$, $x \in [0, +\infty)$ since the proof of the other is analogous. If $\alpha(x) \leq u(x)$ on $[0, +\infty)$, is not true, then there exists $x_0 \in [0, +\infty)$ such that

$$\alpha(x_0) > u(x_0) \text{ and } u(x_0) - \alpha(x_0) = \inf_{x\in[0,+\infty)}(u(x) - \alpha(x)) < 0.$$

Now in view of $\lim_{x\to+\infty}(u'(x) - \alpha'(x)) \geq 0$, there are three cases.

Case 1. If $x_0 \in (0, +\infty)$, then we have $u(x_0) < \alpha(x_0)$, $u'(x_0) = \alpha'(x_0)$ and $u''(x_0) \geq \alpha''(x_0)$. Since $u(x_0) < \alpha(x_0)$, $u'(x_0) = \alpha'(x_0)$ and $\sup_{x\in[0,+\infty)} |\alpha'(x)| < N$, we have $f^*(x_0, u(x_0), u'(x_0)) = f(x_0, \alpha(x_0), \alpha'(x_0))$ and

$$0 \leq u''(x_0) - \alpha''(x_0) \quad < \quad -q(x_0)[f^*(x_0, u(x_0), u'(x_0)) - f(x_0, \alpha(x_0), \alpha'(x_0))] = 0,$$

which is a contradiction.

Case 2. If $x_0 = 0$ and $u(0) - \alpha(0) = \inf_{x\in[0,+\infty)}(u(x) - \alpha(x)) < 0$, then for all $s \in [0, \eta]$, $u(s) - \alpha(s) \geq u(0) - \alpha(0)$ and we have

$$\lambda \int_0^{\eta} (u(s) - \alpha(s))ds \geq \lambda \int_0^{\eta} (u(0) - \alpha(0))ds = \lambda\eta(u(0) - \alpha(0)).$$

Moreover from boundary conditions we obtain

$$\lambda \int_0^{\eta} (u(s) - \alpha(s))ds = u(0) - \lambda \int_0^{\eta} \alpha(s)ds \leq u(0) - \alpha(0),$$

then we have

$$0 \leq (1 - \lambda\eta)(u(0) - \alpha(0)),$$

unfortunately from $0 < \lambda\eta < 1$ and $u(0) - \alpha(0) < 0$, we have a contradiction.

Case 3. If $\lim_{x\to+\infty} u(x) - \alpha(x) = \inf_{x\in[0,+\infty)}(u(x) - \alpha(x)) < 0$, then for $\forall x \in [0, +\infty)$, $u'(x) - \alpha'(x) \leq 0$ and there exists a $x < +\infty$ is big enough such that for $\forall s \in [x, +\infty)$, $u(s) - \alpha(s) < 0$. Obviously,

$$\begin{aligned} u'(x) - \alpha'(x) &= \int_x^{\infty} (q(s)f^*(s, u(s), u'(s)) + \alpha''(s))ds \\ &= \int_x^{\infty} (q(s)f(s, \alpha(s), \alpha'(s)) + \alpha''(s))ds \\ &> 0, \end{aligned}$$

which is also a contradiction. Therefore,

$$\alpha(x) \leq u(x), \text{ for all } x \in [0, +\infty).$$

Step 3. Lastly, we show that $|u'(x)| < N$ for $x \in [0, +\infty)$. Suppose that there is a $x_0 \in [0, +\infty)$ with $|u'(x_0)| \geq N$. Since $\lim_{x\to+\infty} u'(x) = C < N$, there exists a $T > 0$ such that

$$|u'(x)| < N, \ \forall x \geq T.$$

Let $x_1 = \inf\{x \leq T : |u'(s)| < N, \forall s \in [x, +\infty)\}$. Then $|u'(x_1)| = N$ and $|u'(x)| < N$ for all $x > x_1$ and there exists a x_2 such that $|u'(x)| \geq N$ for $x \in [x_2, x_1]$. We have two cases $u'(x_1) = N$ and $u'(x) \geq N$ for $x \in [x_2, x_1]$ or $u'(x_1) = -N$ and $u'(x) \leq -N$ for $x \in [x_2, x_1]$. We assume that $u'(x_1) = N$ and $u'(x) \geq N$ for $x \in [x_2, x_1]$ then we have

$$\begin{aligned}
\int_r^N \frac{s}{h(s)}ds &\leq \int_C^N \frac{s}{h(s)}ds \\
&= -\int_{x_1}^{\infty} \frac{u'(s)}{h(u'(s))}u''(s)ds \\
&= -\int_{x_1}^{\infty} \frac{-q(s)f(s,u(s),u'(s))u'(s)}{h(u'(s))}ds \\
&\leq \int_{x_1}^{\infty} q(s)\phi(s)u'(s)ds \\
&\leq m\int_{x_1}^{\infty} \frac{u'(s)}{(1+s)^{\gamma}}ds \\
&= m\left(\int_{x_1}^{\infty} \left(\frac{u(s)}{(1+s)^{\gamma}}\right)' ds - \int_{x_1}^{\infty} u(s)\left(\frac{1}{(1+s)^{\gamma}}\right)' ds\right) \\
&\leq m\left(\sup_{x\in[0,+\infty)} \frac{\beta(x)}{(1+x)^{\gamma}} - \inf_{x\in[0,+\infty)} \frac{\alpha(x)}{(1+x)^{\gamma}} + \|\beta\|_1 \frac{\gamma}{\gamma-1}\right) \\
&< \int_r^N \frac{s}{h(s)}ds,
\end{aligned}$$

which is a contradiction. If $u'(x_1) = -N$ and $u'(x) \leq -N$ for $x \in [x_2, x_1]$, a similar contradiction can be obtained. Hence, $|u'(x)| < N$ for all $x \in [0, +\infty)$. Consequently,

$$u''(x) = -q(x)f^*(x,u(x),u'(x)) = -q(x)f_1(x,u(x),u'(x)) = -q(x)f(x,u(x),u'(x)).$$

So, u is a solution of (1). □

Before we establish the existence of at least three solutions of the problem (1), we give the following theorem which is important to the strategy to obtain three solutions.

Theorem 3.2 ([17]). *Let X show a Banach space and let $\Omega \subset X$ be an open bounded set. Assume that I be identity function on X and $T : \overline{\Omega} \longrightarrow X$ is a compact function. Let $p \in X$, $p \notin (I-T)(\partial\Omega)$ and $d(I-T, \Omega, p)$ show the degree of $(I-T)$ at p depend on Ω. Then*

(i) (Domain decomposition property) If $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where Ω_i is open sets and mutually disjoint, then
$$d(I-T,\Omega,p) = d(I-T,\Omega_1,p) + d(I-T,\Omega_2,p) + d(I-T,\Omega_3,p).$$

(ii) (Excision property) If $K \subset \overline{\Omega}$ is a compact set such that $p \notin (I-T)(K)$, then

$$d(I-T,\Omega,p) = d(I-T,\Omega\setminus K,p).$$

Theorem 3.3. *Assume that there exist strict lower and upper solutions α_2, β_1 and lower and upper solutions α_1, β_2 of BVP (1) satisfying*

$$\alpha_1(x) \leq \alpha_2(x) \leq \beta_2(x),\ \ \alpha_1(x) \leq \beta_1(x) \leq \beta_2(x),\ \alpha_2(x) \nleq \beta_1(x)\ for\ x \in [0,+\infty).$$

Suppose that $f : [0,+\infty)\times\mathbb{R}^2 \to \mathbb{R}$ is continuous function satisfying the Nagumo's condition with respect to the pair of functions α_1, β_2. If (12) and (13) hold, then (1) has at least three solutions $u_1, u_2, u_3 \in X \cap C^2(0,+\infty)$ satisfying

$$\alpha_1(x) \leq u_1(x) \leq \beta_1(x),\ \alpha_2(x) \leq u_2(x) \leq \beta_2(x),\ u_3(x) \nleq \beta_1(x),\ u_3(x) \ngeq \alpha_2(x),\ x \in [0,+\infty).$$

Proof. We define the auxiliary function f_1^* similar to f^* in Theorem 3.1 such that α and β are replaced with α_1 and β_2, respectively. We consider the modified equation

$$\begin{aligned} &u''(x) + q(x)f_1^*(x, u(x), u'(x)) = 0, \quad x \in (0, +\infty), \\ &u(0) = \lambda \int_0^\eta u(s)ds, \ \ u'(+\infty) = C. \end{aligned} \tag{25}$$

We want to show that (25) has at least three solutions. We define an operator by

$$(T_2u)(x) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + Cx + \int_0^\infty G(x,s)q(s)f_1^*(s, u(s), u'(s))ds.$$

We can prove that $T_2 : X \to X$ is completely continuous as T in Theorem 3.1. By using the degree theory, we will show that T_2 has at least three fixed points which are the solutions of (25). For $x \in \overline{\Omega}$, similar to the above proof in Theorem 3.1, we can find

$$\begin{aligned} &\|T_2u\|_1 \le \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + C + \frac{1}{(1-\lambda\eta)}\int_0^\infty sH_3q(s)\phi(s)ds := k_1, \\ &\|T_2u\|_\infty \le C + \int_0^\infty H_3q(s)\phi(s)ds := k_2, \end{aligned}$$

where $H_3 = \sup\limits_{0\le x\le\|u\|_\infty} h(x) < +\infty$, $\Omega = \{u \in X : \|u\| < K\}$ and $K > \max\{k_1, k_2\}$. Then we obtain $\|T_2u\| < K$, which implies that $T_2\overline{\Omega} \subset \Omega$. Thus $\deg(I - T_2, \Omega, 0) = 1$. We take

$$\Omega_{\alpha_2} = \{u \in \Omega : u(x) > \alpha_2(x),\ x \in [0, +\infty)\},\ \Omega^{\beta_1} = \{u \in \Omega : u(x) < \beta_1(x),\ x \in [0, +\infty)\}.$$

By $\alpha_2(x) \nleq \beta_1(x)$, $\alpha_1(x) \le \alpha_2(x) \le \beta_2(x)$, $\alpha_1(x) \le \beta_1(x) \le \beta_2(x)$, $x \in [0, +\infty)$, we have $\overline{\Omega}_{\alpha_2} \cap \overline{\Omega^{\beta_1}} = \varnothing$ and the set $\Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}} \neq \varnothing$. Because of the strict upper and lower solutions β_1, α_2 and Definition 2.6, T_2 has no solution in $\partial\Omega_{\alpha_2} \cup \partial\Omega^{\beta_1\cdot}$ From Theorem 3.2 (i), we get

$$\deg(I - T_2, \Omega, 0) = \deg(I - T_2, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}, 0) + \deg(I - T_2, \Omega_{\alpha_2}, 0) + \deg(I - T_2, \Omega^{\beta_1}, 0). \tag{26}$$

First, we show that $\deg(I - T_2, \Omega_{\alpha_2}, 0) = 1$. For this, we define completely continuous operator $T_3 : \overline{\Omega} \to \overline{\Omega}$ by

$$(T_3u)(x) = \frac{C\lambda\eta^2}{2(1-\lambda\eta)} + Cx + \int_0^\infty G(x,s)q(s)f_2^*(s, u(s), u'(s))ds,$$

where the function f_2^* is similar to f_1^* except α_1 is replaced with α_2. In a way similar to that the proof of Theorem 3.1 it is easy to prove that T_3 has a fixed point x satisfies $\alpha_2(x) \le u(x) \le \beta_2(x)$, $x \in [0, +\infty)$. Since the lower solution α_2 is strict and Definition 2.6, $u(x) \neq \alpha_2(x)$, $x \in [0, +\infty)$. Therefore, $u \in \Omega_{\alpha_2}$. Hence

$$\deg(I - T_3, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) = 0.$$

Moreover, we can show $T_3\overline{\Omega} \subset \Omega$. Then we obtain

$$\deg(I - T_3, \Omega, 0) = 1. \tag{27}$$

Since $f_2^* = f$ in the region Ω_{α_2}, we have

$$\begin{aligned} \deg(I - T_2, \Omega_{\alpha_2}, 0) &= \deg(I - T_3, \Omega_{\alpha_2}, 0) \\ &= \deg(I - T_3, \Omega_{\alpha_2}, 0) + \deg(I - T_3, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) \\ &= \deg(I - T_3, \Omega, 0) = 1. \end{aligned} \tag{28}$$

Similar to the proof of (28), we have

$$\deg(I - T_2, \Omega^{\beta_1}, 0) = 1. \tag{29}$$

By (26), (27) and (29) we obtain

$$\deg(I - T_2, \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}, 0) = -1.$$

Therefore, T_2 has at least three fixed points $u_1 \in \Omega_{\alpha_2}$, $u_2 \in \Omega^{\beta_1}$ and $u_3 \in \Omega \setminus \overline{\Omega_{\alpha_2} \cup \Omega^{\beta_1}}$ which are solutions of the problem (1). Then the proof is complete. □

Example 3.4. *Consider the second-order three-point boundary value problem*

$$\begin{aligned} &u''(x) + \frac{1}{e^x}(1+x)\arctan(-u(x))((u'(x))^2 + 1) = 0, \quad x \in (0, +\infty), \\ &u(0) = \frac{1}{2}\int_0^1 u(s)ds, \;\; u'(+\infty) = \frac{1}{5}, \end{aligned} \tag{30}$$

where $\lambda = \dfrac{1}{2} > 0$, $\eta = 1$ *and clearly* $0 < \lambda\eta < 1$.

Let

$$q(x) = \frac{1}{e^x}, \;\; f(x, u, v) = (1+x)\arctan(-u)(v^2 + 1).$$

Also, we notice that $C = \dfrac{1}{5}$. We take $\alpha(x) = -x - 1$, $\beta(x) = x + 1$.
Then $\alpha(x), \beta(x) \in C^2[0, +\infty)$ and $\alpha'(x) = -1, \alpha''(x) = 0, \beta'(x) = 1, \beta''(x) = 0$. Moreover, we have

$$\begin{aligned} &\alpha''(x) + q(x)f(x, \alpha(x), \alpha'(x)) = \frac{2(1+x)}{e^x}\arctan(x+1) > 0, \quad x \in (0, +\infty), \\ &\alpha(0) = -1 < \lambda\int_0^\eta \alpha(s)ds = \frac{1}{2}\int_0^1 (-s-1)ds = \frac{-3}{4}, \\ &\alpha'(+\infty) = -1 < \frac{1}{5} = C, \end{aligned}$$

and

$$\begin{aligned} &\beta''(x) + q(x)f(x, \beta(x), \beta'(x)) = \frac{2(1+x)}{e^x}\arctan(-x-1) < 0, \quad x \in (0, +\infty), \\ &\beta(0) = 1 > \lambda\int_0^\eta \beta(s)ds = \frac{1}{2}\int_0^1 (s+1)ds = \frac{3}{4}, \\ &\beta'(+\infty) = 1 > \frac{1}{5} = C. \end{aligned}$$

Thus α, β are lower and upper solutions of (30), respectively. Furthermore, $\alpha, \beta \in X$, $\alpha(x) \le \beta(x)$, $x \in [0, +\infty)$. Clearly, f is continuous, moreover, f satisfies the Naguma's condition with respect to $\alpha(x) = -x - 1$ and $\beta(x) = x + 1$; that is, when $0 \le x < +\infty$, $-x - 1 \le u \le x + 1$ and $v \in \mathbb{R}$, it holds

$$|f(x, u, v)| \le \phi(x)h(|v|),$$

where $\phi(x) = (1 + x)$ and $h(v) = \dfrac{\pi}{2}(v^2 + 1)$ and

$$\int_0^\infty \frac{s}{h(s)}ds = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}ds = +\infty.$$

Also,

$$\int_0^\infty \max\{s, 1\}q(s)ds = \int_0^1 \frac{1}{e^s}ds + \int_1^\infty \frac{s}{e^s}ds = 1 + \frac{1}{e} < +\infty,$$

$$\int_0^\infty \max\{s,1\}q(s)\phi(s)ds = \int_0^1 \frac{1+s}{e^s}ds + \int_1^\infty \frac{s+s^2}{e^s}ds = 2 + \frac{4}{e} < +\infty;$$

and for $\gamma = 2$,

$$\begin{aligned} m = \sup_{x\in[0,+\infty)} (1+x)^\gamma q(x)\phi(x) &= \sup_{x\in[0,+\infty)} (1+x)^2 \frac{1}{e^x}(1+x) \\ &= \sup_{x\in[0,+\infty)} \frac{(1+x)^3}{e^x} \\ &= \frac{(1+2)^3}{e^2} \approx 3.65 < \infty, \end{aligned}$$

that is, (12) and (13) are satisfied. Therefore, from Theorem 3.1, the boundary problem (30) has at least one solution u such that

$$\alpha(x) = -x-1 \le u(x) \le x+1 = \beta(x),\ |u'(x)| < N \text{ for all } x \in [0,+\infty),$$

where $N > \sqrt{e^{\frac{108\pi}{e^2}}(r^2+1)-1}$, $r \ge 1$ with $\gamma = 2$.

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