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Generalized Quasi Power Increasing Sequences and Their Some New Applications

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Abstract. In this paper, we generalize a known theorem by using a general class of power increasing sequences instead of a quasi-δ-power increasing sequence. This theorem also includes some known and new results.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\delta (\log n)^\gamma,\ \gamma \geq 0,\ 0 < \delta < 1\}$ (see [13]). If we take γ=0, then we get a quasi-δ-power increasing sequence. It should be noted that every almost increasing sequence is quasi-δ-power increasing sequence for any nonnegative δ, but the converse need not be true as can be seen by taking an example, say $X_n = n^{-\delta}$ for $\delta > 0$ (see [10]). We write $\mathcal{BV}_O = \mathcal{BV} \cap \mathcal{C}_O$, where $\mathcal{C}_O = \{x = (x_k) \in \Omega : \lim_k |x_k| = 0\}$, $\mathcal{BV} = \{x = (x_k) \in \Omega : \sum_k |x_k - x_{k+1}| < \infty\}$ and Ω being the space of all real or complex- valued sequences. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{3}$$

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The series $\sum a_n$ is said to be summable $\mid C,\alpha \mid_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha} - u_{n-1}^{\alpha} \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha} \mid^k < \infty. \tag{4}$$

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0, \ (n \geq 0). \tag{5}$$

The sequence-to-sequence transformation

$$V_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{6}$$

defines the sequence (V_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $\mid N, p_n \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} \mid V_n - V_{n-1} \mid^k < \infty. \tag{7}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ \alpha \geq 0 \tag{8}$$

the Nörlund mean reduces to the (C,α) mean and $\mid N, p_n \mid_k$ summability becomes $\mid C,\alpha \mid_k$ summability. Also, if we take k=1, then we get $\mid N, p_n \mid$ summability. If we take $p_n = 1$ for all values of n, then we get the $(C,1)$ mean and in this case $\mid N, p_n \mid_k$ summability becomes $\mid C, 1 \mid_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

2. Known result

The following general theorem is known dealing with absolute Nörlund summabiliy factors.

Theorem 2.1 [4] Let $p_0 > 0$, $p_n \geq 0$, (p_n) be a non-increasing sequence and $(\lambda_n) \in \mathcal{BV}_O$. Let (X_n) be a quasi-δ-power increasing sequence for some δ $(0 < \delta < 1)$ and let there be sequences (λ_n) and (β_n) such that

$$\mid \lambda_n \mid X_n = O(1), \tag{9}$$

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{10}$$

$$\beta_n \to 0, \tag{11}$$

$$\sum nX_n \mid \Delta\beta_n \mid < \infty. \tag{12}$$

If the sequence (w_n^{α}) defined by (see [12])

$$w_n^{\alpha} = \begin{cases} \left|t_n^{\alpha}\right|, & \alpha = 1, \\ \max\limits_{1\leq v\leq n} \left|t_v^{\alpha}\right|, & 0 < \alpha < 1 \end{cases} \tag{13}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{14}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$ and $0 < \alpha \leq 1$.

Remark 2. 2 We can take $(\lambda_n) \in \mathcal{BV}$ instead of $(\lambda_n) \in \mathcal{BV}_O$ and it is sufficient to prove Theorem 2.1.

3. The Main Result

The aim of this paper is to generalize Theorem 2.1 by using a quasi-f-power increasing sequence instead of a quasi-δ-power increasing sequence. Now, we shall prove the following theorems.

Theorem 3.1 Let $(\lambda_n) \in \mathcal{BV}$ and let (X_n) be a quasi-f-power increasing sequence. If the conditions (9)-(12) and (14) of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n$ is summable $\mid C, \alpha \mid_k$, $k \geq 1$ and $0 < \alpha \leq 1$.

Theorem 3.2 Let $(\lambda_n) \in \mathcal{BV}$ and let (p_n) be as in Theorem 2.1. Let (X_n) be a quasi-f-power increasing sequence. If the conditions (9)-(12) and (14) of Theorem 2.1 are satisfied, then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$ and $0 < \alpha \leq 1$.

We need the following lemmas for the proof of our theorem.

Lemma 3.3 [5] Except for the condition $(\lambda_n) \in \mathcal{BV}$, under the conditions on (X_n), (β_n) and (λ_n) as as expressed in the statement of Theorem 2.1, we have the following

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty, \tag{15}$$

$$n X_n \beta_n = O(1). \tag{16}$$

Lemma 3.4 [7] If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{17}$$

Lemma 3.5 [11] If $-1 < \alpha \leq \sigma$, $k > 1$ and the series $\sum a_n$ is summable $\mid C, \alpha \mid_k$, then it is also summable $\mid C, \sigma \mid_k$. The case k =1 of Lemma 3.5 is due to Kogbetliantz (see [9]). The case $k > 1$ is a special case of the theorem of Flett (see [8], Theorem 1).

Lemma 3.6 [14] Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C, 1 \mid_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$.

4. Proof of Theorem 3.1 Let (T_n^{α}) be the nth (C, α), with $0 < \alpha \leq 1$, mean of the sequence $(n a_n \lambda_n)$. Then, by (2), we find that

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{18}$$

Thus, by first applying Abel's transformation and then using Lemma 3.4, we have that

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

$$
\begin{aligned}
\mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{\mid \lambda_n \mid}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\
&\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid w_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha.
\end{aligned}
$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{-1} \mid T_{n,r}{}^{\alpha} \mid^k < \infty \quad for \quad r = 1, 2.
$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{-1} \mid T_{n,1}^\alpha \mid^k &\leq \sum_{n=2}^{m+1} n^{-1} (A_n^\alpha)^{-k} \left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \mid \Delta\lambda_v \mid\right\}^k \\
&\leq \sum_{n=2}^{m+1} n^{-1} n^{-\alpha k} \left\{\sum_{v=1}^{n-1} v^{\alpha k} (w_v^\alpha)^k \mid \Delta\lambda_v \mid\right\} \times \left\{\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \beta_v \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k+1}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \beta_v \int_v^\infty \frac{dx}{x^{\alpha k+1}} \\
&= O(1) \sum_{v=1}^{m} v \beta_v \frac{(w_v^\alpha)^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{(w_r^\alpha)^k}{r} \\
&+ O(1) m\beta_m \sum_{v=1}^{m} \frac{(w_v^\alpha)^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 3.3. Again, we have that

$$\begin{aligned}\sum_{n=1}^{m} n^{-1} \mid T_{n,2}^{\alpha} \mid^{k} &= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid \frac{(w_n^{\alpha})^k}{n} \\ &= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{(w_v^{\alpha})^k}{v} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} \\ &= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\ &= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 3.3. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2 In order to prove Theorem 3.2, we need consider only the special case in which (N, p_n) is (C, α). Therefore, Theorem 3.2 will then follow by means of Theorem 3.1, Lemma 3.5 (for $\sigma = 1$) and Lemma 3.6. If we take γ=0, then we get Theorem 2.1. Also if take γ=0 and $\alpha = 1$, then we obtain a known result which was proved in [2]. If we take γ=0, $\alpha = 1$ and k=1, then we obtain another result concerning the $\mid N, p_n \mid$ summability (see [3]). Finally, if we take k=1, then we get a new result dealing with absolute Nörlud summability factors of infinite series.

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